ON SUBALGEBRA LATTICES OF A FINITE UNARY ALGEBRA I

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Abstract. One of the main aims of the present and the next part [15] is to show that the theory of graphs (its language and results) can be very useful in algebraic investigations. We characterize, in terms of isomorphisms of some digraphs, all pairs $\langle \mathcal{A}, \mathcal{L}\rangle$, where $\mathcal{A}$ is a finite unary algebra and $\mathcal{L}$ a finite lattice such that the subalgebra lattice of $\mathcal{A}$ is isomorphic to $\mathcal{L}$. Moreover, we find necessary and sufficient conditions for two arbitrary finite unary algebras to have isomorphic subalgebra lattices. We solve these two problems in the more general case of partial unary algebras.

In the next part [15] we will use these results to describe connections between various kinds of lattices of (partial) subalgebras of a finite unary algebra.

Keywords: unary algebra, partial algebra, subalgebra lattice, directed graph

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One of the important concerns of universal algebra and the theory of partial algebras is the connection between algebras and their subalgebra lattices. For instance, characterizations of subalgebra lattices for algebras in a given variety or of a given type are this kind of problems (see e.g. [10]). Moreover, several results (see e.g. [6], [17], [18]) describe algebras or varieties of algebras which have special subalgebra lattices (i.e. modular, distributive, etc.). For example, T. Evans and B. Ganter proved in [6] that an arbitrary subalgebra modular variety (i.e. a variety in which every algebra has a modular subalgebra lattice) is Hamiltonian (i.e. any subalgebra is a congruence class of a suitable congruence). Hence and by [11], it is Abelian. Moreover, J. Shapiro showed in [17] that every subalgebra distributive variety (i.e. each of its algebras has a distributive subalgebra lattice) is strongly Abelian. Note that some of such results concern also classical algebras—Boolean algebras, groups, modules (see e.g. [12], [16] or [7], [8]). For example, D. Sachs showed in [16] that two Boolean algebras are isomorphic iff their lattices of subalgebras are isomorphic. E. Lukács and P. P. Pálfy proved in [12] that the modularity of the subgroup lattice of the direct square of any group implies that $\mathcal{G}$ is commutative.

In the present part, and also in the next [15], we show that the theory of directed and undirected graphs can be very useful in algebraic investigations of connections between finite unary algebras and their subalgebra lattices. We do not restrict our attention to total algebras only, and we consider also the more general case of partial algebras. More precisely, we use some results of graph theory and also some of universal algebra to characterize, in terms of isomorphisms of some digraphs, all pairs $\langle \mathcal{A}, \mathcal{L} \rangle$, where $\mathcal{A}$ is a finite partial unary algebra and $\mathcal{L}$ a finite lattice, such that the strong subalgebra lattice of $\mathcal{A}$ is isomorphic to $\mathcal{L}$. Moreover, we find necessary and sufficient conditions for two arbitrary finite partial unary algebras to have isomorphic strong subalgebra lattices. Although in this part we use only the classical kind of subalgebras they will be called strong as opposed to the other kinds of partial subalgebras considered in the theory of partial algebras (see [3] or [5]).

In the next part [15] we will apply the solutions of these problems to describe connections between lattices of weak, relative, strong subalgebras and initial segments of the finite unary partial algebra.

For basic notions concerning partial algebras see e.g. [3] or [5], concerning (total) algebras and lattices of subalgebras see e.g. [10], concerning *digraphs* (i.e. directed graphs) and (undirected) graphs see e.g. [4]. For any partial unary algebra $\mathcal{A} = \langle A, (k^{\mathcal{A}})_{k \in K} \rangle$, the complete and algebraic lattice of all strong subalgebras of $\mathcal{A}$ under (strong subalgebra) inclusion $\leqslant_s$ will be denoted by $\mathcal{S}_s(\mathcal{A})$. For any digraph $\mathcal{D}$, by $V^{\mathcal{D}}$ and $E^{\mathcal{D}}$ we denote its sets of vertices and edges, respectively. In the present paper we investigate only finite algebras. Since we use digraphs to represent partial unary algebras, all digraphs considered are also finite, i.e. the sets of vertices and edges are finite. Recall (see [2]) that each partial unary algebra $\mathcal{A} = \langle A, (k^{\mathcal{A}})_{k \in K} \rangle$ can be represented by a digraph $\mathcal{D}(\mathcal{A})$ obtained from $\mathcal{A}$ by omitting the names of all operations. More formally, A is the set of all vertices of $\mathcal{D}(\mathcal{A})$, $\big\{\langle a, k, b \rangle \in A \times K \times A \colon \langle a, b \rangle \in k^{\mathcal{A}}\big\}$ is the set of all edges of $\mathcal{D}(\mathcal{A})$, and for each edge $\langle a, k, b \rangle$, a is its initial vertex and b is its final vertex. Note that this construction is a very particular case of the Grothendieck construction (see [1], section 4.2 and 11.2), but applied to models of digraphs (in the category of sets and partial functions) rather than to functors.

