H-CONVEX GRAPHS

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Abstract. For two vertices u and v in a connected graph G, the set $I(u,v)$ consists of all those vertices lying on a $u-v$ geodesic in G. For a set S of vertices of G, the union of all sets $I(u,v)$ for $u,v \in S$ is denoted by $I(S)$. A set S is convex if $I(S)=S$. The convexity number $\operatorname{con}(G)$ is the maximum cardinality of a proper convex set in G. A convex set S is maximum if $|S| = \operatorname{con}(G)$. The cardinality of a maximum convex set in a graph G is the convexity number of G. For a nontrivial connected graph H, a connected graph G is an H-convex graph if G contains a maximum convex set S whose induced subgraph is $\langle S\rangle = H$. It is shown that for every positive integer k, there exist k pairwise nonisomorphic graphs $H_1, H_2, \ldots, H_k$ of the same order and a graph G that is H_i-convex for all i ($1 \leqslant i \leqslant k$). Also, for every connected graph H of order $k \geqslant 3$ with convexity number 2, it is shown that there exists an H-convex graph of order n for all $n \geqslant k+1$. More generally, it is shown that for every nontrivial connected graph H, there exists a positive integer N and an H-convex graph of order n for every integer $n \geqslant N$.

Keywords: convex set, convexity number, H-convex

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1. Introduction

For two vertices u and v in a connected graph G, the *distance* $d(u,v)$ between u and v is the length of a shortest $u-v$ path in G. A $u-v$ path of length $d(u,v)$ is also referred to as a $u-v$ *geodesic*. The interval $I(u,v)$ consists of all those vertices lying on a $u-v$ geodesic in G. For a set S of vertices of G, the union of all sets $I(u,v)$ for $u,v \in S$ is denoted by $I(S)$. Hence $x \in I(S)$ if and only if x lies on some $u-v$ geodesic, where $u,v \in S$. The intervals $I(u,v)$ were studied and characterized by Nebeský [13, 14] and were also investigated extensively in the book by Mulder [12],

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where it was shown that these sets provide an important tool for studying metric properties of connected graphs. A set S of vertices of G with $I(S) = V(G)$ is called a *geodetic set* of G, and the cardinality of a minimum geodetic set is the *geodetic number* of G. The geodetic number of a graph was studied in [2]; while the geodetic number of an oriented graph was studied in [5].

A set S of vertices in a graph G is *convex* if $I(S) = S$. Certainly, $V(G)$ is convex. The *convex hull* $[S]$ of a set S of vertices of G is the smallest convex set containing S. So S is a convex set in G if and only if $[S] = S$. The smallest cardinality of a set S whose convex hull is $V(G)$ is called the *hull number* of G. The hull number of a graph was introduced by Everett and Seidman [9] and investigated further in [3], [7], and [11].

Convexity in graphs is discussed in the book by Buckley and Harary [1] and studied by Harary and Niemenen [10] and in [8]. For a nontrivial connected graph G, the *convexity number* $\mathrm{con}(G)$ was defined in [4] as the maximum cardinality of a proper convex set of G, that is,

$$\mathrm{con}(G) = \max\{|S|: \ S \text{ is a convex set of } G \text{ and } S \neq V(G)\}.$$

A convex set S in G with $|S| = \mathrm{con}(G)$ is called a *maximum convex set.* A nontrivial connected graph G of order n with $\mathrm{con}(G) = k$ is called a (k, n) *graph.* The convexity number was also studied in [6] and [8].

As an illustration of these concepts, we consider the graph G of Figure 1. Let $S_1 = \{u, v, z\}$, $S_2 = \{u, v, z, s\}$, and $S_3 = \{u, v, z, s, y, t\}$. Since $[S_1] = S_2 \neq S_1$, $[S_2] = S_2$, and $[S_3] = S_3$, it follows that S_1 is not a convex set, while S_2 and S_3 are convex sets. However, S_2 is not a maximum convex set as $4 = |S_2| < |S_3| = 6$. Moreover, it is routine to verify that there is no proper convex set in G containing more than six vertices of G and so $\mathrm{con}(G) = 6$. Therefore, G is a (6, 8) graph.

