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BLOCK DIAGONALIZATION

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Abstract. We study block diagonalization of matrices induced by resolutions of the unit matrix into the sum of idempotent matrices. We show that the block diagonal matrices have disjoint spectra if and only if each idempotent matrix in the inducing resolution double commutes with the given matrix. Applications include a new characterization of an eigenprojection and of the Drazin inverse of a given matrix.

Keywords: eigenprojection, resolutions of the unit matrix, block diagonalization

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1. Introduction and preliminaries

In this paper we are concerned with a block diagonalization of a given matrix A; by definition, A is *block diagonalizable* if it is similar to a matrix of the form

$$\begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \dots & & \dots & \\ 0 & 0 & \dots & A_m \end{bmatrix} = \operatorname{diag}(A_1, \dots, A_m). \tag{1.1}$$

Then the spectrum $\sigma(A)$ of A is the union of the spectra $\sigma(A_1), \dots, \sigma(A_m)$, which in general need not be disjoint. The ultimate such diagonalization is the Jordan form of the matrix; however, it is often advantageous to utilize a coarser diagonalization, easier to construct, and customized to a particular distribution of the eigenvalues. The most useful diagonalizations are the ones for which the sets $\sigma(A_i)$ are pairwise disjoint; it is the aim of this paper to give a full characterization of these diagonalizations.

For any matrix $A \in \mathbb{C}^{n\times n}$ we denote its kernel and image by $\ker A$ and $\operatorname{im} A$, respectively. A matrix E is *idempotent* (or a *projection matrix*) if $E^2 = E$, and *nilpotent* if $E^p = 0$ for some positive integer p. Recall that $\operatorname{rank} E = \operatorname{tr} E$ if E is idempotent. Matrices A, B are *similar*, written $A \sim B$, if $A = Q^{-1}BQ$ for some nonsingular matrix Q. The similarity transformation will be also written explixitly as $\psi(U) = \psi_Q(U) = Q^{-1}UQ$. The *commutant* and the *double commutant* of a matrix $A \in \mathbb{C}^{n\times n}$ are defined by

$$\begin{aligned} \operatorname{comm}(A) &= \{U \in \mathbb{C}^{n\times n} \colon\ AU = UA\},\\ \operatorname{comm}^2(A) &= \{U \in \mathbb{C}^{n\times n} \colon\ UV = VU \text{ for all } V \in \operatorname{comm}(A)\}. \end{aligned}$$

The main result of the present paper is the fact that the block diagonal matrices have disjoint spectra if and only if the idempotent matrices inducing the diagonalization double commute with the given matrix. One way to prove this is to use a powerful theorem of matrix theory which states that the second commutant $\operatorname{comm}^2(A)$ coincides with the set of all matrices of the form $f(A)$, where f is a polynomial ([10, Theorem 1.3.7] or [11, p. 106]). (A simple proof of this result is given by Lagerstrom in [4].) We prefer to give an elementary proof of the disjoint spectra theorem, which provides us with a greater insight. As a corollary we show that an idempotent matrix E is an eigenprojection of A if and only if it is in the double commutant of A and $(A - \mu I)E$ is nilpotent for some μ.

To study block diagonalizations of the form (1.1), it is convenient to use resolutions of the unit matrix whose components are idempotent matrices.

Definiton 1.1. Let $m \geqslant 2$ be an integer. An m-tuple $(E_1, \ldots, E_m)$ of idempotent $n \times n$ matrices is called a *resolution of the unit matrix* if $E_1 + \ldots + E_m = I$. A resolution is called *nontrivial* if each component matrix is nonzero. We say that the resolution $(E_1, \ldots, E_m)$ *commutes with the matrix* A if $E_i \in \operatorname{comm}(A)$ for each i, and that it *double commutes with* A if $E_i \in \operatorname{comm}^2(A)$ for each i.