In [14] we defined a special kind of subdigraphs which correspond to strong subalgebras, and therefore we called them strong. More precisely, a digraph $\mathcal{H}$ is a strong subdigraph of a digraph $\mathcal{D}$ ($\mathcal{H} \leqslant_s \mathcal{D}$) iff $\mathcal{H}$ is an ordinary subdigraph of $\mathcal{D}$ and for each edge e of $\mathcal{D}$, if the initial vertex of e belongs to $\mathcal{H}$, then e belongs to $\mathcal{H}$, in particular, the final vertex of e also belongs to $\mathcal{H}$. It is easy to see that for any two strong subdigraphs $\mathcal{H}$ and $\mathcal{K}$ of $\mathcal{D}$, they are equal iff $V^{\mathcal{H}} = V^{\mathcal{K}}$; $\mathcal{H}$ is a strong subdigraph of $\mathcal{K}$ iff $V^{\mathcal{H}} \subseteq V^{\mathcal{K}}$. It is also easy to verify that for a partial unary algebra $\mathcal{A}$ and its strong subalgebra $\mathcal{B}$, the digraph $\mathcal{D}(\mathcal{B})$ representing $\mathcal{B}$ is indeed a strong subdigraph of $\mathcal{D}(\mathcal{A})$. Moreover, it can be shown, in a similar way as for partial unary algebras (see also [14], where the precise proof is given) that the set of all strong subdigraphs of a digraph $\mathcal{D}$ forms a complete and algebraic lattice $\mathcal{S}_s(\mathcal{D})$ under (strong subdigraph) inclusion $\leqslant_s$. In the same paper we proved the following result ($\simeq$ denotes isomorphisms of algebras, digraphs, graphs, lattices, etc.):

Theorem 1. *For each partial unary algebra $\mathcal{A}$, $\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{S}_s\big(\mathcal{D}(\mathcal{A})\big)$.*

Proof. The proof is obtained by a verification that the function assigning to each strong subalgebra $\mathcal{B}$ of $\mathcal{A}$ its digraph $\mathcal{D}(\mathcal{B})$, which is a strong subdigraph of $\mathcal{D}(\mathcal{A})$, forms this isomorphism, i.e. it is a bijection and preserves the lattice ordering $\leqslant_s$ of $\mathcal{S}_s(\mathcal{A})$ and $\mathcal{S}_s\big(\mathcal{D}(\mathcal{A})\big)$. $\square$

Let $\mathcal{D}$ be a digraph and $W \subseteq V^{\mathcal{D}}$ a subset of vertices. The *contraction* of W (see [4], chapter 3) is an operation on $\mathcal{G}$ defined as follows:

(a) W is replaced by a single point which will be denoted often by $\overline{w}$,

(b) all directed edges with endpoints in W are replaced by a single loop in $\overline{w}$,

(c) each directed edge going into W (or out of W) is replaced by the directed edge with the same initial vertex (or final vertex) ending in $\overline{w}$ (or starting from $\overline{w}$).

The digraph obtained from $\mathcal{D}$ by the contraction of W will be denoted by $\mathcal{D}/W$. We will also use the convention that $\mathcal{D}/\emptyset\colon = \mathcal{D}$. Obviously, if $\mathcal{D}$ is connected, then $\mathcal{D}/W$ is also connected. Moreover, by simple verification we obtain that for each subdigraph $\mathcal{H}$ of $\mathcal{D}$, $\mathcal{H}/(V^{\mathcal{H}} \cap W)$ is a subdigraph of $\mathcal{D}/W$.

The contraction of a set of vertices in a digraph need not preserve the strong subdigraph lattice. For instance, from every non-empty digraph we can obtain a trivial digraph, i.e. with exactly one vertex, by contracting the set of all vertices. It is also easy to show that this operation even need not preserve strong subdigraphs.

Now we prove the first important fact which shows that for a special kind of sets of vertices, the contraction of these sets preserves the strong subdigraph lattices. Recall first (see [4], chapter 3) that a digraph $\mathcal{D}$ is *strongly connected* iff for any two distinct vertices v, w, there is a path going from v to w. Secondly (see also [4]), $\mathcal{D}$ is strongly connected iff $\mathcal{D}$ is connected and every edge lies on a cycle. Here we assume that no path encounters the same vertex twice, and analogously for a cycle except its first and its last vertex which coincide. Each loop forms a cycle, called trivial.