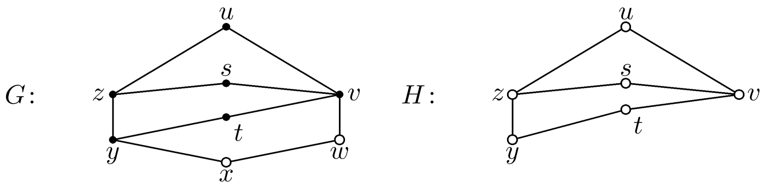


Figure 1. Maximum convex sets

If S is a convex set in a connected graph G, then the subgraph $\langle S\rangle$ induced by S is connected. A goal of this paper is to study the structure of $\langle S\rangle$ for a maximum convex set S in G. For a nontrivial connected graph H, a connected graph G is called an H*-convex graph* if G contains a maximum convex set S such that $\langle S\rangle = H$. (We write $G_1 = G_2$ to indicate that the graphs G_1 and G_2 are isomorphic.) For example, the graph G of Figure 1 is an H-convex graph for the graph H of Figure 1 since S_3 is a maximum convex set in G and $\langle S_3 \rangle = H$. A single graph G can be an H-convex graph for many graphs H, as we now see.

Theorem 1.1. *For each positive integer k, there exist k pairwise nonisomorphic graphs $H_1, H_2, \ldots, H_k$ of the same order and a graph G that is H_i-convex for all i $(1 \leqslant i \leqslant k)$.*

Proof. For k pairwise nonisomorphic graphs F_i $(1 \leqslant i \leqslant k)$ of the same order, say p, let $H_i = \overline{K}_2 + F_i$, where $V(\overline{K}_2) = \{u_i, v_i\}$. We claim that the graphs H_i $(1 \leqslant i \leqslant k)$ are pairwise nonisomorphic graphs. To show this, assume, to the contrary, that H_1 and H_2, say, are isomorphic, and let f be an isomorphism from $V(H_1)$ to $V(H_2)$.

If $\{f(u_1), f(v_1)\} = \{u_2, v_2\}$, then the restriction of f to $V(F_1)$ induces an isomorphism from $V(F_1)$ to $V(F_2)$, a contradiction. If $\{f(u_1), f(v_1)\}$ contains exactly one vertex of $V(F_2)$, say $f(u_1) = u_2$ and $f(v_1) \in V(F_2)$, then the fact that $u_1v_1 \notin E(H_1)$ and $u_2f(v_1) \in E(H_2)$ implies that f is not an isomorphism, again a contradiction. Hence $\{f(u_1), f(v_1)\} \subseteq V(F_2)$. Then $f(u) = u_2$ and $f(v) = v_2$, where $u, v \in V(F_1)$, and $f(u_1) = w$ and $f(v_1) = z$, where $w, z \in V(F_2)$. So $uv \notin E(H_1)$ and $wz \notin E(H_2)$. Since $\deg_{H_1} u = \deg_{H_2} u_2 = p$ and $\deg_{H_1} v = \deg_{H_2} v_2 = p$, it follows that u and v are adjacent to every vertex in $V(H_1) - \{u, v\}$. Similarly, w and z are adjacent to every vertex in $V(H_2) - \{w, z\}$.

Define a mapping g from $V(H_1)$ to $V(H_2)$ by $g(u_1) = u_2$, $g(v_1) = v_2$, $g(u) = w$, $g(v) = z$, and $g(t) = f(t)$ for all $t \in V(H_1) - \{u_1, v_1, u, v\}$. It is routine to verify that g is an isomorphism from $V(H_1)$ to $V(H_2)$. Then the restriction of g to $V(F_1)$ induces an isomorphism from $V(F_1)$ to $V(F_2)$, which is impossible. Therefore, the graphs H_i $(1 \leqslant i \leqslant k)$ are pairwise nonisomorphic, as claimed.

Let G be the graph obtained from the complete bipartite graph $K_{k,k}$, whose partite sets are $V_1 = \{x_1, x_2, \ldots, x_k\}$ and $V_2 = \{y_1, y_2, \ldots, y_k\}$, by replacing the edge x_iy_i by H_i for each i with $1 \leqslant i \leqslant k$, where u_i is identified with x_i and v_i is identified with y_i. (The graph G is shown in Figure 2 for $k = 3$.) The graph G has the desired properties. $\square$

A vertex v in a graph G is called an *extreme vertex* if the subgraph induced by its neighborhood $N(v)$ is complete. Connected graphs of order $n \geqslant 3$ containing an extreme vertex are precisely those having convexity number $n - 1$. The following theorem appeared in [4].

Theorem A. *Let G be a noncomplete connected graph of order n. Then* $\operatorname{con}(G) = n - 1$ *if and only if G contains an extreme vertex.*

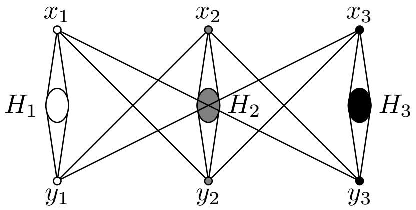


Figure 2. An H_i-convex graph ($i = 1, 2, 3$)

Theorem A implies that if H is a connected graph of order k, then the graph G of order $k+1$ obtained by adding a pendant edge to H is an H-convex graph.

