PARTIALLY IRREGULAR ALMOST PERIODIC SOLUTIONS OF ORDINARY DIFFERENTIAL SYSTEMS

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Abstract. Let $f(t,x)$ be a vector valued function almost periodic in t uniformly for x, and let $\operatorname{mod}(f) = L_1 \oplus L_2$ be its frequency module. We say that an almost periodic solution $x(t)$ of the system

$$\dot{x} = f(t,x), \quad t \in \mathbb{R}, \quad x \in D \subset \mathbb{R}^n$$

is irregular with respect to L_2 (or partially irregular) if $(\operatorname{mod}(x) + L_1) \cap L_2 = \{0\}$.

Suppose that $f(t,x) = A(t)x + X(t,x)$, where $A(t)$ is an almost periodic $(n \times n)$-matrix and $\operatorname{mod}(A) \cap \operatorname{mod}(X) = \{0\}$. We consider the existence problem for almost periodic irregular with respect to $\operatorname{mod}(A)$ solutions of such system. This problem is reduced to a similar problem for a system of smaller dimension, and sufficient conditions for existence of such solutions are obtained.

Keywords: almost periodic differential systems, almost periodic solutions

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1. Introduction

Let D be a compact subset of $\mathbb{R}^n$ and let $AP(D,\mathbb{R}^n)$ be a class of functions $h\colon \mathbb{R} \times D \to \mathbb{R}^n$ such that each $h(t,x) \in AP(D,\mathbb{R}^n)$ is continuous on $\mathbb{R} \times D$ and almost periodic in t uniformly for $x \in D$. By $\operatorname{mod}(h)$ we will denote the frequency module of $h(t,x)$. Consider the system

$$\dot{x} = f(t,x), \quad t \in \mathbb{R}, \quad x \in D, \tag{1}$$

where $f \in AP(D,\mathbb{R}^n)$. The study of almost periodic solutions to (1) is an important problem of the theory of ordinary differential equations. Many authors have investigated this problem, see e.g. [1–7]. It should be noted that most of them have supposed that $\operatorname{mod}(x) \subset \operatorname{mod}(f)$, where $x(t)$ is the solution under consideration. However, there can be various relations between $\operatorname{mod}(x)$ and $\operatorname{mod}(f)$. In [8], J. Kurzweil and O. Vejvoda have shown that there exists a system (1) admitting an almost periodic solution $x(t)$ such that $\operatorname{mod}(x) \cap \operatorname{mod}(f) = \{0\}$. This result enables us to introduce the following

Definition 1. An almost periodic solution $x(t)$ of system (1) is called irregular if $\operatorname{mod}(x) \cap \operatorname{mod}(f) = \{0\}$.

The interest in such solutions is inspired by the analogous problem for periodic systems, see [8–12]. In [13] and [14] we obtained necessary and sufficient conditions of existence of almost periodic irregular solutions to (1). For linear systems, we also studied some properties of such solutions in these papers. The same problems are investigated for quasiperiodic systems in [15]. For solutions of such systems, irregularity is equivalent to linear independence of the solution's and the system's frequency bases over $\mathbb{Q}$.

In [16] we have shown that some classes of quasiperiodic systems admit quasiperiodic solutions that have some of the right hand part base frequencies. It is interesting to investigate similar phenomena for the almost periodic system.

Definition 2. Let $\operatorname{mod}(f) = L_1 \oplus L_2$ be the frequency module of the right hand part of system (1). An almost periodic solution $x(t)$ of the system (1) is called irregular with respect to L_2 (or partially irregular) if $(\operatorname{mod}(x) + L_1) \cap L_2 = \{0\}$.

The aim of this paper is to study the conditions of existence of almost periodic partially irregular solutions of system (1), where $f(t,x) = A(t)x + X(t,x)$. To this effect we reduce the existence problem of almost periodic irregular with respect to $\operatorname{mod}(A)$ solutions to a similar problem for a system of smaller dimension.

2. Preliminaries

Definition 3. A real number γ is called a Fourier exponent (or frequency) of $h(t,x) \in AP(D, \mathbb{R}^n)$, if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T h(t,x) \exp(-\mathrm{i}\gamma t)\, \mathrm{d}t \not\equiv 0, \quad x \in D.$$

The set Γ of all Fourier exponents of $h(t,x)$ is called the frequency set of this function.

It is well known that Γ is countable [3].