Next, for each digraph $\mathcal{D}$ and its subset $W \subseteq V^{\mathcal{D}}$, by $[W]_{\mathcal{D}}$ we denote the subdigraph spanned on W, i.e. W is its set of vertices and all edges of $\mathcal{D}$ with endpoints in W form its set of edges.

Theorem 2. *Let $\mathcal{D}$ be a digraph and let $W \subseteq V^{\mathcal{D}}$ be a set such that $[W]_{\mathcal{D}}$ is strongly connected. Then*

$$\mathcal{S}_s(\mathcal{D}) \simeq \mathcal{S}_s(\mathcal{D}/W).$$

Proof. Observe that for each strong subdigraph $\mathcal{H} \leqslant_s \mathcal{D}$, $\mathcal{H}$ and W are disjoint or $\mathcal{H}$ contains W. Indeed, if v is a common vertex of $\mathcal{H}$ and W, then for each $w \in W \setminus \{v\}$ there is a path $(e_1, \ldots, e_n)$ going from v to w. Using a simple induction we obtain that $e_1, \ldots, e_n$ belong to $\mathcal{H}$, so $w \in V^{\mathcal{H}}$.

Further, by simple verification we obtain that for each $\mathcal{H} \leqslant_s \mathcal{D}$, if W and $\mathcal{H}$ are disjoint, then $\mathcal{H}$ is a strong subdigraph of $\mathcal{D}/W$. On the other hand, if W is contained in $\mathcal{H}$, then $\mathcal{H}/W$ is a strong subdigraph of $\mathcal{D}/W$.

The above facts imply that

$$\varphi(\mathcal{H}) = \mathcal{H}/(V^{\mathcal{H}} \cap W) \text{ for each } \mathcal{H} \leqslant_s \mathcal{D}$$

is a well-defined function of the set of all strong subdigraphs of $\mathcal{D}$ into the set of all strong subdigraphs of $\mathcal{D}/W$. Of course, $\varphi(\mathcal{H}) = \mathcal{H}$ or $\varphi(\mathcal{H}) = \mathcal{H}/W$.

Take $\mathcal{H}, \mathcal{K} \leqslant_s \mathcal{D}$ and assume $\varphi(\mathcal{H}) = \varphi(\mathcal{K})$. Let $\overline{w}$ be the vertex of $\mathcal{D}/W$ corresponding to W. If $\overline{w}$ does not belong to $\varphi(\mathcal{H})$, then W is disjoint with $\mathcal{H}$ and $\mathcal{K}$, so $\mathcal{H} = \mathcal{K}$. If $\overline{w}$ belongs to $\varphi(\mathcal{H})$, then W must have common vertices with $\mathcal{H}$ and $\mathcal{K}$. Hence, W is contained in $\mathcal{H}$ and in $\mathcal{K}$. This fact easily implies that the sets of vertices of these digraphs coincide. Because they are strong subdigraphs, it follows that $\mathcal{H} = \mathcal{K}$. Thus φ is an injection.

Take a strong subdigraph $\mathcal{K}$ of $\mathcal{D}/W$. If $\overline{w}$ does not belong to $\mathcal{K}$, then obviously $\mathcal{K}$ is also a strong subdigraph of $\mathcal{D}$ and $\varphi(\mathcal{K}) = \mathcal{K}$. Assume that $\overline{w}$ belongs to $\mathcal{K}$ and take $[U]_{\mathcal{D}}$, where U is the set of all vertices of $\mathcal{K}$ without $\overline{w}$ and all vertices of W, i.e. $U = \big(V^{\mathcal{K}} \setminus \{\overline{w}\}\big) \cup W$. Since $\mathcal{K} \leqslant_s \mathcal{D}/W$, it is easy to show that $[U]_{\mathcal{D}}$ is a strong subdigraph of $\mathcal{D}$. Hence, $[U]_{\mathcal{D}}/W$ is also a strong subdigraph of $\mathcal{D}/W$. This implies $[U]_{\mathcal{D}}/W = \mathcal{K}$, because their vertex sets are, of course, equal. Thus φ is also surjective.

Now we must only prove that φ and its inverse preserve the inclusion $\leqslant_s$. Take $\mathcal{H}, \mathcal{K} \leqslant_s \mathcal{D}$ and recall that $\mathcal{H} \leqslant_s \mathcal{K}$ iff $V^{\mathcal{H}} \subseteq V^{\mathcal{K}}$. Analogously $\varphi(\mathcal{H}) \leqslant_s \varphi(\mathcal{K})$ iff $V^{\varphi(\mathcal{H})} \subseteq V^{\varphi(\mathcal{K})}$. Thus it is enough to show that the vertex set of $\mathcal{H}$ is contained in $\mathcal{K}$ iff $\varphi(\mathcal{K})$ contains the vertex set of $\varphi(\mathcal{H})$. This fact easily follows from the definition of the contraction, because each of these two digraphs contains W or is disjoint with W. Thus φ is the desired lattice isomorphism. $\square$

Note (see [4], chapter 3) that for a digraph $\mathcal{D}$, *strongly connected components* can be considered. More precisely, for each vertex $v \in V^{\mathcal{D}}$ we take all vertices w such that $w = v$ or there is a path going from v to w and also a path going from w to v.