2. The cartesian product of graphs

We now consider the relationship between $\operatorname{con}(H)$ and $\operatorname{con}(H \times K_2)$ for a connected graph H. Let $H \times K_2$ be formed from two copies H_1 and H_2 of H, where corresponding vertices of H_1 and H_2 are adjacent. Let $S_i \subseteq V(H_i)$ for $i = 1, 2$. Then S_2 is called the *projection* of S_1 onto H_2 if S_2 is the set of vertices in H_2 corresponding to the vertices of H_1 that are in S_1. We begin with a lemma concerning convex sets in $H \times K_2$.

Lemma 2.1. *For a nontrivial connected graph H, let $H \times K_2$ be formed from two copies H_1 and H_2 of H, where corresponding vertices of H_1 and H_2 are adjacent. Then every convex set of $H \times K_2$ is either*
(1) *a convex set in H_1,*
(2) *a convex set in H_2, or*
(3) *$S_1 \cup S_2$, where S_1 is convex in H_1 and S_2 is the projection of S_1 onto H_2.*

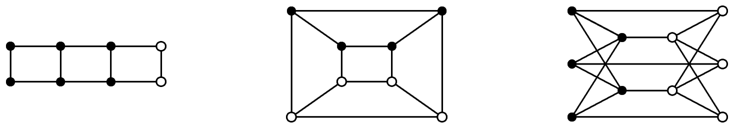
Proof. Let S be a convex set in $H \times K_2$. If $S \subseteq V(H_i)$, $i = 1, 2$, then S is a convex set of H_i, implying that (1) or (2) holds. Otherwise, $S_i = S \cap V(H_i) \neq \emptyset$, $i = 1, 2$, and $S = S_1 \cup S_2$. Assume, to the contrary, that S_2 is not the projection of S_1 onto H_2. Then there exist corresponding vertices $x \in V_1$ and $x' \in V_2$ such that exactly one of these belongs to $S_1 \cup S_2$, say $x \notin S_1$ and $x' \in S_2$. Let $y \in S_1$ and let P be an $x-y$ geodesic in H_1. Then the $x'-y$ path Q beginning at x' and followed by P is a geodesic, implying that $V(Q) \subseteq S_1 \cup S_2$. So $x \in S_1$, a contradiction. Therefore, (3) holds. $\square$

Theorem 2.2. *If H is a connected graph of order at least 2, then*

$$\operatorname{con}(H \times K_2) = \max\{|V(H)|,\ 2\operatorname{con}(H)\}.$$

P r o o f. Let S be a maximum convex set in $H \times K_2$, where $H \times K_2$ is formed from two copies H_1 and H_2 of H. If $S \cap V(H_i) = \emptyset$ for some i $(i = 1, 2)$, say $S \cap V(H_2) = \emptyset$, then $S = V(H_1)$ since S is a maximum convex set. Hence $|S| = \mathrm{con}(H \times K_2) = |V(H_1)| = |V(H)|$. Otherwise, $S_i = S \cap V(H_i) \neq \emptyset$ for $i = 1, 2$, and $S = S_1 \cup S_2$, where by Lemma 2.1, S_2 is the projection of S_1 onto H_2. Again, since S is a maximum convex set in $H \times K_2$, it follows that S_i is a maximum convex set in H_i for $i = 1, 2$. Thus $|S| = \mathrm{con}(H \times K_2) = |S_1 \cup S_2| = 2\,\mathrm{con}(G)$. Therefore, $\mathrm{con}(H \times K_2) = \max\{|V(H)|,\ 2\,\mathrm{con}(H)\}$. $\square$

As an illustration of Theorem 2.2, for $H = P_4, C_4, K_{2,3}$, the graphs $H \times K_2$ are shown of Figure 3. Now $|V(P_4)| = 4$ and $\mathrm{con}(P_4) = 3$, so $\mathrm{con}(P_4 \times K_2) = 2\,\mathrm{con}(P_4) = 6$. Also, $|V(C_4)| = 4$ and $\mathrm{con}(C_4) = 2$, so $\mathrm{con}(C_4 \times K_2) = |V(C_4)| = 2\,\mathrm{con}(C_4) = 4$. Moreover, $|V(K_{2,3})| = 5$ and $\mathrm{con}(K_{2,3}) = 2$, so $\mathrm{con}(K_{2,3} \times K_2) = |V(K_{2,3})| = 5$. A maximum convex set is indicated in each graph in Figure 3.



