THE INTERVAL FUNCTION OF A CONNECTED GRAPH AND A CHARACTERIZATION OF GEODETIC GRAPHS

Ladislav Nebeský, Praha

(Received July 20, 1999)

Abstract. The interval function (in the sense of H. M. Mulder) is an important tool for studying those properties of a connected graph that depend on the distance between vertices. An axiomatic characterization of the interval function of a connected graph was published by Nebeský in 1994. In Section 2 of the present paper, a simpler and shorter proof of that characterization will be given. In Section 3, a characterization of geodetic graphs will be established; this characterization will utilize properties of the interval function.

Keywords: graphs, distance, interval function, geodetic graphs

MSC 2000: 05C12, 05C75

By a graph we mean here a finite undirected graph (without loops and multiple edges). If G is a graph, then $V(G)$ and $E(G)$ denote its vertex set and its edge set, respectively. Moreover, if G is connected and $u, v \in V(G)$, then $d_G(u, v)$ denotes the distance between u and v in G.

The letters $f, g, \ldots, n$ will be used for denoting integers.

1. Let G be a connected graph. Put $W = V(G)$. Following Mulder's book [4], by the *interval function* of G we mean the mapping I_G of $W \times W$ into 2^W (i.e. into the set of all subsets of W) defined as follows:

$$I_G(r, s) = \{t \in W;\ d_G(r, s) = d_G(r, t) + d_G(t, s)\}$$

for each ordered pair $r, s \in W$.

Lemma 1. *Let G be a connected graph. Put $W = V(G)$ and $J = I_G$. Then J satisfies the following axioms* (i1)–(i7)*:*

(i1) $J(v, u) = J(u, v)$ *for all* $u, v \in W$*;*

(i2) $u \in J(u,v)$ *for all* $u, v \in W$*;*
(i3) *if* $x \in J(u,v)$*, then* $|J(u,x) \cap J(x,v)| = 1$ *for all* $u, v, x \in W$*;*
(i4) *if* $x \in J(u,v)$ *and* $y \in J(x,v)$*, then* $y \in J(u,v)$ *for all* $u, v, x, y \in W$*;*
(i5) *if* $x \in J(u,v)$ *and* $y \in J(x,v)$*, then* $x \in J(u,y)$ *for all* $u, v, x, y \in W$*;*
(i6) *if* $|J(u,x)| = 2 = |J(v,y)|$*,* $x, y \in J(u,v)$ *and* $u \in J(x,y)$*, then* $v \in J(x,y)$ *for all* $u, v, x, y \in W$*;*
(i7) *if* $|J(u,x)| = 2 = |J(v,y)|$*,* $x \in J(u,v)$*,* $x \notin J(u,y)$ *and* $y \notin J(u,v)$*, then* $v \in J(x,y)$ *for all* $u, v, x, y \in W$*.*

P r o o f. Axioms (i1)–(i5) can be verified easily. It is also not difficult to verify axioms (i6) and (i7); their verification can be found in [6]. □

R e m a r k 1. Properties of the interval function of a connected graph that are very similar to axioms (i1)–(i5) were presented in [4, 1.1.2.].

Let W be a finite nonempty set, and let J be a mapping of $W \times W$ into 2^W. We denote by $\mathbb{G}_J$ the graph H with $V(H) = W$ and

$$E(H) = \{rs;\ r, s \in W,\ r \neq s \text{ and } J(r,s) = \{r,s\} = J(s,r)\}.$$

If $\mathbb{G}_J$ is connected and $n \geqslant 0$, then we denote by J_n the mapping of

$$Z_n \stackrel{\text{df}}{=} \{(u,v) \in W \times W;\ d_{\mathbb{G}_J}(u,v) = n\}$$

into 2^W such that $J_n(x,y) = J(x,y)$ for each $(x,y) \in Z_n$.

Lemma 2. *If* G *is a connected graph and* $J = I_G$*, then* $G = \mathbb{G}_J$*.*

P r o o f is obvious. □

In Lemmas 3–5 and in Corollary 1 we will assume that a finite nonempty set W and a mapping J of $W \times W$ into 2^W are given.