Note 1.2. If $(E_1, \ldots, E_m)$ is a nontrivial resolution of the unit $n \times n$ matrix, then $n = \operatorname{tr} I = \operatorname{tr}(E_1 + \ldots + E_m) = \operatorname{tr} E_1 + \ldots + \operatorname{tr} E_m = \operatorname{rank} E_1 + \ldots + \operatorname{rank} E_m$, which means that $\operatorname{im} E_i \cap \operatorname{im} E_j = \{0\}$ if $i \neq j$. Therefore

$$E_iE_j = 0 \text{ if } i \neq j. \tag{1.2}$$

We will need the following result on projections associated with direct sums of subspaces, which can be deduced from [7, Theorem 214B].

Lemma 1.3. *$\mathbb{C}^n$ is the direct sum $\mathbb{C}^n = X_1 \oplus \ldots \oplus X_m$ of nonzero subspaces X_i if and only if there is a nontrivial resolution $(E_1, \ldots, E_m)$ of the unit $n \times n$ matrix* *such that $X_i = \operatorname{im} E_i$, $i = 1, \dots, m$. There are nonsingular matrices $Q, P \in \mathbb{C}^{n\times n}$ and consistent partitions $Q = [Q_1 \dots Q_m]$, $P = [P_1 \dots P_m]$ such that*

$$E_i = Q_i P_i^T, \quad i = 1, \dots, m.$$

A matrix A leaves the subspaces X_i invariant if and only if A commutes with the resolution $(E_1, \dots, E_m)$.

The next lemma, the well known primary decomposition theorem that can be found, for instance in [6], is stated in the form suitable for the future development.

Lemma 1.4. *Let $A \in \mathbb{C}^{n\times n}$. If $\sigma(A) = \{\mu_1, \dots, \mu_s\}$, then $\mathbb{C}^n = X_1 \oplus \dots \oplus X_s$, where $X_i = \ker(\mu_i I - A)^{p_i}$ and $p_i = \operatorname{ind}(\mu_i I - A)$, $i = 1, \dots, s$. If $(E(\mu_1), \dots, E(\mu_s))$ is the resolution of the unit $n \times n$ matrix associated with this direct sum, then $E(\mu_i) \in \operatorname{comm}^2(A)$ for $i = 1, \dots, s$.*

The matrices $E(\mu_i)$ from the preceding lemma are called the *eigenprojections* of A, and can be constructed by the method described in Lemma 1.3, provided the spaces $\ker(\mu_i I - A)^{p_i}$ are known. If μ is not an eigenvalue of A, we define $E(\mu) = 0$. We can also express the eigenprojection $E(\mu) \neq 0$ through the holomorphic calculus for A, described in [6]. We have

$$E(\mu) = f(A), \tag{1.3}$$

where f is equal to 1 in an open set M containing the eigenvalue μ and to 0 in an open set N containing all other eigenvalues of A, while M and N are disjoint.

For the proof of the main theorem we will need a result which is a special case of [5, Theorem 5.8.1]. We give a proof for completeness.

Lemma 1.5. *Let the matrices $A_1 \in \mathbb{C}^{p\times p}$, $A_2 \in \mathbb{C}^{q\times q}$ be such that the equation $A_1 U - U A_2 = 0$ has only the trivial solution. Then A_1, A_2 cannot be simultaneously singular.*

P r o o f. Let $u \in \mathbb{C}^p$ and $v \in \mathbb{C}^q$ be such that $A_1 u = 0$ and $v^T A_2 = 0$. Set $U = uv^T$, so that $U \in \mathbb{C}^{p\times q}$. Then

$$\begin{aligned} A_1 U &= A_1(uv^T) = (A_1 u)v^T = 0v^T = 0, \\ UA_2 &= (uv^T)A_2 = u(v^T A_2) = u0 = 0. \end{aligned}$$

By hypothesis, $U = uv^T = 0$. Hence either $u = 0$ or $v = 0$; equivalently, either A_1 or A_2 is nonsingular. $\square$

2. Resolutions of the unit matrix

In this section we describe block diagonalizations (1.1) in terms of resolutions of the unit matrix. For reader's convenience we summarize basic facts about block diagonalizations in the following proposition.