Next we take the subdigraph spanned on this set. Obviously these subdigraphs are pairwise disjoint and strongly connected and maximal.

Let $\mathcal{D}$ be a finite digraph and $W_1, W_2, \ldots, W_n$ the family of all non-trivial strongly connected components of $\mathcal{D}$. Then we can consider the digraph $\mathcal{T}(\mathcal{D})$ obtained from $\mathcal{D}$ by contracting these sets. More precisely, we first contract W_1 (then, of course, $W_2, \ldots, W_n$ are all non-trivial strongly connected components of $\mathcal{D}/W_1$). Secondly, we contract W_2 in $\mathcal{D}/W_1$, and so on. We repeat this procedure n times to obtain our digraph. It is easy to see that the order of the contraction of these sets is not important, so $\mathcal{T}(\mathcal{D})$ is uniquely determined.

It is well-known (see [4], chapter 3) that $\mathcal{T}(\mathcal{D})$ has no non-trivial cycles. By Theorem 2 we also have that for each $1 \leqslant i \leqslant n-1$,

$$\mathcal{S}_s\big((\ldots(\mathcal{D}/W_1)/\ldots)/W_i\big) \simeq \mathcal{S}_s\big(((\ldots(\mathcal{D}/W_1)/\ldots)/W_i\big)/W_{i+1}\big),$$

in particular we obtain $\mathcal{S}_s(\mathcal{D}) \simeq \mathcal{S}_s\big(\mathcal{T}(\mathcal{D})\big)$.

Summarizing, for a given finite digraph $\mathcal{D}$ we have constructed the digraph $\mathcal{T}(\mathcal{D})$ without non-trivial cycles but with the same strong subdigraph lattice. Now we show that this digraph can be simplified further. We start with the following auxiliary

Lemma 3. *Let $\mathcal{D}$ be a finite and simple digraph without cycles. Then for every edge e, there is a path $(f_1, f_2, \ldots, f_n)$ going from the initial vertex of e to the final vertex of e and $f_1, \ldots, f_n$ are isthmi.*

Recall that a digraph is simple iff it has not loops and for any two distinct vertices v, w there is at most one edge from v to w. Moreover, (see [4]) an edge e is an *isthmus* iff e is regular (i.e. e is not a loop) and e is the only path from its initial vertex to its final vertex.

P r o o f. Take an edge e and let v, w be its initial and its final vertex, respectively. Note that each edge is regular, because $\mathcal{D}$ is simple. Observe that, because e forms a one-element path from v to w, the family of all paths going from v to w is non-empty. It is also finite, because $\mathcal{D}$ is a finite digraph. Thus we can choose in this family a path $p = (f_1, \ldots, f_n)$ with maximal length. Assume now that for some $1 \leqslant i \leqslant n$, f_i is not an isthmus. Then there is a path $(g_1, \ldots, g_k)$ going from the initial vertex of f_i to the final vertex of f_i. First, this path has at least two edges, because $\mathcal{D}$ is simple. Secondly, $(f_1, \ldots, f_{i-1}, g_1, \ldots, g_k, f_{i+1}, \ldots, f_n)$ is a path, since $\mathcal{D}$ has no cycles. Thus there is a path going from v to w with length greater than p. This contradiction shows that $f_1, \ldots, f_n$ are isthmi. $\square$

Lemma 4. *Let $\mathcal{D}$ be a finite and simple digraph without cycles and let $\mathcal{H}$ be its subdigraph consisting of all vertices of $\mathcal{D}$ and all isthmi of $\mathcal{D}$. Then $\mathcal{S}_s(\mathcal{H}) \simeq \mathcal{S}_s(\mathcal{D})$.*

P r o o f. Observe first that, because $\mathcal{H}$ is a subdigraph of $\mathcal{D}$, for each strong subdigraph $\mathcal{K} \leqslant_s \mathcal{D}$, $[V^{\mathcal{K}}]_{\mathcal{H}}$ is a strong subdigraph of $\mathcal{H}$.