Lemma 3. *Assume that* J *satisfies axioms* (i1), (i2) *and* (i3)*. Let* $u_0, \ldots, u_n \in W$*, where* $n \geqslant 1$*, and let*

$$|J(u_0,u_1)| = \ldots = |J(u_{n-1},u_n)| = 2. \tag{1}$$

Then $u_0u_1, \ldots, u_{n-1}u_n \in E(\mathbb{G}_J)$*.*

P r o o f is very easy. □

Lemma 4. *Assume that J satisfies axioms* (i1)–(i4). *Consider arbitrary distinct $u, v \in W$. Then* (a) *u and v belong to the same component of $\mathbb{G}_J$ and* (b) *there exists $w \in J(u, v)$ such that $|J(u, w)| = 2$.*

Proof. We proceed by induction on $|J(u, v)|$. By (i1) and (i2), $|J(u, v)| \geqslant 2$. If $|J(u, v)| = 2$, then, by virtue of (i3), u and v are adjacent and thus they belong to the same component; we put $w = v$. Now, let $|J(u, v)| > 2$. There exists $x \in J(u, v), u \neq x \neq v$. By (i1) and (i2), $u \in J(u, x)$ and $v \in J(x, v)$. According to (i1) and (i4), $J(u, x), J(x, v) \subseteq J(u, v)$. By virtue of (i3), $|J(u, x)|, |J(x, v)| < |J(u, v)|$. By the induction hypothesis, (a$'$) x belongs to the same component as u and to the same component as v, and (b$'$) there exists $w \in J(u, x)$ such that $|J(u, w)| = 2$. Obviously, u and v belong to the same component. Combining (i1) and (i4), we get $w \in J(u, v)$. Hence (a) and (b) hold. $\square$

Corollary 1. *If J satisfies axioms* (i1)–(i4), *then $\mathbb{G}_J$ is connected.*

Lemma 5. *Assume that J satisfies axioms* (i2), (i4) *and* (i5). *Let $u_0, \ldots, u_n, v \in W$, where $n \geqslant 1$, and let*

$$u_{i+1} \in J(u_i, v) \tag{2_i}$$

for each i, $0 \leqslant i \leqslant n - 1$. Then

$$u_j \in J(u_0, v) \text{ and } u_{j-1} \in J(u_0, u_j) \tag{3_j}$$

for each j, $1 \leqslant j \leqslant n$.

Proof. We proceed by induction on j. The case $j = 1$ is trivial. Let $j \geqslant 2$. By the induction hypothesis, $u_{j-1} \in J(u_0, v)$. By (2_{j-1}), $u_j \in J(u_{j-1}, v)$. As follows from (i4), $u_j \in J(u_0, v)$. As follows from (i5), $u_{j-1} \in J(u_0, u_j)$. $\square$

2. The interval function of a connected graph G plays a very important role in studying those structural properties of G that depend on distance between vertices. Cf. Mulder [4] or, for example, Bandelt and Mulder [1] and [2], and Bandelt, Mulder and Wilkeit [3].

However, the concept of the interval function of a connected graph is not only well-motivated; it is also transparently characterizable. Nebeský [6] proved a theorem which can be reformulated as follows: If W is a finite nonempty set, J is a mapping of $W \times W$ into 2^W and $\mathbb{G}_J$ is connected, then J is the interval function of $\mathbb{G}_J$ if and only if J satisfies axioms (i1)–(i7).

The proof given in [6] was unnecessarily complicated. A new proof will be presented here. It will utilize some ideas of the original proof but it will be shorter and significantly simpler. We will formulate a theorem slightly stronger than the one mentioned above:

Theorem 1. *Let W be a finite nonempty set, and let J be a mapping of $W \times W$ into 2^W. Then J is the interval function of a connected graph if and only if J satisfies axioms* (i1)–(i7).

P r o o f. If J is the interval function of a connected graph G, then, by virtue of Lemma 2, $V(G) = W$ and thus, by Lemma 1, J satisfies axioms (i1)–(i7).