Proposition 2.1. (i) *If $A \in \mathbb{C}^{n\times n}$ and $(E_1, \ldots, E_m)$ is a nontrivial resolution of the unit $n \times n$ matrix commuting with A, then there is a similarity transformation ψ such that*

$$
\begin{aligned}
\psi(A) &= \operatorname{diag}(A_1, \ldots, A_m),\\
\psi(E_1) &= \operatorname{diag}(I, \ldots, 0),\\
&\;\;\ldots\\
\psi(E_m) &= \operatorname{diag}(0, \ldots, I).
\end{aligned}
\tag{2.1}
$$

(ii) *Conversely, every block diagonalization $\psi(A) = \operatorname{diag}(A_1, \ldots, A_m)$ gives rise to a nontrivial resolution $(E_1, \ldots, E_m)$ of the unit matrix satisfying* (2.1).

(iii) *If $A \sim \operatorname{diag}(A_1, \ldots, A_m)$, then $\sigma(A) = \sigma(A_1) \cup \ldots \cup \sigma(A_m)$.*

(iv) *Let $(E(\mu_1), \ldots, E(\mu_s))$ be the resolution of the unit matrix associated with the primary decomposition of Lemma* 1.4, *and let $\operatorname{diag}(A_1, \ldots, A_s)$ be the induced block diagonalization. Then $\sigma(A_i) = \{\mu_i\}$ for $i = 1, \ldots, s$.*

P r o o f. (i) Let $(E_1, \ldots, E_m)$ be a nontrivial resolution of the unit matrix commuting with A. According to Lemma 1.3, the subspaces $X_i = \operatorname{im} E_i$ are invariant under A. If Q, P are the matrices of Lemma 1.3, then the similarity transformation $\psi = \psi_Q$ has the required properties.

(ii) If $\psi(A) = \operatorname{diag}(A_1, \ldots, A_m)$, then the required resolution of the unit matrix consists of the idempotents $E_i = \psi^{-1}(\operatorname{diag}(0, \ldots, I, \ldots, 0))$, $i = 1, \ldots, m$.

(iii) Follows from observing that $\lambda I - A$ is singular if and only if at least one of the matrix blocks $\lambda I_i - A_i$ is singular.

(iv) Let $Q = [Q_1 \ldots Q_s]$ and $P = [P_1 \ldots P_s]$ be the partitioned matrices from Lemma 1.3 associated with the primary decomposition $\mathbb{C}^n = \ker(\mu_1 I - A)^{p_1} \oplus \ldots \oplus \ker(\mu_s I - A)^{p_s}$. If $Q^{-1}AQ = \operatorname{diag}(A_1, \ldots, A_s)$, then $AQ_i = Q_i A_i$ for $i = 1, \ldots, s$. Let $A_i w = \mu w$ for some $w \neq 0$. Then $\mu \in \{\mu_1, \ldots, \mu_s\}$,

$$Q_i A_i w = \mu Q_i w \Longrightarrow AQ_i w = \mu Q_i w \Longrightarrow Ax = \mu x,$$

where $x = Q_i w$ is a linear combination of the columns of Q_i, and $x \in \ker(\mu_i I - A)^{p_i}$. Since $\ker(\mu_i I - A) \subset \ker(\mu_i I - A)^{p_i}$ and since $\ker(\mu_i I - A)^{p_i} \cap \ker(\mu_j I - A)^{p_j} = \{0\}$ for $i \neq j$, we have $\mu = \mu_i$. Hence $\sigma(A_i) = \{\mu_i\}$. $\square$

We say that the resolution $(E_1, \ldots, E_m)$ of the preceding proposition *induces* the block diagonalization $A \sim \operatorname{diag}(A_1, \ldots, A_m)$. In general, the sets $\sigma(A_i)$ need not be disjoint. However, if they are, the block diagonalization has special properties, in particular, the (disjoint) partition of the spectrum determines uniquely the projections $E_1, \ldots, E_m$.