Secondly, if $\mathcal{K}$ is a strong subdigraph of $\mathcal{H}$, then $[V^{\mathcal{K}}]_{\mathcal{D}}$ is a strong subdigraph of $\mathcal{D}$. To see this take an edge e of $\mathcal{D}$ starting from $V^{\mathcal{K}}$. Then by Lemma 3 there is a path $(f_1, \ldots, f_n)$ of isthmi in $\mathcal{D}$ (thus, in particular, $f_1, \ldots, f_n$ belong to $\mathcal{H}$) going from the initial vertex of e to the final vertex of e. Hence, using a simple induction and the fact that $\mathcal{K}$ is a strong subdigraph of $\mathcal{H}$, we obtain that this path belong to $\mathcal{K}$. Thus, in particular, the final vertex of e belongs to $V^{\mathcal{K}}$, so e is an edge of $[V^{\mathcal{K}}]_{\mathcal{D}}$.

Having the above facts it is sufficient to assign the subdigraph $[V^{\mathcal{K}}]_{\mathcal{H}}$ of $\mathcal{H}$ to each $\mathcal{K} \leqslant_s \mathcal{D}$ to obtain a well-defined function from the set of all strong subdigraphs of $\mathcal{D}$ into the set of all strong subdigraphs of $\mathcal{H}$. Moreover, the function assigning $[V^{\mathcal{K}}]_{\mathcal{D}}$ to each $\mathcal{K} \leqslant_s \mathcal{H}$ is its inverse. It is also easy to see that these two functions preserve the strong subdigraph inclusion $\leqslant_s$. This completes the proof. □

Let $\mathcal{D}$ be an arbitrary finite digraph and take $\mathcal{T}(\mathcal{D})$. Observe first that we can remove all loops in $\mathcal{T}(\mathcal{D})$. The digraph $\mathcal{H}$ so obtained has its strong subdigraph lattice isomorphic to $\mathcal{S}_s(\mathcal{T}(\mathcal{D}))$, and consequently, isomorphic to $\mathcal{S}_s(\mathcal{D})$. $\mathcal{H}$ has no cycles and is finite.

Secondly, let $\mathcal{K}$ be a simple digraph containing all vertices of $\mathcal{H}$ and for any two distinct vertices v and w, let there be an edge in $\mathcal{K}$ from v to w iff there is an edge in $\mathcal{H}$ from v to w. Then the strong subdigraph lattices of $\mathcal{H}$ and $\mathcal{K}$ are isomorphic. Note that this isomorphism is given by assigning to each $\mathcal{M} \leqslant_s \mathcal{H}$ the digraph $[V^{\mathcal{M}}]_{\mathcal{K}}$ which is, of course, a strong subdigraph of $\mathcal{K}$. Hence, $\mathcal{S}_s(\mathcal{K}) \simeq \mathcal{S}_s(\mathcal{D})$.

Thirdly, take the subdigraph of $\mathcal{K}$ consisting of all its vertices and all its isthmi. This digraph (observe that it is constructed from $\mathcal{D}$ in four steps) will be denoted by $\mathrm{T{\small IS}}(\mathcal{D})$.

By the above definitions, results and Lemma 4 we obtain the following important

Theorem 5. *Let $\mathcal{D}$ be a finite digraph. Then $\mathcal{S}_s(\mathcal{D}) \simeq \mathcal{S}_s(\mathrm{T{\small IS}}(\mathcal{D}))$.*

Now take a finite partial unary algebra $\mathcal{A}$ and set $\mathrm{T{\small IS}}(\mathcal{A})\colon = \mathrm{T{\small IS}}(\mathcal{D}(\mathcal{A}))$. Theorems 1 and 5 imply

Corollary 6. *For each finite partial unary algebra $\mathcal{A}$, $\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{S}_s(\mathrm{T{\small IS}}(\mathcal{A}))$.*

Now we show that $\mathrm{T{\small IS}}(\mathcal{A})$ uniquely determines the strong subalgebra lattice $\mathcal{S}_s(\mathcal{A})$ for any finite algebra $\mathcal{A}$. Recall (see [9] or [10]) that a lattice $\mathcal{L} = \langle L, \leqslant_{\mathcal{L}} \rangle$ is isomorphic to the strong subalgebra lattice for some partial unary algebra iff $\mathcal{L}$ is algebraic, distributive and every element of $\mathcal{L}$ is the join of completely join-irreducible elements. $l \in L$ is *completely join-irreducible* iff for any subset K of $\mathcal{L}$, $l = \bigvee K$ implies $l \in K$; l is *join-irreducible*, if the condition holds for all two-element subsets K. A partial unary algebra $\mathcal{A} = \langle (A, (k^{\mathcal{A}})_{k\in K}\rangle$ such that $\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{L}$ can be constructed (see also [10]) as follows: the set $CI(\mathcal{L})$ of all completely join-irreducible elements of $\mathcal{L}$ is its carrier, i.e. $A = CI(\mathcal{L})$, and $CI(\mathcal{L})$ is its unary type, i.e. $K = CI(\mathcal{L})$, and for all $a \in A$, $k \in K$, if $k \leqslant_{\mathcal{L}} a$ and $k \neq a$, then $k^{\mathcal{A}}(a) = k$; otherwise the unary partial operation $k^{\mathcal{A}}$ on a is not defined. Of course, $\mathcal{A}$ can be completed to a total algebra $\overline{\mathcal{A}}$ of type K (for $a \in A$ and $k \in K$, if k is not less than a, we set $k^{\mathcal{A}}(a) = a$), but then the digraph corresponding to this algebra has loops. Note that $\mathcal{D}(\mathcal{A})$ is a simple digraph and has no cycles.