Conversely, let J satisfy axioms (i1)–(i7). Put $G = \mathbb{G}_J$. By Corollary 1, G is connected. Put $d = d_G$ and $I = I_G$. We will prove that $J = I$.

Suppose, to the contrary, that $I \neq J$. Then there exists $n \geqslant 0$ such that $J_n \neq I_n$ and

$$J_f = I_f \text{ for all } f,\ 0 \leqslant f < n. \tag{4}$$

It is easy to see that $n \geqslant 2$. We distinguish two cases.

C a s e 1. Let $I_n \setminus J_n \neq \emptyset$. There exist $u, v, w \in W$ such that $d(u, v) = n$ and $w \in I(u, v) \setminus J(u, v)$. Thus, there exist $v_0, \ldots, v_n \in W$ and g, $0 < g < n$, such that $v_0 = v, v_n = u, v_g = w$ and $(v_n, \ldots, v_0)$ is a path from u to v of length $d(u, v)$ in G. Let $v_{n-1} \in J(u, v)$; clearly, $d(v_{n-1}, v) = n - 1$ and $v_g \in I(v_{n-1}, v)$; by (4), $v_g \in J(v_{n-1}, v)$ and by (i4), $v_g \in J(u, v)$; a contradiction. Hence $v_{n-1} \notin J(u, v)$. By (i1) and (i2), $v \in J(u, v)$. Recall that $d(u, v) = n$. Lemmas 3 and 4 imply that there exist $u_0, \ldots, u_n \in W$ such that $u_0 = u$, (1) holds and (2_i) holds for each i, $0 \leqslant i \leqslant n - 1$. By Lemma 5, ($3_n$) holds. As follows from Lemma 3,

$$d(u_j, v_j) \leqslant n \text{ for each } j,\ 0 \leqslant j \leqslant n. \tag{5}$$

Put $u_{-1} = v_{n-1}$. The following three statements hold for $i = 0$:

$$d(u_i, v_i) = n, \tag{6_i}$$
$$v \in J(u_i, v_i), \tag{7_i}$$
$$u_{i-1} \notin J(u_i, v_i). \tag{8_i}$$

By (3_n) and (i1), $u_{n-1} \in J(u_n, u_0)$. Since $v_n = u_0$, (8_n) does not hold.

There exists h, $0 \leqslant h \leqslant n - 1$, such that ($6_h$), ($7_h$) and ($8_h$) hold but at least one of (6_{h+1}), (7_{h+1}) and (8_{h+1}) does not. Combining (2_h), (7_h) and (i1) with (i4) and (i5), we get

$$u_{h+1} \in J(u_h, v_h), \tag{9}$$
$$v \in J(u_{h+1}, v_h). \tag{10}$$

As follows from (6_h),

$$d(u_h, v_{h+1}) = n - 1. \tag{11}$$

Clearly, $u_{h-1} \in I(u_h, v_{h+1})$. By (11) and (4), $u_{h-1} \in J(u_h, v_{h+1})$. Let $v_{h+1} \in J(u_h, v_h)$; by (i1) and (i4) we get $u_{h-1} \in J(u_h, v_h)$, which contradicts (8_h). Hence

$$v_{h+1} \notin J(u_h, v_h). \tag{12}$$

Let $u_{h+1} \in J(u_h, v_{h+1})$. By (11) and (4), $u_{h+1} \in I(u_h, v_{h+1})$. Thus $d(u_{h+1}, v_{h+1}) = n - 2$. As follows from (6_h), $d(u_{h+1}, v_h) = n - 1$ and $v_{h+1} \in I(u_{h+1}, v_h)$. By (4), $v_{h+1} \in J(u_{h+1}, v_h)$. Combining (9) and (i4), we see that $v_{h+1} \in J(u_h, v_h)$, which contradicts (12). Hence $u_{h+1} \notin J(u_h, v_{h+1})$. Thus, combining (9), (12) and (i7), we get

$$v_h \in J(u_{h+1}, v_{h+1}). \tag{13}$$

Let $d(u_{h+1}, v_{h+1}) < n$. By (13) and (4), $v_h \in I(u_{h+1}, v_{h+1})$. Therefore, $d(u_{h+1}, v_h) < n - 1$. This means that $d(u_h, v_h) < n$, which contradicts (6_h). Thus, by virtue of (5), we get (6_{h+1}).