$P_4 \times K_2$ $\qquad$ $C_4 \times K_2$ $\qquad$ $K_{2,3} \times K_2$

Figure 3. The graphs $P_4 \times K_2$, $C_4 \times K_2$, and $K_{2,3} \times K_2$

The following corollaries are immediate consequences of Theorem 2.2.

Corollary 2.3. *If H is a nontrivial connected graph of order k with $\mathrm{con}(H) \leqslant k/2$, then there exists an H-convex graph of order $2k$.*

Corollary 2.4. *If H is a nontrivial connected graph, then for $n \geqslant 2$,*

$$\mathrm{con}(H \times Q_{n-1}) = 2^{n-2} \max\{|V(H)|,\ 2\,\mathrm{con}(H)\}.$$

In particular, for $n \geqslant 2$, $\mathrm{con}(Q_n) = 2^{n-1}$.

P r o o f. We proceed by induction on n. If $n = 2$, then $H \times Q_1 = H \times K_2$ and the result is trivial. Assume that $\mathrm{con}(H \times Q_{k-1}) = 2^{k-2} \max\{|V(H)|,\ 2\,\mathrm{con}(H)\}$ for some $k \geqslant 2$. Since $H \times Q_k = (H \times Q_{k-1}) \times K_2$, it follows by Theorem 2.2 and the induction hypothesis that

$$\begin{aligned}
\mathrm{con}(H \times Q_k) &= \max\{|V(H \times Q_{k-1})|,\ 2\,\mathrm{con}(H \times Q_{k-1})\}\\
&= \max\{2^{k-1}|V(H)|,\ 2[2^{k-2}\max\{|V(H)|,\ 2\,\mathrm{con}(H)\}]\}\\
&= 2^{k-1}\max\{|V(H)|,\ \max\{|V(H)|,\ 2\,\mathrm{con}(H)\}\}\\
&= 2^{k-1}\max\{|V(H)|,\ 2\,\mathrm{con}(H)\}.
\end{aligned}$$

Therefore, $\operatorname{con}(H \times Q_{n-1}) = 2^{n-2} \max\{|V(H)|,\ 2 \operatorname{con}(H)\}$. For $H = K_2$, $H \times Q_{n-1} = Q_n$ and $H \times K_2 = C_4$. Thus $\operatorname{con}(Q_n) = 2^{n-2} \operatorname{con}(C_4) = 2^{n-2} \cdot 2 = 2^{n-1}$. $\square$

Corollary 2.5. *For $n \geqslant 2$, Q_{n+1} is a Q_n-convex graph. Indeed, Q_n is the unique graph H such that Q_{n+1} is H-convex.*

By an argument similar to that employed in the proof of Theorem 2.2, we have the following result.

Theorem 2.6. *If H is a connected graph of order at least 2, then*

$$\operatorname{con}(H \times K_n) = \max\{(n-1)|V(H)|,\ n \operatorname{con}(H)\}.$$

3. H-convex graphs of large order

We have seen that if H is a connected graph of order k, then there exists an H-convex graph of order $k+1$. If H is complete, however, then there exists an H-convex graph of order n for all $n \geqslant k+1$.

Theorem 3.1. *For $k \geqslant 2$, there exists a K_k-convex graph of order n for all $n \geqslant k+1$.*

P r o o f. For vertices x and y in the complete graph K_{k+1}, let $F = K_{k+1} - xy$. Clearly, F is a K_k-convex graph of order $k+1$. Thus we may assume that $n \geqslant k+2$. Let G be the graph obtained from F by adding $n-k-1$ ($\geqslant 1$) new vertices $v_1, v_2, \ldots, v_{n-k-1}$ and the $2(n-k-1)$ edges xv_i and yv_i, $1 \leqslant i \leqslant n-k-1$. The graph G is shown in Figure 4. Let $S = V(F) - \{x\}$. Since $\langle S \rangle = K_k$, it follows that S is convex. It remains to show that S is a maximum convex set in G.

