ON THE VOLTERRA INTEGRAL EQUATION AND AXIOMATIC MEASURES OF WEAK NONCOMPACTNESS

Dariusz Bugajewski, Poznań

(Received April 22, 1999)

Abstract. We prove that a set of weak solutions of the nonlinear Volterra integral equation has the Kneser property. The main condition in our result is formulated in terms of axiomatic measures of weak noncompactness.

Keywords: measure of weak noncompactness, Volterra integral equation

MSC 2000: 45D05, 47H09

1. Introduction

The notion of the measure of weak noncompactness was introduced by De Blasi in 1977 ([6]). This index has found applications in fixed point theorems (cf. [7]) and many existence results for weak solutions of differential and integral equations in Banach spaces (cf. [3], [4], [5] and other). Recall that weak solutions of the Cauchy problem in reflexive Banach spaces were investigated by Szép ([11]) and weak solutions of nonlinear integral equations in these spaces by O'Regan ([10]). But, it is not easy to construct some formulas which allow to express the measure of weak noncompactness in a convenient form. This was the reason for introducing the notion of axiomatic measures of weak noncompactness, see [2]. In that paper several examples of axiomatic measures of weak noncompactness in Banach spaces were constructed.

The aim of this paper is to investigate weak solutions (more precisely: weakly continuous solutions) of the nonlinear Volterra integral equation by using this axiomatic index. Our method of proving is more sophisticated then in the case when one applies the classical measure of weak noncompactness. As a corollary of our main theorem we obtain a similar result for the Cauchy problem.

2. Preliminaries

Denote by $\mathcal{M}_E$ the family of all bounded subsets of a given Banach space E, and by $\mathcal{W}_E$ the family of all weakly relatively compact subsets of E (shortly: $\mathcal{M}$ and $\mathcal{W}$, respectively).

Definition 1 ([2]). A function $\gamma\colon \mathcal{M} \to [0,+\infty)$ is said to be an axiomatic measure of weak noncompactness if it satisfies the following conditions:

1° the family $\ker\gamma = \{X \in \mathcal{M}\colon \gamma(X) = 0\}$ is nonempty and $\ker\gamma \subset \mathcal{W}$;
2° $X \subset Y \Rightarrow \gamma(X) \leqslant \gamma(Y)$;
3° $\gamma(\overline{\mathrm{conv}}X) = \gamma(X)$, where $\overline{\mathrm{conv}}X$ denotes the closed convex hull of X;
4° $\gamma(\lambda X + (1-\lambda)Y) \leqslant \lambda\gamma(X) + (1-\lambda)\gamma(Y)$ for $\lambda \in [0,1]$;
5° if $X_n \in \mathcal{M}^{wc}$, where $\mathcal{M}^{wc}$ denotes the family of all weakly closed subsets of E, $X_{n+1} \subset X_n$, $n = 1,2,\ldots$, and $\lim\limits_{n\to\infty} \gamma(X_n) = 0$, then

$$X_\infty = \bigcap_{n=1}^{\infty} X_n \neq \emptyset.$$

The family $\ker\gamma$ described in 1° is called the kernel of the measure γ. It can be easily verified that the measure γ satisfies

$$\gamma(\overline{X}^w) = \gamma(X),$$

where $\overline{X}^w$ denotes the weak closure of X.

Definition 2. If

6° $\gamma(X \cup Y) = \max\{\gamma(X), \gamma(Y)\}$ for any $X, Y \in \mathcal{M}$, then we say that γ has the maximum property.

Notice that if $\gamma\colon \mathcal{M} \to [0,+\infty)$ satisfies
1°, 2°, 6° and $\gamma(\{x\}) = 0$ for any $x \in E$, then γ satisfies 5° (see [2]).

By using similar arguments as in [1] and by Th. 3 ([2]) one can prove the following Ambrosetti's type lemma which will be useful in the sequel.

Lemma. *Assume that γ has the maximum property and V is a uniformly bounded and strongly uniformly equicontinuous subset of the space $C_w(A,E)$, where A is a compact interval in $\mathbb{R}^n$ and $C_w(A,E)$ denotes the space of all weakly continuous functions $A \to E$ with the topology of weak uniform convergence. Then for every compact subset $T \subset A$ we have*

$$\gamma(V(T)) = \sup_{t\in T} \gamma(V(t)),$$

where $V(t) = \{x(t)\colon\ x \in V\}$, $V(T) = \{x(t)\colon\ x \in V,\ t \in T\}$.