Combining (10), (13), (i1) and (i4), we get (7_{h+1}).

Assume that $u_h \in J(u_{h+1}, v_{h+1})$. Combining (9), (13) and (i6), we see that $v_{h+1} \in J(u_h, v_h)$, which contradicts (12). We get (8_{h+1}), which is a contradiction with the definition of h.

Case 2. Let $I_n \subseteq J_n$. Then $J_n \setminus I_n \neq \emptyset$. There exist $u, v, z \in W$ such that $d(u, v) = n$ and $z \in J(u, v) \setminus I(u, v)$. By (i2), $z \neq u$. By Lemma 3, there exists $t \in J(u, z)$ such that $|J(u, t)| = 2$. By (i1), (i4) and (i5), $t \in J(u, v)$ and $z \in J(t, v)$. If $d(t, v) < n$, then $d(t, v) = n - 1$, $t \in I(u, v)$ and, by (4), $z \in I(t, v) \subseteq I(u, v)$; a contradiction. Hence $d(t, v) \geqslant n$. Lemmas 3 and 4 imply that there exist $u_0, \ldots, u_n \in W$ such that $u_0 = u$, $u_1 = t$, (1) holds and (2_i) holds for each $i, 0 \leqslant i \leqslant n - 1$. Since $d(u_1, v) \geqslant n$ and $u_1u_2, \ldots, u_{n-1}u_n \in E(G)$, we have

$$d(u_{i+1}, v) \geqslant n - i \tag{14_i}$$

for each i, $0 \leqslant i \leqslant n - 1$. Thus $u_n \neq v$. By Lemma 5, (3_n) holds. Since $d(u, v) = n$, there exist $v_0, \ldots, v_n \in W$ such that $v_0 = v, v_n = u$ and $(v_n, \ldots, v_0)$ is a path from u to v of length $d(u, v)$ in G. Thus

$$d(v, v_i) = i \tag{15_i}$$

for each i, $0 \leqslant i \leqslant n - 1$. Moreover, (5) holds.

Obviously, both (6_0) and (7_0) hold. By (3_n), $u_n \in J(v_n, v)$. Since $u_n \neq v$, (i1), (i2) and (i3) imply that (7_n) does not hold.

There exists h, $0 \leqslant h \leqslant n-1$, such that (6_h) and (7_h) hold but at least one of (6_{h+1}) and (7_{h+1}) does not. Similarly as in Case 1, we get (9), (10) and (11).

Let $d(u_{h+1}, v_h) < n$. Combining (4) and (10), we get $v \in I(u_{h+1}, v_h)$. By virtue of (14_h), we have $d(v, v_h) < h$, which contradicts (15_h). Hence $d(u_{h+1}, v_h) \geqslant n$.

Let $d(u_{h+1}, v_{h+1}) < n$. Since $d(u_{h+1}, v_h) \geqslant n$, we have $d(u_{h+1}, v_h) = n$ and $d(u_{h+1}, v_{h+1}) = n-1$. Therefore, $v_{h+1} \in I(u_{h+1}, v_h)$. Since $I_n \subseteq J_n$, we get $v_{h+1} \in J(u_{h+1}, v_h)$. Thus, combining (9) and (i5), we see that $u_{h+1} \in J(u_h, v_{h+1})$. By (11) and (4), $u_{h+1} \in I(u_h, v_{h+1})$ and therefore, $d(u_{h+1}, v_{h+1}) = n-2$; a contradiction. Thus, by virtue of (5), (6_{h+1}) holds.