Definition 4. The set $\omega = \{\omega_1, \omega_2, \ldots\}$ is called a frequency base for $h(t,x)$ if the following conditions hold: i) ω is linearly independent over the rationals; ii) any frequency $\gamma \in \Gamma$ may be written in the form $\gamma = a_1\omega_1 + \ldots + a_k\omega_k$, where $a_1, \ldots, a_k$ are rational numbers.

Note that ω need not be a part of Γ in general [4].

Definition 5. The frequency module $\operatorname{mod}(h_1, \ldots, h_k)$ of $h_j(t,x) \in AP(D, \mathbb{R}^n)$ ($j = \overline{1,k}$; $1 \leqslant k < +\infty$) is the smallest additive group of real numbers that contains all Fourier exponents of this functions.

Now let us consider the system

$$\dot{x} = F(t,x) + G(t,x), \quad t \in \mathbb{R}, \quad x \in D, \tag{2}$$

where $F(t,x),\ G(t,x) \in AP(D, \mathbb{R}^n)$.

By $\widehat{F}$ we denote the mean value of F, i.e.

$$\widehat{F}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(t,x)\,\mathrm{d}t.$$

Lemma 1. *Suppose that* $\operatorname{mod}(F) \cap \operatorname{mod}(G) = \{0\}$. *System* (2) *has an almost periodic irregular with respect to* $\operatorname{mod}(F)$ *solution* $x(t)$ *iff* $x(t)$ *is a solution to*

$$\dot{x} = \widehat{F}(x) + G(t,x), \quad F(t,x) - \widehat{F}(x) = 0. \tag{3}$$

Proof. Let $x(t)$ be an almost periodic solution of system (2) such that $\operatorname{mod}(x,G) \cap \operatorname{mod}(F) = \{0\}$ and let $\{\nu_1, \nu_2, \ldots\}$ be the frequency set of $F(t,x)$. Let

$$F(t,x) - \widehat{F}(x) \sim \sum_{k,\nu_k \neq 0} f_k(x) \exp(\mathrm{i}\nu_k t)$$

be the Fourier-series expansion of $F(t,x)$ in t, where

$$f_k(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T F(t,x) \exp(-\mathrm{i}\nu_k t)\,\mathrm{d}t \quad (k = 1, 2, \ldots; \nu_k \neq 0).$$

Put

$$f_k(x(t)) \sim \sum_m f_{km} \exp(\mathrm{i}\mu_m t),$$

where

$$f_{km} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f_k(x(t)) \exp(-\mathrm{i}\mu_m t)\,\mathrm{d}t,\ \ \mu_m \in \operatorname{mod}(x)\ \ (k, m = 1, 2, \ldots; \nu_k \neq 0).$$

Then

$$F(t,x(t)) - \widehat{F}(x(t)) \sim \sum_{k,\nu_k \neq 0} \sum_m f_{km} \exp(\mathrm{i}(\nu_k + \mu_m)t).$$

Put

$$-\dot{x}(t) + \widehat{F}(x(t)) + G(t, x(t)) \equiv f_0(t).$$

Note that $f_0(t)$ is almost periodic by [4, p. 27] and $\operatorname{mod}(f_0) \subset \operatorname{mod}(x, G)$. Let $\{\tilde{\mu}_1, \tilde{\mu}_2, \ldots\}$ be the frequency set of $f_0(t)$. We can write

$$f_0(t) \sim \sum_s f_{0s} \exp(\mathrm{i}\tilde{\mu}_s t), \quad f_{0s} = \lim_{T\to\infty} \frac{1}{T} \int_0^T f_0(t) \exp(-\mathrm{i}\tilde{\mu}_s t)\,\mathrm{d}t.$$

Since $x(t)$ is a solution to (2), we have

$$0 \equiv f_0(t) - \widehat{F}(x(t)) + F(t, x(t)) \sim \sum_s f_{0s} \exp(\mathrm{i}\tilde{\mu}_s t) \\ + \sum_{k,\nu_k\neq 0} \sum_m f_{km} \exp((\mathrm{i}(\nu_k + \mu_m)t).$$

Since $\operatorname{mod}(f_0) \subset \operatorname{mod}(x, G)$, $\operatorname{mod}(x, G) \cap \operatorname{mod}(F) = \{0\}$, we obtain $\operatorname{mod}(f_0) \cap \operatorname{mod}(F) = \{0\}$. Hence, $\tilde{\mu}_s \neq \mu_m + \nu_k$ ($\nu_k \neq 0$; $s, m, k = 1, 2, \ldots$). In this case all coefficients will be zero in this Fourier-series expansion. By the uniqueness theorem for almost periodic functions, we have $f_0(t) \equiv 0$, $F(t, x(t)) - \widehat{F}(x(t)) \equiv 0$, i.e. $x(t)$ satisfies (3).