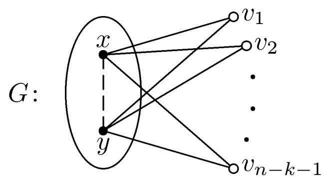


Figure 4. A K_k-convex graph of order n

Let S' be a convex set of G with $|S'| = \operatorname{con}(G) \geqslant k$. Since $I(x,y) = V(G)$, it follows that S' contains at most one of x and y. Let $X = \{v_1, v_2, \ldots, v_{n-k-1}\}$. We claim that $S' \cap X = \emptyset$. Assume, to the contrary, that this is not the case. First assume that S' contains two vertices of X, say $v_1, v_2 \in S'$. Then $x, y \in I(v_1, v_2)$ and so $I(S') = V(G)$, a contradiction. Hence S' contains exactly one vertex of X, say v_1. Since $k \geqslant 3$, it follows that S' contains at least two distinct vertices $u, v \in V(F)$. We may assume, without loss of generality, that $u \neq x, y$ as S' contains at most one of x and y. Since x and y lie on a $u - v_1$ geodesic, it follows that $x, y \in I(u, v_1)$ and so $I(u, v) = V(G)$, again a contradiction. Hence $S' \cap X = \emptyset$, as claimed. Because S' contains at most one of x and y, $\mathrm{con}(G) = |S'| \leqslant k$ and so $\mathrm{con}(G) = k$. □

We next show that for every connected graph H of order k with convexity number 2, there exists an H-convex graph of order n for all $n \geqslant k + 1$. First note that if u, v, w is a path of length 2 in a connected graph G of order at least 4, then $\{u, v, w\}$ is convex if either $uw \in E(G)$ or v is the unique vertex mutually adjacent to u and w. We summarize this observation below.

Lemma 3.2. *If G is a connected graph of order $n \geqslant 4$ with $\mathrm{con}(G) = 2$, then every path of length 2 lies on a 4-cycle in G but on no 3-cycle.*

The converse of Lemma 3.2 is not true since, for example, every path of length 2 in the n-cube Q_n, $n \geqslant 3$, lies on a 4-cycle but on no 3-cycle, while $\mathrm{con}(Q_n) = 2^{n-1}$.

Theorem 3.3. *For every connected graph H of order $k \geqslant 3$ with convexity number 2, there exists an H-convex graph of order n for all $n \geqslant k + 1$.*

Proof. If $k = 3$, then $H = K_3$ or $H = P_3$. If $H = K_3$, then there exists an H-convex graph of order n for all $n \geqslant k + 1$ by Theorem 3.1. For $H = P_3$, the cycles C_5 and C_6 are P_3-convex graphs of orders 5 and 6, respectively, so we may assume that $n \geqslant 7$. Let G be an elementary subdivision of $K_{3,n-4}$ (shown in Figure 5). Since $S = \{u_1, v_1, w\}$ is a maximum convex set of G and $\langle S \rangle = P_3$, it follows that G is a P_3-convex graph of order n.

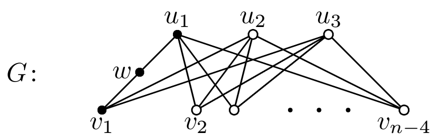


Figure 5. A P_3-convex graph of order n

Assume next that $k = 4$. Since $\mathrm{con}(H) = 2$, it follows that H contains neither triangles nor extreme vertices. This implies that $H = C_4$. For each $n \geqslant 5$, a C_4-convex graph of order n is shown in Figure 6.

We now assume that $k \geqslant 5$. Since there always exists an H-convex graph of order $k + 1$, we assume that $n \geqslant k + 2$. Again, H contains no triangles. If $n = k + 2$,

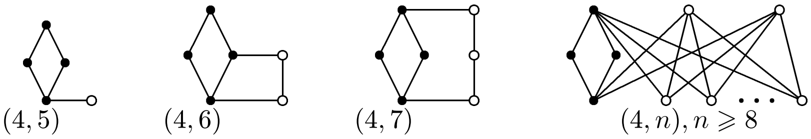


Figure 6. C_4-convex graphs

then the graph G obtained from H by adding two new vertices x, y and the edges ux, xy, yv, where $uv \in E(H)$, has the desired properties. So we may assume that $n = k + l$, where $l \geqslant 3$. Let x, z, y be a path of length 2 in H. Thus $xy \notin E(H)$. Let $F = K_{2,l-1}$ whose partite sets are $V_1 = \{u_1, u_2\}$ and $V_2 = \{v_1 = z, v_2, \ldots, v_{l-1}\}$ such that $V(H) \cap V(F) = \{z\}$. The graph G is constructed from H and F by adding the edges (1) yv_i $(2 \leqslant i \leqslant l-1)$ and (2) xu_j for $j = 1, 2$. Thus $yv_i \in E(G)$ for $1 \leqslant i \leqslant l-1$ and $xv_i \in E(G)$ if and only if $i = 1$. The graphs H and G are shown in Figure 7. The order of G is $k + l = n$. Since $S = V(H)$ is convex and $\langle S \rangle = H$, it remains to show that S is a maximum convex set in G.