3. Main result

In this section we investigate topological structure of the set of weakly continuous solutions of the nonlinear Volterra integral equation

$$x(t) = g(t) + \int\limits_{A(t)} f(t,s,x(s))\,\mathrm{d}s, \qquad t \in A, \tag{1}$$

where $A = [0,a_1] \times [0,a_2] \times \ldots \times [0,a_n]$ $(a_i > 0, i = 1,\ldots,n)$, $A(t) = \{s \in \mathbb{R}^n\colon\ 0 \leqslant s_i \leqslant t_i, i = 1,\ldots,n\}$ and the symbol "$\int$" stands for the weak Riemann integral.

Assume that E is a weakly sequentially complete Banach space and

$\overline{1}$) $g\colon\ A \to E$ is a weakly continuous function;

$\overline{2}$) $f\colon\ A^2 \times E \to E$

is a weakly-weakly continuous function such that

i) for every $r > 0$ there exists $m_r > 0$ such that $\|f(t,s,x)\| \leqslant m_r$ for all $t,s \in A$ and $\|x\| \leqslant r$;

ii) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $t,\tau \in A$, $\|t-\tau\| < \delta \Rightarrow \|f(t,s,x) - f(\tau,s,x)\| < \varepsilon$ whenever $(s,x) \in A \times E$.

The main result of our paper is given by the following Kneser type

Theorem. *Suppose that $\overline{1}$), $\overline{2}$) are satisfied. If the measure γ has the maximum property and*

7° $\gamma(Y) \leqslant$*diam* Y *for every* $Y \in \mathcal{M}$,

8° $\gamma(g(T)+Y) \leqslant \gamma(Y)$ *for every compact subset* $T \subset A$ *and every* $Y \in \mathcal{M}$,

9° $\gamma(X+Y) \leqslant \gamma(Y)$ *for all* $X \in$ *ker*γ *and* $Y \in \mathcal{M}$,

and there exists a continuous function $h\colon\ I \times \mathbb{R}_+ \to \mathbb{R}_+$ which is nondecreasing in the second variable and such that the function identically equal to zero is the unique continuous solution of the inequality

$$u(t) \leqslant \int\limits_{A(t)} h(s,u(s))\,\mathrm{d}s, \qquad t \in A,$$

and

$$\gamma(f(t, T \times X)) \leqslant h(t,\gamma(X)) \tag{2}$$

for $t \in A$ and for all bounded subsets $T \subset A$ and $X \subset E$, then there exists a set $J = [0, d_1] \times [0, d_2] \times \ldots \times [0, d_n] \subset A$ such that the set S of all weakly continuous solutions of (1)*, defined on J, is nonempty, compact and connected in $C_w(J, E)$.*

P r o o f. Let $c = \sup\limits_{t \in A} \|g(t)\|$ and $\varrho = \sup\limits_{r>0} \frac{r-c}{m_r}$. Choose positive numbers $e < \varrho$ and b in such a way that $c + m_b e < b$. Then choose numbers d_i, $i = 1, \ldots, n$, such that $0 < d_i \leqslant a_i$, $i = 1, \ldots, n$ and $d_1 d_2 \ldots d_n < \min(e, 1)$. Set $J = [0, d_1] \times [0, d_2] \times \ldots \times [0, d_n]$. In the space $\mathbb{R}^n$ we introduce a norm defined by the formula

$$\|t\| = \max\left(|t_1|, \frac{d_1}{d_2}|t_2|, \ldots, \frac{d_1}{d_n}|t_n|\right) \qquad \text{for } t = (t_1, t_2, \ldots, t_n).$$

Then $J = \{t \in \mathbb{R}^n \colon t \geqslant 0 \text{ and } \|t\| \leqslant d_1\}$. Denote by $\widetilde{B}$ the set of all weakly continuous functions $J \to B_b$, where $B_b = \{z \in E \colon \|z\| \leqslant b\}$. We shall consider $\widetilde{B}$ as a topological subspace of $C_w(J, E)$. Define $G(x)(t) = g(t) + F(x)(t)$, where

$$F(x)(t) = \int\limits_{A(t)} f(t, s, x(s))\,\mathrm{d}s, \qquad t \in J,\ x \in \widetilde{B}.$$