Conversely, let $x(t)$ be an almost periodic irregular with respect to $\operatorname{mod}(F)$ solution of system (3). Then $F(t, x(t)) - \widehat{F}(x(t)) \equiv 0$ and, therefore, $x(t)$ satisfies (2). This completes the proof of Lemma 1. □

Corollary. *System (2) has an almost periodic irregular with respect to* $\operatorname{mod}(F)$ *solution* $x(t)$ *iff* $x(t)$ *satisfies the conditions*

$$\dot{x} = F(t_0, x) + G(t, x), \quad F(t, x) - F(t_0, x) = 0, \quad t_0 \in \mathbb{R}.$$

Let $H(t)$ be $(m\times n)$-matrix function (i.e. the elements of $H(t)$ are some real-valued function on $\mathbb{R}$). By $\operatorname{rank} H$ we denote the column rank of H.

Lemma 2. *Let* $P(t)$ *be an almost periodic* $(m \times n)$*-matrix with*

$$\widehat{P} = \lim_{T\to\infty} \frac{1}{T} \int_0^T P(t)\,\mathrm{d}t = 0.$$

The system

(4) $$P(t)z = 0 \quad (z \in D \subset \mathbb{R}^n)$$

has a nontrivial almost periodic irregular solution iff rank$P < n$.

P r o o f. Let $z(t)$ be a nontrivial almost periodic irregular solution to (4) and let $\{\mu_1, \mu_2, \ldots\}$ be the frequency set of $z(t)$. Let

$$z(t) \sim \sum_s z_s \exp(\mathrm{i}\mu_s t), \quad z_s = \lim_{T\to\infty} \frac{1}{T} \int_0^T z(t) \exp(-\mathrm{i}\mu_s t)\,\mathrm{d}t \quad (s = 1, 2, \ldots),$$

be the Fourier-series expansion of $z(t)$. Then

$$\sum_s P(t) z_s \exp(\mathrm{i}\mu_s t) \sim 0. \tag{5}$$

Denote by $\nu = \{\nu_1, \nu_2, \ldots\}$ the Fourier exponents of $P(t)z_s$. Note that $\nu_j \neq 0$; $j = 1, 2, \ldots$; and $\nu \subset \mathrm{mod}(P)$. Then

$$P(t)z_s \equiv p_s(t) \sim \sum_k p_{sk} \exp(\mathrm{i}\nu_k t),$$
$$p_{sk} = \lim_{T\to\infty} \frac{1}{T} \int_0^T p_s(t) \exp(-\mathrm{i}\nu_k t)\,\mathrm{d}t \quad (k = 1, 2, \ldots).$$

Now we rewrite (5) as

$$\sum_s \sum_k p_{sk} \exp[\mathrm{i}(\mu_s + \nu_k)t] \sim 0.$$

Since $\mathrm{mod}(z) \cap \mathrm{mod}(P) = \{0\}$ and $\nu_k \neq 0$, we have $\mu_s + \nu_k \neq \mu_p + \nu_r$ ($s \neq p$; $k \neq r$; $s, k, p, r = 1, 2, \ldots$). Hence $p_{sk} = 0$ for all s, k. By the uniqueness theorem for almost periodic functions [2], we obtain

$$P(t) z_s \equiv 0 \quad (s = 1, 2, \ldots). \tag{6}$$

Since $z(t) \not\equiv 0$, there exist r, j such that $z_r^{(j)} \neq 0$; $z_r = (z_r^{(1)}, \ldots, z_r^{(j)}, \ldots, z_r^{(n)})$ $(1 \leqslant j \leqslant n)$. By (6) we have

$$p_{l1}(t) z_r^{(1)} + \ldots + p_{lj}(t) z_r^{(j)} + \ldots + p_{ln}(t) z_r^{(n)} \equiv 0 \quad (l = \overline{1, m};\ z_r^{(j)} \neq 0).$$

This implies that rank$P < n$.