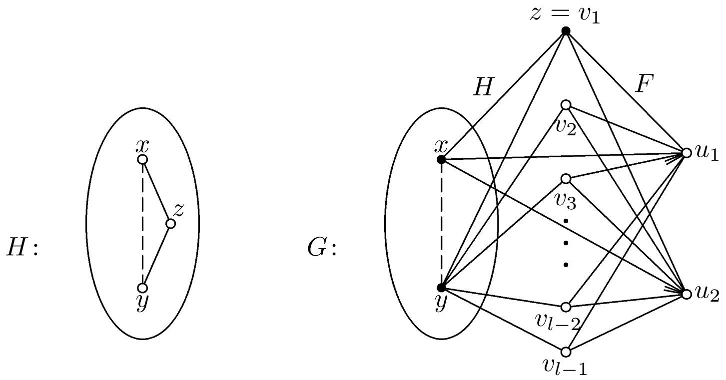


Figure 7. Graphs H and G

First we make an observation. For any two nonadjacent vertices z', z'' of F, it follows that $u_1, u_2 \in [\{z', z''\}]$, implying that $\{x, y, z = v_1\} \subseteq [\{z', z''\}]$. Since $\mathrm{con}(H) = 2$, it follows that $V(H) \subseteq [\{x, y, z\}]$ and so $[\{z', z''\}] = V(G)$. Hence if S_0 is a set of vertices containing two nonadjacent vertices of F, then $[S_0] = V(G)$. Thus there is no maximum convex set in G containing two nonadjacent vertices of F.

Assume, to the contrary, that there exists a convex set S' in G, where $k + 1 \leqslant |S'| < n$. Then $S' \cap (V(G) - S) = S' \cap (V(F) - \{z\}) \neq \emptyset$. Assume first that $z \in S'$. Then S' contains exactly one of u_1 and u_2, say u_1, and, in fact, $S' = S \cup \{u_1\}$. Since $d(y, u_1) = 2$, it follows that $\{v_2, v_3, \ldots, v_{l-1}\} \subseteq [\{u_1, y\}] \subseteq S'$, and so $S' = V(G)$, a contradiction. Hence $z \notin S'$. Since S' does not contain two nonadjacent vertices of F, it follows that S' contains exactly two (necessarily adjacent) vertices of $V(F) - \{z\}$ and that $V(H) - \{z\} \subseteq S'$. Hence $y \in S'$ and S' contains either u_1 or u_2, say

u_1. Again, $\{v_2, v_3, \ldots, v_{l-1}\} \subseteq [\{u_1, y\}] \subseteq S'$ and once again $S' = V(G)$, which is impossible. $\square$

Since the complete bipartite graphs $K_{r,s}$, where $2 \leqslant r \leqslant s$, have convexity number 2, we have the following corollary.

Corollary 3.4. *For $2 \leqslant r \leqslant s$, there exists a $K_{r,s}$-convex graph of order n for all $n \geqslant r + s + 1$.*

We have seen that for some graphs H of order $k \geqslant 2$, there exist H-convex graphs of order n for all $n \geqslant k+1$. However, there are graphs H such that H-convex graphs of order n exist for some integers $n \geqslant k+1$ but not for all such integers n. For example, for each tree T of order $k \geqslant 4$, there is no T-convex graph of order $k + 2$. To see this, first let $T = P_k$, where $k \geqslant 4$, and assume, to the contrary, that there exists a connected graph G of order $k + 2$ with $\mathrm{con}(G) = k$ and having a maximum convex set $S = \{v_1, v_2, \ldots, v_k\}$ such that $E(\langle S \rangle) = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$. Necessarily, G contains no complete vertices. Let $V(G) - S = \{x, y\}$. Since G contains no end-vertices, v_1 and v_k are adjacent to at least one of x and y. If v_1 and v_k are both adjacent to one of x and y, say x, then x lies on a $v_1 - v_k$ geodesic in G and so S is not convex. So we may assume that $v_1x, v_ky \in E(G)$ and $v_1y, v_kx \notin E(G)$. If $xy \in E(G)$, then x and y lie on the $v_1 - v_k$ geodesic v_1, x, y, v_k, which is impossible. Hence $xy \notin E(G)$. Since x is not an extreme vertex, $v_ix \notin E(G)$ for some i with $3 \leqslant i \leqslant k - 1$. But then x lies on a $v_1 - v_i$ geodesic, a contradiction. Therefore, there is no P_k-convex graph of order $k + 2$.

Assume now that $T \neq P_k$. Thus T has at least three end-vertices. Assume, to the contrary, that there exists a connected graph G of order $k + 2$ with $\mathrm{con}(G) = k$ and G contains a maximum convex set S such that $\langle S \rangle = T$, where $V(G) - S = \{x, y\}$. Necessarily, at least one of x and y is adjacent to at least two end-vertices of T, which is impossible. In fact, this argument implies that if T is a tree of order k with p end-vertices, then there exists no T-convex graph of order n with $k+2 \leqslant n \leqslant k+p-1$.