Proposition 2.2. *Let the spectrum $\sigma(A)$ of $A \in \mathbb{C}^{n\times n}$ be a disjoint union of the sets $S_1, \ldots, S_m$. Then A is similar to a block diagonal matrix $\operatorname{diag}(A_1, \ldots, A_m)$ with $\sigma(A_i) = S_i$ for $i = 1, \ldots, m$. The resolution $(E_1, \ldots, E_m)$ of the unit matrix inducing this diagonalization is uniquely determined by the sets S_i, $i = 1, \ldots, m$, and double commutes with A.*

P r o o f. The idea is to construct a resolution of the unit matrix from the eigenprojections and apply Proposition 2.1. For each $i \in \{1, \ldots, m\}$ define

$$E_i = \sum_{\mu \in S_i} E(\mu). \tag{2.2}$$

From Lemma 1.4 we deduce that each matrix E_i is idempotent, and that $E_1 + \ldots + E_m = I$. Since the $E(\mu)$ double commute with A, so do the E_i. By Proposition 2.1 (iv) we have $\sigma(A_i) = S_i$.

Suppose $(F_1, \ldots, F_m)$ is another nontrivial resolution of the unit matrix inducing the block diagonalization $\psi(A) = \operatorname{diag}(B_1, \ldots, B_m)$ with $\sigma(B_i) = S_i$, $i = 1, \ldots, m$. We show that $F_k = E_k$ for $k \in \{1, \ldots, m\}$ by an application of the holomorphic calculus whose definition and properties can be found in [6]. If f is a function holomorphic in an open set containing $\sigma(A)$, then

$$f(A) = \psi^{-1}(\operatorname{diag}(f(B_1), \ldots, f(B_m))).$$

Let $k \in \{1, \ldots, m\}$. Since the sets S_i are pairwise disjoint, there exists a function g holomorphic in an open neighbourhood of $\sigma(A)$ such that g is equal to 1 in a neighbourhood of S_k and to 0 in an open neighbourhood of $\bigcup_{j\neq k} S_j$. Then, by Proposition 2.1 (i),

$$g(A) = \psi^{-1}(\operatorname{diag}(0, \ldots, I, \ldots, 0)) = F_k.$$

On the other hand, $g(A) = \sum_{\mu \in S_k} E(\mu) = E_k$ by (1.3). $\square$

Example 3.1. There are resolutions of the unit matrix commuting with a given matrix A, which do not double commute with A. Define

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and $E_2 = I - E_1$. A direct verification shows that (E_1, E_2) is a resolution of I commuting with A. Further, U commutes with A, but does not commute with E_i, $i = 1, 2$. Hence $E_i \in \operatorname{comm}(A)$ but $E_i \notin \operatorname{comm}^2(A)$, $i = 1, 2$. In agreement with Proposition 2.2, the sets $\sigma(A_1) = \{0\}$ and $\sigma(A_2) = \{0, 1\}$ overlap where A_1, A_2 are the 2×2 block diagonal matrices of A.

Example 3.2. We give a characterization of the Drazin inverse through the construction described in Proposition 2.2. Let $A \in \mathbb{C}^{n\times n}$ be a singular matrix, and let $S_1 = \{0\}$, $S_2 = \sigma(A) \setminus \{0\}$. Let $\psi(A) = \operatorname{diag}(A_1, A_2)$ be a block diagonalization of A such that $\sigma(A_1) = S_1$ and $\sigma(A_2) = S_2$, where ψ is a similarity transformation. Then $0 \notin \sigma(A_2)$, and

$$A^D = \psi^{-1}(\operatorname{diag}(0, A_2^{-1}))$$

is the Drazin inverse of A [1, Theorem 7.2.1].