Obviously if $\mathcal{L}$ is finite, then $\mathcal{A}$ is also finite. It is also well-known that a finite lattice is algebraic and each of its elements is the join of completely join-irreducible elements. Thus *a lattice $\mathcal{L}$ is isomorphic to the strong subalgebra lattice for some finite unary algebra iff $\mathcal{L}$ is finite and distributive.*

With each finite and distributive lattice $\mathcal{L} = \langle L, \leqslant_{\mathcal{L}}\rangle$ we can associate a digraph $\mathcal{D}(\mathcal{L})$ in the following way: We first consider the finite partial unary algebra $\mathcal{A}$ defined above. Secondly, we take $\mathcal{D}(\mathcal{A})$. Next, $\mathcal{D}(\mathcal{L})$ is the subdigraph of $\mathcal{D}(\mathcal{A})$ consisting of all vertices and all isthmi. Observe also that $\mathcal{D}(\mathcal{A})$ can be constructed from $\mathcal{L}$ as follows: $CI(\mathcal{L})$ is its set of all vertices, and $\{\langle p, q\rangle \in CI(\mathcal{L}) \times CI(\mathcal{L})\colon\ q \leqslant_{\mathcal{L}} p\}$ is its set of all edges, and for any edge $\langle p, q\rangle$, p is its initial vertex and q is its final vertex. Moreover, an edge $\langle p, q\rangle$ is an isthmus in $\mathcal{D}(\mathcal{A})$ iff $q \prec p$, where $\prec$ is the covering relation on $CI(\mathcal{L})$, i.e. $q \prec p$ iff $q \leqslant_{\mathcal{L}} p$ and for any $z \in CI(\mathcal{L})$, $q \leqslant_{\mathcal{L}} z \leqslant_{\mathcal{L}} p$ implies $z = q$ or $z = p$. These two facts imply that the digraph $\mathcal{D}(\mathcal{L})$ can be constructed from $\mathcal{L}$ as follows: $CI(\mathcal{L})$ is its set of all vertices, and the set of all pairs $\langle p, q\rangle$ such that $p, q \in CI(\mathcal{L})$ and $q \prec p$ is its set of all edges, and for each edge $\langle p, q\rangle$ of $\mathcal{D}(\mathcal{L})$, p is its initial vertex and q is its final vertex. Note also that in the case of finite lattices $CI(\mathcal{L})$ is just the set of all non-zero join-irreducible elements.

Theorem 7. *For each finite and distributive lattice $\mathcal{L}$, $\mathcal{S}_s(\mathcal{D}(\mathcal{L})) \simeq \mathcal{L}$.*

Proof. Take the algebra $\mathcal{A}$ constructed above such that $\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{L}$. Then $\mathcal{S}_s(\mathcal{D}(\mathcal{A})) \simeq \mathcal{L}$ by Theorem 1. By Lemma 4 we deduce $\mathcal{S}_s(\mathcal{D}(\mathcal{L})) \simeq \mathcal{S}_s(\mathcal{D}(\mathcal{A}))$, because $\mathcal{D}(\mathcal{A})$ is simple and without cycles. This two facts complete the proof. □

Proposition 8. *Let $\mathcal{K}$ and $\mathcal{L}$ be finite and distributive lattices. Then*

$$\mathcal{K} \simeq \mathcal{L} \text{ iff } \mathcal{D}(\mathcal{K}) \simeq \mathcal{D}(\mathcal{L}).$$

Proof. Observe that if two finite and distributive lattices $\mathcal{K}$ and $\mathcal{L}$ are isomorphic, then also their digraphs $\mathcal{D}(\mathcal{K})$ and $\mathcal{D}(\mathcal{L})$ also isomorphic. This follows from the facts that non-zero join-irreducible elements and the covering relation on the set of such elements are preserved by a lattice isomorphism. On the other hand, isomorphic digraphs have isomorphic strong subdigraph lattices. Thus by Theorem 7 we obtain $\mathcal{K} \simeq \mathcal{S}_s(\mathcal{D}(\mathcal{K})) \simeq \mathcal{S}_s(\mathcal{D}(\mathcal{L})) \simeq \mathcal{L}$. $\square$

Now we prove an important technical fact needed in the sequel.