From what we have seen, there exist connected graphs H of order $k \geqslant 2$ such that for many integers $n \geqslant k + 1$, no H-convex graph of order n exist. However, any such integers n with this property must be finite in number, as we now show.

Theorem 3.5. *For every nontrivial connected graph H, there exists a positive integer N and an H-convex graph of order n for every integer $n \geqslant N$.*

Proof. If H is a complete graph, then the result follows by Theorem 3.1. So we may assume that H is not complete and that $W = \{w_1, w_2, \ldots, w_p\}$ is a minimum geodetic set in H. Since H is not complete, W contains some pairs of nonadjacent vertices. We first construct a graph F_q for each integer $q \geqslant 3$. Let P and Q be two copies of the path P_q of order q, where $P\colon x_1, x_2, \ldots, x_q$ and $Q\colon y_1, y_2, \ldots, y_q$. Then the graph F_q is obtained from P and Q by adding the edges $x_i y_{i+1}$ and $y_i x_{i+1}$ for $1 \leqslant i \leqslant q-1$. The graph F_4 is shown in Figure 8.

F_4:

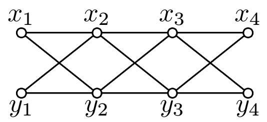


Figure 8. The graph F_4

We next construct a graph F by adding a copy of F_q, for some $q \geqslant 3$, for each pair w_i, w_j, $1 \leqslant i < j \leqslant p$, of nonadjacent vertices of W as well as certain edges between this pair of vertices and F_q. If $d(w_i, w_j) = 2$, then we add a copy F_{ij} of F_3 to H, where $V(F_{ij}) = \{x_{ij}(1), x_{ij}(2), x_{ij}(3)\} \cup \{y_{ij}(1), y_{ij}(2), y_{ij}(3)\}$, and the edges $w_i x_{ij}(1)$, $w_i y_{ij}(1)$, $w_j x_{ij}(3)\}$, $w_j y_{ij}(3)$ (see Figure 9 (a)). If $d(w_i, w_j) = l_{ij} \geqslant 3$, then we add a copy F_{ij} of $F_{l_{ij}}$ to H, where $V(F_{ij}) = \{x_{ij}(1), x_{ij}(2), \ldots, x_{ij}(l_{ij})\} \cup \{y_{ij}(1), y_{ij}(2), \ldots, y_{ij}(l_{ij})\}$, and the edges $w_i x_{ij}(1)$, $w_i y_{ij}(1)$, $w_j x_{ij}(l_{ij})$, $w_j y_{ij}(l_{ij})$ (see Figure 9 (b) for the case $l_{ij} = 4$). The resulting graph is F. Let

$$Y = \bigcup \{y_{ij}\left(\lceil l_{ij}/2 \rceil - 1\right), y_{ij}\left(\lceil l_{ij}/2 \rceil\right), y_{ij}\left(\lceil l_{ij}/2 \rceil + 1\right)\}$$

where the union is taken over all pairs i, j with $1 \leqslant i < j \leqslant p$ for which $w_i w_j \notin E(G)$. Then Y is a subset of $V(F)$. Define $N = 2 + |V(F)|$ and let n be an integer such that $n \geqslant N$. Then $n = k + |V(F)|$ for some integer $k \geqslant 2$. We next construct a graph G from F by adding k new vertices $u_1, u_2, \ldots, u_k$ and the edges $u_i y$ for all $y \in Y$ and $1 \leqslant i \leqslant k$. Thus G has order n. Observe that if G contains four mutually adjacent vertices, then these four vertices must belong to H.

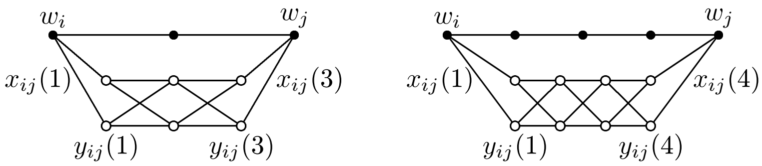


Figure 9. Constructing the graph G

Next we show that G is an H-convex graph. Let $S = V(H)$ and $\overline{S} = V(G) - V(H)$. Let $u, v \in S$. Observe that every $u - v$ geodesic in G contains only vertices of H. Hence S is convex in G and $\langle S \rangle = H$. It remains to show that S is a maximum convex set in G.