Let (E_1, E_2) be the resolution of the unit matrix inducing the above block diagonalization of A. The Drazin inverse A^D can be expressed in terms of E_1, E_2: First we observe that $\operatorname{diag}(0, A_2^{-1}) = \operatorname{diag}(I, A_2)^{-1} \operatorname{diag}(0, I)$, which under the similarity transformation ψ^{-1} becomes

$$A^D = (AE_2 + E_1)^{-1}E_2. \tag{3.1}$$

This is essentially the representation of A^D obtained by Rothblum [8]. By Proposition 2.2, the resolution (E_1, E_2) double commutes with A; consequently, the Drazin inverse also double commutes with A.

Example 3.3. The construction of Example 3.2 can be generalized when we partition the spectrum of $A \in \mathbb{C}^{n\times n}$ into two disjoint sets S_1 and S_2 such that $0 \notin S_2$. Let $\psi(A) = \operatorname{diag}(A_1, A_2)$ be a block diagonalization of A of Proposition 2.2 with $\sigma(A_i) = S_i$, $i = 1, 2$, and let (E_1, E_2) be the resolution of the unit matrix which induces it. The matrix $AE_2 + E_1$ is nonsingular being similar to

$$\operatorname{diag}(A_1, A_2)\ \operatorname{diag}(0, I) + \operatorname{diag}(I, 0) = \operatorname{diag}(I, A_2),$$

and we can define a matrix A^{Δ} by

$$A^{\Delta} = (AE_2 + E_1)^{-1}E_2. \tag{3.2}$$

If $S_1 = \{0\}$, then $A^{\Delta} = A^D$. The matrix A^{Δ} defined by (3.2) has many useful properties, and may be regarded as a generalization of the Drazin inverse.

Example 3.4. Let $A \in \mathbb{C}^{n\times n}$ be a power bounded matrix. Then the spectrum is known to lie in the closed unit disc, and the eigenvalues with the unit modulus have index 1. We partition the spectrum $\sigma(A)$ into two parts:

$$S_1 = \{\lambda \in \sigma(A)\colon\ |\lambda| < 1\}, \qquad S_2 = \{\lambda \in \sigma(A)\colon\ |\lambda| = 1\}.$$

Assuming that the sets S_1, S_2 are nonempty, there is a resolution (E_1, E_2) of the unit matrix corresponding to the sets S_1, S_2, respectively. (Explicitly, $E_1 = \sum\limits_{|\mu|<1} E(\mu)$ and $E_2 = \sum\limits_{|\mu|=1} E(\mu)$.) Suppose that $A \sim \operatorname{diag}(A_1, A_2)$ is the induced block diagonalization. Then (A_1^k) converges to 0 while (A_2^k) converges or boundedly oscillates as $k \to \infty$ depending on whether $\sigma_{\mathrm{per}} \subset \{1\}$ or not. (Here σ_{per} is the peripheral spectrum of A.)

4. Main results

We are now ready to present the main theorem of this paper, namely the characterization of those block diagonalization of a matrix for which the block diagonal matrices have disjoint spectra.

Theorem 4.1. *Let $(E_1, \ldots, E_m)$ be a nontrivial resolution of the unit $n \times n$ matrix commuting with a given matrix $A \in \mathbb{C}^{n\times n}$, and let $A \sim \operatorname{diag}(A_1, \ldots, A_m)$ be the induced block diagonalization. Then the union*

$$\sigma(A) = \sigma(A_1) \cup \ldots \cup \sigma(A_m)$$

is disjoint if and only if $(E_1, \ldots, E_m)$ double commutes with A.

Proof. If the union is disjoint, Proposition 2.2 gives the answer.

Let $m = 2$, and let (E_1, E_2) be a nontrivial resolution of the unit $n \times n$ matrix double commuting with A. Applying a similarity transformation if necessary we may assume that

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}$$

with $A_1 \in \mathbb{C}^{p\times p}$ and $A_2 \in \mathbb{C}^{q\times q}$, where $p+q=n$.