By virtue of (6_h), $v_{h+1} \in I(u_h, v_h)$; by (6_{h+1}), $u_h \in I(u_{h+1}, v_{h+1})$. Recall that $I_n \subseteq J_n$. We have $v_{h+1} \in J(u_h, v_h)$ and $u_h \in J(u_{h+1}, v_{h+1})$. Thus, (9) and (i6) imply (13). Similarly as in Case 1, we get (7_{h+1}), which is a contradiction with the definition of h.

Thus $J = I$, which completes the proof. □

R e m a r k 2. An extension of Theorem 1 (with a different and rather long proof) was presented in Nebeský [8].

3. A graph G is said to be *geodetic* if it is connected and for each pair $r, s \in V(G)$, there exists exactly one path from r to s of length $d_G(r, s)$. The problem to characterize geodetic graphs was stated in Ore's book [10].

The next theorem gives a characterization of geodetic graphs based on properties of the interval function.

Theorem 2. *Let G be a graph. Put $W = V(G)$. Then G is geodetic if and only if there exists a mapping J of $W \times W$ into 2^W such that $G = \mathbb{G}_J$ and J satisfies axioms* (i1), (i2), (i3) *and the following axioms* (g1) *and* (g2)*:*

(g1) *if $x \in J(u, v)$, then $J(u, v) = J(u, x) \cup J(x, v)$ for all $u, v, x \in W$;*

(g2) *if $|J(u, x)| = 2 = |J(v, y)|$ and $x \in J(u, v)$, then $x \in J(u, y)$ or $v \in J(x, y)$ for all $u, v, x, y \in W$.*

P r o o f. I. Assume that G is geodetic. Then G is connected. Let J denote its interval function. By Lemma 1, J satisfies (i1), (i2) and (i3). As immediately follows from the definition of a geodetic graph, J satisfies (g1). We will show that J satisfies (g2).

Consider arbitrary $u, v, x, y \in W$. Assume that $|J(u, x)| = 2 = |J(v, y)|$ and $x \in J(u, v)$. If $y = u$, then $x = v$ and thus $v \in J(x, y)$. Let $y \neq u$. Since G is geodetic, there exists exactly one path P from u to v of length $d_G(u, v)$ in G. If $y \in J(u, v)$, then y belongs to P and thus $x \in J(u, y)$. Let $y \notin J(u, v)$. Then

$d_G(x,v) \leqslant d_G(x,y) \leqslant d_G(x,v)+1$. If $d_G(x,y) = d_G(x,v)+1$, then $v \in J(x,y)$. If $d_G(x,y) = d_G(x,v)$, then $x \in J(u,y)$. Thus J satisfies (g2).

II. Conversely, assume that J satisfies (i1), (i2), (i3), (g1) and (g2). The fact that J satisfies (g2) implies that J satisfies (i7). First, we will show that J also satisfies (i4), (i5) and (i6).

Consider arbitrary $u,v,x,y \in W$.

Let $x \in J(u,v)$ and $y \in J(x,v)$. By (g1),

$$J(u,v) = J(u,x) \cup J(x,v) \text{ and } J(x,v) = J(x,y) \cup J(y,v). \tag{16}$$

Combining (i2) and (16), we get $y \in J(u,v)$. By (g1) again,

$$J(u,v) = J(u,y) \cup J(y,v). \tag{17}$$

If $x=y$, then (i1), (i2) and (17) imply that $x \in J(u,y)$. Let $x \neq y$. By (i2), $x \in J(x,y)$. Since $y \in J(x,v)$, it follows from (i3) that $J(x,y) \cap J(y,v) = \{y\}$, and therefore, $x \notin J(y,v)$. Since $x \in J(u,v)$, (17) implies that $x \in J(u,y)$. We see that J satisfies (i4) and (i5).