First we make some observations. Let $U = \{u_1, u_2, \ldots, u_k\}$. If $u_i, u_j \in U$ and $u_i \neq u_j$, then $[\{u_i, u_j\}] = V(G)$. For any two nonadjacent vertices z', z'' of $\overline{S}$,

$U \subseteq [\{z', z''\}]$, implying that $[\{z', z''\}] = V(G)$. Also, if $z \in \overline{S}$, then $[S \cup \{z\}] = V(G)$. Hence if S_0 is a set of vertices containing either (1) two nonadjacent vertices of $\overline{S}$ or (2) $S \cup \{z\}$ for some $z \in \overline{S}$, then $[S_0] = V(G)$.

Assume, to the contrary, that there exists a proper convex set S' of G with $|S'| \geqslant |S|+1$. Then S' contains at least one and at most three vertices of $\overline{S}$ since no vertices of $\overline{S}$ belong to a subgraph isomorphic to K_4. By the observations above, we have two cases.

Case 1. $(S - \{x\}) \cup \{z_1, z_2\} \subseteq S'$, where $x \in S$, $z_1, z_2 \in \overline{S}$, and $z_1 z_2 \in E(G)$. Since W is a geodetic set of H, it follows that x lies on a $w_a - w_b$ geodesic P' in H, where $w_a, w_b \in W$ and $1 \leqslant a < b \leqslant p$. If $z_1, z_2 \in V(F_{ab})$, then $[(V(P') - \{x\}) \cup \{z_1, z_2\}] = V(G)$. Since $(V(P') - \{x\}) \cup \{z_1, z_2\} \subseteq S'$, it follows that $S' = V(G)$, a contradiction. Thus at least one of z_1 and z_2 does not belong to $V(F_{ab})$, say $z_1 \notin V(F_{ab})$. Assume first that $z_1 \in V(F_{st})$, where $\{s, t\} \neq \{a, b\}$. Then $w_s, w_t \in S'$ and $[\{w_s, w_t, z_1\}] = V(G)$. Otherwise, $z_1 \in U$. Then $[\{w_i, w_j, z_1\}] = V(G)$ for every two nonadjacent vertices $w_i, w_j \in W$. This implies that $S' = V(G)$, again a contradiction.

Case 2. $(S - \{x, x'\}) \cup \{z_1, z_2, z_3\} \subseteq S'$, where $x, x' \in S$, $z_1, z_2, z_3 \in \overline{S}$, and $\langle \{z_1, z_2, z_3\} \rangle = K_3$. This implies that at least one of z_1, z_2, z_3 belongs to U, say $z_1 = u_1$. Since $[(V(H) - \{x, x'\}) \cup \{u_1\}] = V(G)$ and $(V(H) - \{x, x'\}) \cup \{u_1\} \subseteq S'$, it follows that $S' = V(G)$, which is impossible.

Therefore, G is H-convex. □

References

[1] *F. Buckley, F. Harary*: Distance in Graphs. Addison-Wesley, Redwood City, 1990.

[2] *G. Chartrand, F. Harary, P. Zhang*: On the geodetic number of a graph. To appear in Networks.

[3] *G. Chartrand, F. Harary, P. Zhang*: On the hull number of a graph. To appear in Ars Combin.

[4] *G. Chartrand, C. E. Wall, P. Zhang*: The convexity number of a graph. Preprint.

[5] *G. Chartrand, P. Zhang*: The geodetic number of an oriented graph. Eur. J. Comb. *21* (2000), 181–189.

[6] *G. Chartrand, P. Zhang*: The forcing convexity number of a graph. To appear in Czech. Math. J.

[7] *G. Chartrand, P. Zhang*: The forcing hull number of a graph. To appear in J. Combin. Math. Combin. Comput.

[8] *G. Chartrand, P. Zhang*: Convex sets in graphs. To appear in Congress. Numer.

[9] *M. G. Everett, S. B. Seidman*: The hull number of a graph. Discrete Math. *57* (1985), 217–223.

[10] *F. Harary, J. Nieminen*: Convexity in graphs. J. Differential Geom. *16* (1981), 185–190.

[11] *H. M. Mulder*: The expansion procedure for graphs. Contemporary Methods in Graph Theory (R. Bodendiek, ed.). Wissenschaftsverlag, Mannheim, 1990, pp. 459–477.

[12] *H. M. Mulder*: The Interval Function of a Graph. Mathematisch Centrum, Amsterdam, 1980.

[13] *L. Nebeský*: A characterization of the interval function of a connected graph. Czech. Math. J. *44* (1994), 173–178.
[14] *L. Nebeský*: Characterizing of the interval function of a connected graph. Math. Bohem. *123* (1998), 137–144.

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