It is clear from the inequalities

$$\|F(x)(t) - F(x)(\tau)\| \leqslant \int\limits_{A(t)} \|f(t, s, x(s)) - f(\tau, s, x(s))\|\,\mathrm{d}s + m_b d_2 d_3 \ldots d_n \|t - \tau\|,$$

$$\|F(x)(t)\| \leqslant m_b d_2 d_3 \ldots d_n \|t\| \qquad (x \in \widetilde{B}, t, \tau \in J)$$

that $G(\widetilde{B}) \subset \widetilde{B}$ and $F(\widetilde{B})$ is strongly equiuniformly continuous. In view of a Krasnosel'skij-Krein-type lemma (cf. [9]), we infer that G is continuous.

Further, for any number $\eta > 0$ put $J_\eta = \{t \in J \colon \|t\| \leqslant \eta\}$. For any positive integer k define

$$r_k(t) = \begin{cases} 0, & \text{if } t \in J_{1/k}, \\ (1 - \frac{1}{k\|t\|})t, & \text{if } t \in J \backslash J_{1/k}. \end{cases}$$

It can be easily verified that $\|r_k(t) - t\| \leqslant 1/k$, $\|r_k(t)\| \leqslant t$ $(t \in J)$ and $r_k(J_{(i+1)/k}) \subset J_{i/k}$ for $i = 0, 1, \ldots, k-1$. Next, for any positive integer k define $G_k(x)(t) = g(t) + F_k(x)(t)$, where

$$F_k(x)(t) = \int\limits_{A(r_k(t))} f(t, s, x(s))\,\mathrm{d}s, \qquad x \in \widetilde{B},\ t \in J.$$

Analogously to the case of G, the mappings G_k map continuously $\widetilde{B}$ into itself. Moreover,

$$\|G_k(x)(t) - G(x)(t)\| \leqslant \frac{1}{k} m_b n d_2 d_3 \ldots d_n,\ x \in \widetilde{B},\ t \in J. \tag{3}$$

Further, it can be easily verified that there exists a unique element $x_k \in \widetilde{B}$ such that $x_k = G_k(x_k)$. From the above it is clear that there exists a sequence (u_n) such that $u_n \in \widetilde{B}$ for $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \sup_{t\in J} \|u_n(t) - G(u_n)(t)\| = 0. \tag{4}$$

Let $V = \{u_n \colon n \in \mathbb{N}\}$, $W = F(V)$ and $w(t) = \gamma(W(t))$ for $t \in J$. We have $u_n = G_n(u_n) = g + F_n(u_n)$, $n \in \mathbb{N}$ and therefore, by (4),

$$\lim_{n\to\infty} \sup_{t\in J} \|u_n(t) - G(u_n)(t)\| = \lim_{n\to\infty} \|F_n(u_n) - F(u_n)\|_c = 0,$$

where $\|\cdot\|_c$ denotes the supremum norm in the classical space $C(J, E)$ of all continuous functions $J \to E$ with the topology of uniform convergence. Hence the family $(I - G)(V)$, where I denotes the identity map, is strongly relatively compact.

Now, we prove that

$$\gamma((I - G)(V)(T)) = 0 \text{ for every compact subset } T \subset J. \tag{5}$$

Fix $t \in J$ and $\varepsilon > 0$. By (4) there exists $k \in \mathbb{N}$ such that $\|u_n(t) - G(u_n)(t)\| < \varepsilon/2$ for every $k \geqslant n$. In view of the property 6° and 7° we have

$$\gamma((I - G)(V)(t)) = \gamma((I - G)(V_k)(t)) \leqslant \operatorname{diam} \|(I - G)(V_k)(t)\| < \varepsilon.$$

Since $\varepsilon > 0$ has been arbitrary, we obtain $\gamma((I - G)(V)(t)) = 0$. Because $(I - G)(V)$ is strongly uniformly equicontinuous, by Ambrosetti's type lemma we infer that (5) holds.