First we characterize the general form of a matrix $B \in \mathrm{comm}(A)$. Let

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

be the partition of B consistent with that of A, E_1 and E_2. The equality $AB = BA$ is equivalent to

$$A_1B_{11} = B_{11}A_1, \qquad A_2B_{22} = B_{22}A_2, \tag{4.1}$$

$$A_1B_{12} = B_{12}A_2, \qquad A_2B_{21} = B_{21}A_1. \tag{4.2}$$

Since $E_i \in \mathrm{comm}^2(A)$ for $i = 1, 2$, we have $BE_i = E_iB$ for $i = 1, 2$; this implies

$$B_{12} = 0 = B_{21}. \tag{4.3}$$

Let $\mu \in \mathbb{C}$. Suppose that $U \in \mathbb{C}^{p\times q}$ is any matrix satisfying

$$(A_1 - \mu I_1)U - U(A_2 - \mu I_2) = 0.$$

Then $A_1U - UA_2 = 0$, so that

$$B = \begin{bmatrix} 0 & U \\ 0 & 0 \end{bmatrix}$$

satisfies (4.1) and (4.2). Hence $B \in \mathrm{comm}(A)$. By (4.3), $U = 0$. According to Lemma 1.5, the matrices $A_1 - \mu I_1$ and $A_2 - \mu I_2$ cannot be simultaneously singular. Hence

$$\sigma(A_1) \cap \sigma(A_2) = \emptyset.$$

For a general m we proceed by induction, observing that if $(E_1, \ldots, E_m)$ is a nontrivial resolution of the unit matrix and if $F_1 = E_1$, $F_2 = E_2 + \ldots + E_m$, then (F_1, F_2) is also a nontrivial resolution of the unit matrix. This completes the proof. □

The following theorem gives a new characterization of an eigenprojection of A.

Theorem 4.2. *Let $A \in \mathbb{C}^{n\times n}$ and let $E \in \mathbb{C}^{n\times n}$ be a nonzero idempotent matrix. Then E is an eigenprojection of A if and only if*

$$E \in \mathrm{comm}^2(A) \quad \text{and} \quad \sigma((A - \mu I)E) = \{0\} \tag{4.4}$$

for some $\mu \in \mathbb{C}$.

P r o o f. We may assume that $\mu = 0$; the case of a general eigenvalue μ is obtained when A is replaced by $A-\mu I$. If E is the eigenprojection of A corresponding to 0, then E double commutes with A and AE is nilpotent by definition.

Conversely, assume that E satisfies the conditions of the theorem. If $E = I$, then $\sigma(A) = \{0\}$, and the eigenprojection of A at 0 is the unit matrix. If $E \neq I$, then $(E, I - E)$ is a nontrivial resolution of the unit matrix which double commutes with A. Let $A \sim \operatorname{diag}(A_1, A_2)$ be a block diagonalization of A induced by $(E, I - E)$. By Theorem 4.1, the spectra $\sigma(A_1)$, $\sigma(A_2)$ are disjoint. Since $E \sim \operatorname{diag}(I, 0)$, we have $AE \sim \operatorname{diag}(A_1, 0)$, which implies $\sigma(A_1) = \{0\}$. By Proposition 2.2, E is the eigenprojection of A at 0. $\square$

E x a m p l e 4.3. The preceding theorem enables us to give a new characterization of the Drazin inverse A^D of $A \in \mathbb{C}^{n\times n}$. Let E be a nonzero idempotent matrix double commuting with A, and let $A^q E = 0$. Theorem 4.2 quarantees that E is an eigenprojection of A at 0 as $A^q E = (AE)^q = 0$ is equivalent to $\sigma(AE) = \{0\}$. Consequently,

$$A^D = (A + \xi E)^{-1}(I - E)$$

for any nonzero $\xi \in \mathbb{C}$ by Rothblum's formula [8].