Lemma 9. *Let $\mathcal{D}$ be a finite digraph without cycles such that each of its edges is isthmus. Then*

$$\mathcal{D}(\mathcal{S}_s(\mathcal{D})) \simeq \mathcal{D},$$

i.e. the digraph obtained from the strong subdigraph lattice of $\mathcal{D}$ is isomorphic to $\mathcal{D}$.

Proof. First, since $\mathcal{S}_s(\mathcal{D})$ is a complete lattice, for every set $W \subseteq V^{\mathcal{D}}$ there is the least strong subdigraph containing W which will be denoted by $\langle W\rangle^s_{\mathcal{D}}$. It is a simple graph-theoretical generalization of the classical result on the generation of strong subalgebras (its precise proof is given in [14]) that any vertex v of $\mathcal{D}$ belongs to $\langle W\rangle^s_{\mathcal{D}}$ iff $v \in W$ or there is a path going from W to v. Hence, in particular, if there is a path going from v to u, then the vertex set of $\langle u\rangle^s_{\mathcal{D}}$ is contained in $\langle v\rangle^s_{\mathcal{D}}$. Since they are strong subdigraphs, it follows that $\langle u\rangle^s_{\mathcal{D}} \leqslant_s \langle v\rangle^s_{\mathcal{D}}$.

Secondly, in the same way as for unary (total) algebras (see e.g. [10]) we obtain that a strong subdigraph $\mathcal{H}$ of $\mathcal{D}$ is a completely join-irreducible element of $\mathcal{S}_s(\mathcal{D})$ iff $\mathcal{H} = \langle v\rangle^s_{\mathcal{D}}$ for some vertex of $\mathcal{D}$.

Thirdly, take two vertices v and w of $\mathcal{D}$ and assume that $\langle v\rangle^s_{\mathcal{D}} = \langle w\rangle^s_{\mathcal{D}}$. Then $v = w$ or there is a path going from v to w and a path going from w to v, but the latter case implies that $\mathcal{D}$ has cycles, which is impossible. Thus $v = w$.

Let v and w be vertices of $\mathcal{D}$ such that there is an edge going from v to w. Then $\langle w\rangle^s_{\mathcal{D}} \leqslant_s \langle v\rangle^s_{\mathcal{D}}$, because this edge forms a path from v to w. Assume that $\langle w\rangle^s_{\mathcal{D}} \leqslant_s \langle u\rangle^s_{\mathcal{D}} \leqslant_s \langle v\rangle^s_{\mathcal{D}}$ for some vertex u different from w and v. Then there is a path going from v to u and a path going from u to w. Since $\mathcal{D}$ has no cycles, these two paths form another path, with at least three elements, from v to w. But this is impossible, because each edge of $\mathcal{D}$ is an isthmus. Thus $u = w$ or $u = v$, which implies that $\langle v\rangle^s_{\mathcal{D}}$ covers $\langle w\rangle^s_{\mathcal{D}}$, of course, in the set $CI(\mathcal{S}_s(\mathcal{D}))$ of all completely join-irreducible elements of $\mathcal{S}_s(\mathcal{D})$.

Now take two completely join-irreducible elements $\langle v\rangle^s_{\mathcal{D}}$ and $\langle w\rangle^s_{\mathcal{D}}$ of $\mathcal{S}_s(\mathcal{D})$ such that $\langle w\rangle^s_{\mathcal{D}}$ is covered by $\langle v\rangle^s_{\mathcal{D}}$, again in $CI(\mathcal{S}_s(\mathcal{D}))$. In particular, w belongs to $\langle v\rangle^s_{\mathcal{D}}$, so there is a path going from v to w. Assume that this path has a vertex u different from v and w. Then there is a path going from v to u and a path going from u to w, so $\langle w\rangle^s_{\mathcal{D}} \leqslant_s \langle u\rangle^s_{\mathcal{D}} \leqslant_s \langle v\rangle^s_{\mathcal{D}}$ and these three strong subdigraphs of $\mathcal{D}$ are pairwise distinct. This contradiction proves that this path is an edge from v to w.