To conclude the proof, it remains to note that if rank$P = k < n$, then system (5) has $n - k$ linear independent constant solutions $z_1, \ldots, z_{n-k}$. Let $h(t)$ be an almost periodic function such that $\mathrm{mod}(h) \cap \mathrm{mod}(P) = \{0\}$. It can be easily seen that $z_1 h(t), \ldots, z_{n-k} h(t)$ satisfy (5) too. Lemma 2 is proved. □

Further, we consider the system

$$\dot{x} = A(t)x + X(t,x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \tag{7}$$

where $A(t)$ is an almost periodic $(n \times n)$-matrix; $X(t,x) \in AP(\mathbb{R}^n)$.

Assume that

$$\operatorname{mod}(A) \cap \operatorname{mod}(X) = \{0\}. \tag{8}$$

We shall study the conditions of existence of almost periodic irregular with respect to $\operatorname{mod}(A)$ solutions of system (7).

By Lemma 1, system (7) has an almost periodic irregular with respect to $\operatorname{mod}(A)$ solution $x(t)$ iff $x(t)$ satisfies the system

$$\dot{x} = \widehat{A}x + X(t,x), \quad [A(t) - \widehat{A}]x = \widetilde{A}(t)x = 0, \quad \widehat{A} = \lim_{T\to\infty} \frac{1}{T}\int_0^T A(t)\,\mathrm{d}t. \tag{9}$$

If all columns of the matrix $\widetilde{A}(t)$ are linearly independent over $\mathbb{R}$, then system (9) has no nontrivial almost periodic irregular with respect to $\operatorname{mod}(A)$ solution by Lemma 2. Therefore, we suppose that

$$\operatorname{rank}\widetilde{A} = n_1 < n. \tag{10}$$

Denote $n - n_1$ by m. Then there exists an $(n \times n)$-matrix S, $\det S \neq 0$, such that the transformation

$$x = Sy \tag{11}$$

reduces (9) to the form

$$\dot{y} = By + Y(t,y), \quad \widetilde{B}(t)y = 0, \tag{12}$$

where $B = S^{-1}\widehat{A}S = [b_{ij}]_1^n$; $Y(t,y) = S^{-1}X(t,Sy) = \operatorname{col}(Y_1,\dots,Y_n)$; $\widetilde{B}(t) = \widetilde{A}(t)S$, the first m columns of $\widetilde{B}(t)$ are zero and the remaining $n-m$ columns are linearly independent over $\mathbb{R}$. From Lemma 2 we have that the last $n-m$ components of the almost periodic irregular with respect to $\operatorname{mod}(A)$ solution y to (12) are zero, i.e. y has the following structure: $y = (\tilde{y}, 0, \dots 0)$, $\tilde{y} = (y_1, \dots, y_m)$. Substituting $y = (\tilde{y}, 0, \dots 0)$ into (12), we obtain

$$\dot{\tilde{y}} = B_{m,m}\tilde{y} + Y'(t,\tilde{y}), \quad 0 = B_{n-m,m}\tilde{y} + Y''(t,\tilde{y}), \tag{13}$$

where $B_{m,m} = [b_{ij}]_1^m$; $B_{n-m,m} = [b_{ij}]$ $(i = \overline{m+1,n},\ j = \overline{1,m})$; $\mathrm{col}(Y', Y'') = Y(t, y_1, \dots, y_m, 0, \dots, 0)$.

Thus, the problem of existence of an almost periodic irregular with respect to mod(A) solution of (7) is reduced to a similar problem for system (13).

Consider the system

$$\dot{\tilde{y}} = B_{m,m}\tilde{y} + Y'(t, \tilde{y}). \tag{14}$$

Assume that all eigenvalues $\lambda_1(B_{m,m}), \dots, \lambda_m(B_{m,m})$ of $B_{m,m}$ have nonzero real parts

$$\mathrm{Re}\lambda_j(B_{m,m}) \neq 0 \quad (j = \overline{1,m}) \tag{15}$$

and $Y'(t, \tilde{y})$ satisfies the Lipschitz condition

$$\|Y'(t, \tilde{y}'') - Y'(t, \tilde{y}'')\| \leqslant L\|\tilde{y}'' - \tilde{y}'\|, \quad \tilde{y}', \tilde{y}'' \in \mathbb{R}^m, \quad L = \mathrm{const}. \tag{16}$$