Since $u_n(t) = (u_n(t) - G(u_n)(t)) + G(u_n)(t)$, $V(t) \subset (I - G)(V)(t) + G(V)(t)$. Now we verify that

$$\gamma(V(t)) \leqslant \gamma(W(t)) \text{ for every } t \in J. \tag{6}$$

Fix $t \in J$. By (5), 9° and 8° we have

$$\gamma(V(t)) \leqslant \gamma((I - G)(V)(t) + G(V)(t)) \leqslant \gamma(G(V)(t)) \leqslant \gamma(F(V)(t)) = \gamma(W(t)).$$

Analogously, we obtain

$$\begin{aligned} \gamma(V(T)) &\leqslant \gamma((I - G)(V)(T) + G(V)(T)) \leqslant \gamma(G(V)(T)) \\ &= \gamma(g(T) + F(V)(T)) \leqslant \gamma(F(V)(T)) = \gamma(W(T)). \end{aligned} \tag{7}$$

Since W is strongly uniformly equicontinuous and uniformly bounded, the function $s \to w(s)$ is continuous on J. Indeed, fix $\varepsilon > 0$ and choose $\delta > 0$ in such a way that

$$\|F(u_n)(t) - F(u_n)(\tau)\| < \varepsilon$$

for $t, \tau \in J$ such that $\|t - \tau\| < \delta$ and $n \in \mathbb{N}$. Since $W(t) \subset W(\tau) + \{F(u_n)(t) - F(u_n)(\tau)\colon n \in \mathbb{N}\}$, in view of Th. 3 [2] we obtain

$$\begin{aligned}\gamma(W(t)) &\leqslant \gamma(W(\tau)) + \sup_{n\in\mathbb{N}} \|F(u_n)(t) - F(u_n)(\tau)\|\gamma(K(W(J), 1))\\ &\leqslant \gamma(W(\tau)) + \varepsilon\gamma(K(W(J), 1)),\end{aligned}$$

where $K(W(J), 1) = \bigcup\limits_{n\in\mathbb{N}} \bigcup\limits_{t\in J} K(F(u_n)(t), 1)$ and $K(F(u_n)(t), 1)$ denotes the open ball centered at $F(u_n)(t)$ and with radius 1. Hence

$$\gamma(W(t)) - \gamma(W(\tau)) \leqslant \varepsilon\gamma(K(W(J), 1)),$$

and analogously

$$\gamma(W(\tau)) - \gamma(W(t)) \leqslant \varepsilon\gamma(K(W(J), 1)).$$

This proves the continuity of w.

Fix $t \in J$ and $\eta > 0$, and choose $\delta > 0$ in such a way that

$$|h(t, w(s)) - h(t, w(\tau))| \leqslant \eta \tag{8}$$

for $t, s \in J$ such that $\|s - \tau\| \leqslant \delta$.

Divide the rectangle $A(t)$ into m rectangles $P_1, \ldots, P_m$ such that $A(t) = \bigcup\limits_{i=1}^{m} P_i$, diam $P_i \leqslant \delta$ and $\mu(P_i \cap P_j) = 0$ for $i, j = 1, \ldots, m$, $i \neq j$ (here μ denotes the Lebesgue measure in $\mathbb{R}^n$). By Ambrosetti's type lemma and by the continuity of w, there exists $\tau_i \in P_i$ such that

$$\gamma(W(P_i)) = w(\tau_i), \qquad i = 1, \ldots, m. \tag{9}$$

By the mean value theorem, we have

$$F(x)(t) = \sum_{i=1}^{m} \int_{P_i} f(t, s, x(s))\,\mathrm{d}s \subset \sum_{i=1}^{m} \mu(P_i)\overline{\mathrm{conv}}(f(t, P_i \times V(P_i))).$$

Further, by the properties of γ, Th. 2 [2], (2), (7) and (9), we obtain

$$\begin{aligned} w(t) &\leqslant \sum_{i=1}^{m} \mu(P_i)\gamma(\overline{\operatorname{conv}}(f(t, P_i \times V(P_i)))) \\ &\leqslant \sum_{i=1}^{m} \mu(P_i)\gamma(f(t, P_i \times V(P_i))) \leqslant \sum_{i=1}^{m} \mu(P_i)h(t, \gamma(V(P_i))) \\ &\leqslant \sum_{i=1}^{m} \mu(P_i)h(t, \gamma(W(P_i))) = \sum_{i=1}^{m} \mu(P_i)h(t, w(\tau_i)). \end{aligned}$$

On the other hand, (8) implies that

$$\mu(P_i)h(t, w(\tau_i)) \leqslant \int_{P_i} h(t, w(s))\,\mathrm{d}s + \eta\mu(P_i).$$