Summarizing, the function

$$\varphi(v) = \langle v\rangle^s_{\mathcal{D}} \quad \text{for each } v \in V^{\mathcal{D}}$$

is a bijection of $V^{\mathcal{D}}$ onto the set of all completely join-irreducible elements of $\mathcal{S}_s(\mathcal{D})$. Moreover, for any two different vertices v, w, there is an edge in $\mathcal{D}$ from v to w iff there is an edge in $\mathcal{D}(\mathcal{S}_s(\mathcal{D}))$ from $\langle v\rangle^s_{\mathcal{D}}$ to $\langle w\rangle^s_{\mathcal{D}}$, i.e. $\langle v\rangle^s_{\mathcal{D}}$ covers $\langle w\rangle^s_{\mathcal{D}}$. Hence, φ induces the required digraph isomorphism, because $\mathcal{D}$ and $\mathcal{D}(\mathcal{S}_s(\mathcal{D}))$ are simple digraphs. $\square$

Theorem 10. *Let $\mathcal{D}$ be a finite digraph. Then $\mathcal{D}(\mathcal{S}_s(\mathcal{D})) \simeq \mathrm{T{\scriptsize IS}}(\mathcal{D})$.*

P r o o f. By Theorem 5 $\mathcal{S}_s(\mathcal{D})$ and $\mathcal{S}_s(\mathrm{T{\scriptsize IS}}(\mathcal{D}))$ are isomorphic, so $\mathcal{D}(S_s(\mathcal{D}))$ and $\mathcal{D}(\mathcal{S}_s(\mathrm{T{\scriptsize IS}}(\mathcal{D})))$ are isomorphic. Thus by Lemma 9 we obtain our assertion. $\square$

Corollary 11. *For each finite partial unary algebra $\mathcal{A}$, $\mathcal{D}(\mathcal{S}_s(\mathcal{A})) \simeq \mathrm{T{\scriptsize IS}}(\mathcal{A})$.*

P r o o f. By Theorems 1 and 10 we have $\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{S}_s(\mathcal{D}(\mathcal{A}))$ and $\mathcal{D}(\mathcal{S}_s(\mathcal{D}(\mathcal{A}))) \simeq \mathrm{T{\scriptsize IS}}(\mathcal{D}(\mathcal{A}))$. Hence, $\mathcal{D}(\mathcal{S}_s(\mathcal{A})) \simeq \mathrm{T{\scriptsize IS}}(\mathcal{D}(\mathcal{A})) = \mathrm{T{\scriptsize IS}}(\mathcal{A})$. $\square$

Having the above results we can formulate our two main algebraic results:

Theorem 12. *Let $\mathcal{A}$ and $\mathcal{B}$ be finite partial unary algebras (which can be of different types). Then*

$$\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{S}_s(\mathcal{B}) \ \text{ iff } \ \mathrm{T{\scriptsize IS}}(\mathcal{A}) \simeq \mathrm{T{\scriptsize IS}}(\mathcal{B}).$$

P r o o f. By Proposition 8 and Corollary 11, we have the following two equivalences: $\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{S}_s(\mathcal{B})$ iff $\mathcal{D}(\mathcal{S}_s(\mathcal{A})) \simeq \mathcal{D}(\mathcal{S}_s(\mathcal{B}))$ iff $\mathrm{T{\scriptsize IS}}(\mathcal{A}) \simeq \mathrm{T{\scriptsize IS}}(\mathcal{B})$. Hence we obtain our result. $\square$

Theorem 13. *Let $\mathcal{A}$ be a finite partial unary algebra and $\mathcal{L}$ a finite and distributive lattice. Then*

$$\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{L} \text{ iff } \mathrm{T{\scriptsize IS}}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{L}).$$

P r o o f. By Proposition 8 and Corollary 11, $\mathcal{S}_s(\mathcal{A}) \simeq \mathcal{L}$ iff $\mathcal{D}(\mathcal{S}_s(\mathcal{A})) \simeq \mathcal{D}(\mathcal{L})$ iff $\mathrm{T{\scriptsize IS}}(\mathcal{A}) \simeq \mathcal{D}(\mathcal{L})$. $\square$

R e m a r k 1. Obviously we can also formulate and prove (in the same way) analogous results for finite digraphs and their finite strong subdigraph lattices.

R e m a r k 2. Theorems 2, 5, 7 and 13 (formulated for digraphs), Lemma 4 and remarks before Theorem 5 imply that for each finite and distributive lattice $\mathcal{L}$, an arbitrary digraph with its strong subdigraph lattice isomorphic to $\mathcal{L}$ can be constructed as follows: First, we take $\mathcal{D}(\mathcal{L})$. Secondly, we add edges in such a way that for each additional edge there is a path in $\mathcal{D}(\mathcal{L})$ going from its initial vertex to its final vertex. In the third step we insert arbitrary strongly connected digraphs in the place of some vertices.

Observe also that having this construction and Theorem 1 we can build an arbitrary finite partial unary algebra with its strong subalgebra lattice isomorphic to $\mathcal{L}$. First, we take a digraph $\mathcal{D}$ such that $\mathcal{S}_s(\mathcal{D}) \simeq \mathcal{L}$, and next we construct (details of this simple construction can be found in [14]) a finite partial unary algebra $\mathcal{A}$ such that its digraph $\mathcal{D}(\mathcal{A})$ is isomorphic to $\mathcal{D}$.

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