Let $|J(u,x)| = 2 = |J(v,y)|$ and let $x,y \in J(u,v)$. As follows from (i1), (i2) and (i3), we have $x \neq u \neq v \neq y$, $J(u,x) = \{u,x\}$, $J(v,y) = \{v,y\}$ and $u,v \in J(u,v)$. Hence $|J(u,v)| \geqslant 2$. First, let $|J(u,v)| = 2$. Then $u=y$ and $v=x$. We have $u,v \in J(u,v)$. Now, let $|J(u,v)| > 2$. Then $x \neq v$ and $y \neq u$. It follows from (i3) and (g1) that $|\{u,x\} \cap J(x,v)| = 1$ and $J(u,v) = J(u,x) \cup J(x,v)$. Thus $u \notin J(x,v)$. As follows from (i3) and (g1) again, $|J(x,y) \cap \{y,v\}| = 1$ and $J(x,v) = J(x,y) \cup \{y,v\}$. By (i1) and (i2), $y \in J(x,y)$, Thus $v \notin J(x,y)$. Since $u \notin J(x,v)$, we get $u \notin J(x,y)$. We see that J satisfies also (i6).

Put $G = \mathbb{G}_J$. By Corollary 1, G is connected. By Theorem 1 and Lemma 2, J is the interval function of G. Recall that J satisfies (g1). We will show that

$$\text{there exists exactly one path from } u \text{ to } v \text{ of length } d_G(u,v) \text{ in } G \tag{18}$$

for each pair of distinct $u,v \in W$.

Consider arbitrary distinct $u,v \in W$. Put $n = d_G(u,v)$. To prove (18), we will proceed by induction on n. Obviously, $n \geqslant 1$. There exist $u_0, u_1, \ldots, u_n \in W$ such that $u_0 = u$, $u_n = v$ and $(u_0, u_1, \ldots, u_n)$ is a path in G. Since $d_G(u,v) = n$, we see that u_i and u_j are adjacent in G if and only if $|i-j| = 1$ for all i and j, $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$. Clearly, if $n=1$, then (18) holds. Let $n>1$. Then $u_1 \neq v$. Since $d_G(u_1,v) = n-1$, the induction hypothesis implies that $(u_1, \ldots, u_n)$ is the only path from u_1 to v of length $n-1$ in G. Thus $J(u_1,v) = \{u_1, \ldots, u_n\}$. By virtue of (g1), $J(u,v) = \{u_0, u_1, \ldots, u_n\}$. This means that $(u_0, u_1, \ldots, u_n)$ is the only path from u to v of length n in G.

Thus G is geodetic. $\square$

R e m a r k 3. A characterization of geodetic graphs utilizing properties of the set of all shortest paths was given in Nebeský [5] and [7]. A characterization of geodetic graphs based on a binary operation on the vertex set was given in Nebeský [9].

References

[1] *H.-J. Bandelt, H. M. Mulder*: Regular pseudo-median graphs. J. Graph Theory *12* (1988), 533–549.
[2] *H.-J. Bandelt, H. M. Mulder*: Three interval conditions for graphs. Ars Combin. *29B* (1990), 213–223.
[3] *H.-J. Bandelt, H. M. Mulder, E. Wilkeit*: Quasi-median graphs and algebras. J. Graph Theory *18* (1994), 681–703.
[4] *H. M. Mulder*: The Interval Function of a Graph. Mathematical Centre Tracts 132, Mathematisch Centrum, Amsterdam, 1980.
[5] *L. Nebeský*: A characterization of the set of all shortest paths in a connected graph. Math. Bohem. *119* (1994), 15–20.
[6] *L. Nebeský*: A characterization of the interval function of a connected graph. Czechoslovak Math. J. *44* (1994), 173–178.
[7] *L. Nebeský*: A characterization of geodetic graphs. Czechoslovak Math. J. *45* (1995), 491–493.
[8] *L. Nebeský*: Characterizing the interval function of a connected graph. Math. Bohem. *123* (1998), 137–144.
[9] *L. Nebeský*: An algebraic characterization of geodetic graphs. Czechoslovak Math. J. *48* (1998), 701–710.
[10] *O. Ore*: Theory of Graphs. Amer. Math. Soc. Colloq. Publ. 38, Providence, R. I., 1962.

Author's address: *Ladislav Nebeský*, Univerzita Karlova v Praze, Filozofická fakulta, nám. J. Palacha 2, 116 38 Praha 1, Czech Republic, e-mail: ladislav.nebesky@ff.cuni.cz.