Thus

$$w(t) \leqslant \int_{A(t)} h(t, w(s))\,\mathrm{d}s + \eta\mu(A(t)).$$

Since the above inequality holds for every $\eta > 0$, we infer that

$$w(t) \leqslant \int_{A(t)} h(t, w(s))\,\mathrm{d}s, \text{ for } t \in J.$$

By the assumption on h, it follows from the above inequality that $w(t) = 0$ for $t \in J$. Hence, by (6), $V(t)$ is weakly relatively compact for $t \in J$ and therefore by Ascoli's theorem ([8], pp. 80–81) V is relatively compact in $C_w(J, E)$. Hence the sequence (u_n) has a limit point u. In view of (4) and the continuity of G it is clear that $u = G(u)$. This proves that the set S is nonempty.

Further, since G is continuous, S is closed. Because $S = G(S)$, so $w(S(t)) \leqslant w(F(S)(t))$. Hence, by similar arguments as above we can show that S is a compact subset of $C_w(J, E)$.

To prove that S is connected it is enough to apply a similar method as in Th. 3 [3]. The proof of our theorem is complete. □

Corollary. *Let $I = [t_0, t_0 + a] \subset \mathbb{R}$ be a compact interval, E a sequentially complete Banach space, and let $f\colon I \times E \to E$ be a weakly-weakly continuous and locally bounded function. Assume that a measure γ satisfies 6°, 7°, 9° and there exists a continuous nondecreasing function $h\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that the function identically equal to zero is the unique continuous solution of the inequality*

$$u(t) \leqslant \int_0^t h(u(s))\,\mathrm{d}s, \quad t \in I,$$

and

$$\gamma(f(T \times X)) \leqslant h(\gamma(X))$$

for any bounded subsets $T \subset I$ and $X \subset E$. Then there exists an interval $J \subset I$ such that the set of all weak solutions (see [11] *for the definition) of the Cauchy problem*

$$x' = f(t,x), \qquad x(t_0) = x_0,$$

defined on J, is nonempty, compact and connected in $C_w(J,E)$.

The above result with an axiomatic measure of weak noncompactness can be illustrated by the main theorem from [5].

References

[1] *A. Ambrosetti*: Un teorema di esistenza per le equazioni differenziali negli spazi di Banach. Rend. Sem. Mat. Univ. Padova *39* (1967), 349–360.

[2] *J. Banaś, J. Rivero*: On measures of weak noncompactness. Ann. Mat. Pura Appl. *151* (1988), 213–224.

[3] *D. Bugajewski*: On the existence of weak solutions of integral equations in Banach spaces. Comment. Math. Univ. Carolin. *35* (1994), 35–41.

[4] *D. Bugajewski, S. Szufla*: Kneser's theorem for weak solutions of the Darboux problem in Banach spaces. Nonlinear Anal. *20* (1993), 169–173.

[5] *E. Cramer, V. Lakshmikantham, A. R. Mitchell*: On the existence of weak solutions of differential equations in nonreflexive Banach spaces. Nonlinear Anal. *2* (1978), 169–177.

[6] *F. S. De Blasi*: On a property of the unit sphere in Banach spaces. Bull. Math. Soc. Sci. Math. Roum. *21* (1977), 259–262.

[7] *G. Emanuelle*: Measures of weak noncompactness and fixed point theorems. Bull. Math. Soc. Sci. Math. Roum. *25* (1981), 353–358.

[8] *J. L. Kelley, I. Namioka*: Linear Topological Spaces. Van Nostrand, Princeton, 1963.

[9] *M. A. Krasnosel'skij, S. G. Krein*: To the theory of ordinary differential equations in Banach spaces. Trudy Sem. Funk. Anal. Voronezh. Univ. *2* (1956), 3–23. (In Russian.)

[10] *D. O'Regan*: Integral equations in reflexive Banach spaces and weak topologies. Proc. Amer. Math. Soc. *124* (1996), 607–614.

[11] *A. Szép*: Existence theorem for weak solutions of ordinary differential equations in reflexive Banach spaces. Studia Sci. Math. Hungarica *6* (1971), 197–203.

Author's address: *Dariusz Bugajewski*, Faculty of Mathematics and Computer Science, A. Mickiewicz University, Matejki 48/49, 60-769 Poznań, Poland, e-mail: `ddbb@main.amu.edu.pl`.