Then by [4, p. 143] the system (14) has an almost periodic solution $\tilde{y}(t)$ such that $\mathrm{mod}(\tilde{y}) \subset \mathrm{mod}(Y')$. Since $\mathrm{mod}(Y') \subset \mathrm{mod}(X)$, we have $\mathrm{mod}(\tilde{y}) \cap \mathrm{mod}(A) = \{0\}$. It is clear that $\tilde{y}(t)$ is a solution to (13) if

$$B_{n-m,m}\tilde{y}(t) + Y''(t, \tilde{y}(t)) \equiv 0. \tag{17}$$

Let (17) be true. Taking into account (11), we obtain an almost periodic irregular with respect to $\mathrm{mod}(A)$ solution of (7)

$$x(t) = S\mathrm{col}[\tilde{y}(t), 0, \dots, 0], \quad \mathrm{mod}(x, X) \cap \mathrm{mod}(A) = \{0\}, \tag{18}$$

where $\tilde{y}(t)$ is an almost periodic solution to (14).

Thus, we have proved

Theorem. *Suppose system* (7) *satisfies conditions* (8), (10), (15), (16), (17)*; then* (7) *has the almost periodic irregular with respect to* $\mathrm{mod}(A)$ *solution* (18).

R e m a r k. By Lemma 2 we can state that system (7) has an almost periodic irregular with respect to $\mathrm{mod}(A)$ solution $x(t)$ iff $x(t)$ satisfies the system

$$\dot{x} = A(t_0)x + X(t, x), \quad [A(t) - A(t_0)]x = 0, \quad t_0 \in \mathbb{R}.$$

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References

[1] *Samoilenko, A. M.*: The Elements of Mathematical Theory of Multifrequency Oscillations. Nauka, Moskva, 1987. (In Russian.)

[2] *Levitan, B. M., Zhikov, V. V.*: Almost Periodic Functions and Differential Equations. Izdatelstvo Moskovskogo Universiteta, Moskva, 1978. (In Russian.)

[3] *Hale, J. K.*: Oscillations in Nonlinear Systems. Mc Graw-Hill, New York, 1963.

[4] *Fink, A. M.*: Almost Periodic Differential Equations. Lecture Notes in Mathematics 377, Springer, Berlin, 1974.

[5] *Demidovich, B. P.*: Lectures on Mathematical Stability Theory. Nauka, Moskva, 1967. (In Russian.)

[6] *Bibikov, Y. N.*: Multifrequency Nonlinear Oscillations and Their Bifurcations. Izdatelstvo Leningradskogo Universiteta, Leningrad, 1991.

[7] *Zubov, V. I.*: Oscillation Theory. Nauka, Moskva, 1979. (In Russian.)

[8] *Kurzweil, J., Vejvoda, O.*: On periodic and almost periodic solutions of the ordinary differential systems. Czechoslovak Math. J. *5* (1955), 362–370.

[9] *Massera, J. L.*: Observationes sobre les soluciones periodicas de ecuaciones differentiales. Boletin de la Facultad de Ingenieria *4* (1950), 37–45.

[10] *Erugin, N. P.*: On periodic solutions of the linear homogeneous system of differential equations. Doklady Akademii Nauk BSSR *6* (1962), 407–410. (In Russian.)

[11] *Grudo, E. I.*: On periodic solutions with incommensurable periods of periodic differential systems. Differentsial'nye uravneniya *22* (1986), 1409–1506 (In Russian.); English translation: Differential Equations *22* (1986), 1036–1041.

[12] *Grudo, E. I., Demenchuk, A. K.*: On periodic solutions with incommensurable periods of linear nonhomogeneous periodic differential systems. Differentsial'nye uravneniya *23* (1987), 409–416 (In Russian.); English translation: Differential Equations *23* (1987), 284–289.

[13] *Demenchuk, A. K.*: On almost periodic solutions of ordinary differential systems. Izvestiya Akademii Nauk BSSR. Ser. fiz.-mat. nauk. *4* (1987), 16–22. (In Russian.)

[14] *Grudo, E. I., Demenchuk, A. K.*: On periodic and almost periodic solutions of ordinary differential systems. Issues of qualitative theory of differential equations. Novosibirsk (1987), 23–29. (In Russian.)

[15] *Demenchuk, A. K.*: Quasiperiodic solutions of differential systems with frequency bases of solutions and right sides that are linearly independent over $\mathbb{Q}$. Differentsial'nye uravneniya *27* (1991), 1673–1679 (In Russian.); English translation: Differential Equations *27* (1991), 1176–1181.

[16] *Demenchuk, A. K.*: On a class of quasiperiodic solutions of differential systems. Doklady Akademii Nauk Belarusi *36* (1992), 14–17. (In Russian.)

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