Theorem 4.2 can be used to recover the following characterizations of eigenprojections. The equivalence of (i), (ii) and (iii) was proved in [3] by a different method. Condition (iv) was explored in [2] for bounded linear operators acting on a Banach space.

Proposition 4.4. *Let $A \in \mathbb{C}^{n\times n}$ and let $E \in \mathbb{C}^{n\times n}$ be a nonzero idempotent matrix. Then the following conditions are equivalent.*

(i) *E is the eigenprojection of A at 0.*

(ii) (Rothblum [9]) *$E \in \operatorname{comm}(A)$ and $A + \xi E$ is nonsingular for all $\xi \neq 0$.*

(iii) (Koliha and Straškraba [3]) *$E \in \operatorname{comm}(A)$, $\sigma(AE) = \{0\}$ and $A + E$ is nonsingular.*

(iv) (Harte [2]) *$E \in \operatorname{comm}(A)$, $\sigma(AE) = \{0\}$ and $AU = I - E = VA$ for some $U, V \in \mathbb{C}^{n\times n}$.*

P r o o f. (i) $\Longrightarrow$ (ii) was established by Rothblum [9, Theorem 4.2].

(ii) $\Longrightarrow$ (iii) If $\xi \neq 0$, then

$$\left[(A + \xi E)^{-1}E + \xi^{-1}(I - E)\right](AE + \xi I) = I.$$

Hence $AE + \xi I$ is nonsingular for all $\xi \neq 0$, and $\sigma(AE) = \{0\}$. Setting $\xi = 1$, we get the third condition in (iii).

(iii) $\implies$ (iv) Set $S = (A+E)^{-1}(I-E)$. Then $SA = I - E = AS$, and (iv) holds with $U = V = S$.

(iv) $\implies$ (i) Set $S = VAU$ and $F = I - E$. Then we calculate that

$$SF = FS = S, \qquad SA = AS = F.$$

Let $B \in \operatorname{comm}(A)$. If q is such that $(AE)^q = A^q E = 0$, then

$$FB - FBF = FBE = F^q BE = S^q A^q BE = S^q BA^q E = 0.$$

Similarly, $BF - FBF = 0$, and $FB = BF$. This implies $F, E \in \operatorname{comm}^2(A)$, and E is the eigenprojection of A at 0 by Theorem 4.2. $\square$

References

[1] *S. L. Campbell, C. D. Meyer*: Generalized Inverses of Linear Transformations. Dover (reprint), New York, 1991.

[2] *R. E. Harte*: Spectral projections. Irish Math. Soc. Newsletter *11* (1984), 10–15.

[3] *J. J. Koliha, I. Straškraba*: Power bounded and exponentially bounded matrices. Appl. Math. *44* (1999), 289–308.

[4] *P. Lagerstrom*: A proof of a theorem on commutative matrices. Bull. Amer. Math. Soc. *51* (1945), 535–536.

[5] *P. Lancaster*: Theory of Matrices. New York, 1969.

[6] *I. Marek, K. Žitný*: Matrix Analysis for Applied Sciences. vol. 1; 2, Teubner-Texte zur Mathematik 60; 84, Teubner, Leipzig, 1983, 1986.

[7] *J. W. Robbin*: Matrix Algebra Using MINImal MATlab. A. K. Peters, Wellesley, 1995.

[8] *U. G. Rothblum*: A representation of the Drazin inverse and characterizations of the index. SIAM J. Appl. Math. *31* (1976), 646–648.

[9] *U. G. Rothblum*: Resolvent expansions of matrices and applications. Linear Algebra Appl. *38* (1981), 33–49.

[10] *D. A. Suprunenko, R. I. Tyshkevich*: Commutative Matrices. Academic Press, New York, 1968.

[11] *J. H. M. Wedderburn*: Lectures on Matrices. AMS Colloq. Publ. 17, Amer. Math. Soc., Providence, 1934.

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