ON THE OSCILLATION OF VOLTERRA SUMMATION EQUATIONS

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Abstract. The asymptotic and oscillatory behavior of solutions of Volterra summation equations

$$y_n = p_n \pm \sum_{s=0}^{n-1} K(n,s) f(s, y_s),\ n \in \mathbb{N}$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$, are studied. Examples are included to illustrate the results.

Keywords: Volterra summation equations, oscillation, asymptotic behavior

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1. Introduction

Qualitative properties of solutions of difference equations assume importance in the absence of closed form solutions. In case the solutions are not expressible in terms of the usual known functions, an analysis of the equation is necessary to find the facets of the solutions. One such qualitative property, which has wide applications, is the oscillation of solutions. It is but natural to expect to know the solution in an explicit form which unfortunately is not always possible. Hence a rewarding alternative is to resort to the qualitative study.

In the qualitative theory of difference equation oscillatory and asymptotic behavior of solutions play an important role. This is apparent from a large number of research papers dedicated to it. The references [1, 2, 4] present a fairly exhaustive list for the interested reader. However, oscillation results for summation equations of the Volterra type are scant eventhough such equations arise in the study of mathematical biology, engineering etc. in which discrete models are used (see, for example, the model of the spread of an epidemic [p. 99, 4]). Some recent results on Volterra summation equations can be found in [3, 5, 6, 7]. In this paper we establish sufficient conditions for all solutions of the equations

$$y_n = p_n - \sum_{s=0}^{n-1} K(n,s) f(s, y_s), \ n \in \mathbb{N} \tag{1}$$

and

$$y_n = p_n + \sum_{s=0}^{n-1} K(n,s) f(s, y_s), \ n \in \mathbb{N} \tag{2}$$

where $\mathbb{N} = \{0, 1, 2, \ldots\}$ to be oscillatory. Further we obtain growth estimates on solutions of equations (1) and (2).

One may easily observe that the conditions presented in the paper guarantee, roughly speaking, that the solutions $\{y_n\}$ behave like the sequence $\{p_n\}$.

2. Assumptions and definitions

(C_1) $\{p_n\}$ is a sequence of real numbers;
(C_2) $K\colon \mathbb{N} \times \mathbb{N} \to \mathbb{R}^+$ and $K(n,s) = 0$, $s > n$;
(C_3) $f\colon \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\varphi f(n, \varphi) > 0$ for $\varphi \neq 0$.
By a solution of equations (1) and (2) we mean a real sequence $\{y_n\}$ satisfying equations (1) and (2) for all $n \in \mathbb{N}$. A solution of equations (1) and (2) is said to be oscillatory if it is neither eventually positive nor eventually negative, and nonoscillatory otherwise. An oscillatory solution $\{y_n\}$ is said to be properly unbounded if $\lim\limits_{n\to\infty} \sup(y_n) = +\infty$ and $\lim\limits_{n\to\infty} \inf(y_n) = -\infty$. A nonoscillatory solution is properly unbounded if it is unbounded.

3. Main results

We begin with the following theorem.

Theorem 1. *Assume that*

$$\frac{f(n,u)}{u} \leqslant M \tag{3}$$

for some $M > 0$, $n \in \mathbb{N}$ and $u \neq 0$. Further assume that there exist positive real sequences $\{q_n\}, \{h_n\}$, $n \in \mathbb{N}$, such that $h_s = 0$ for $s > n$,

$$K(n,s) \leqslant q_n h_s \tag{4}$$

$$\sum_{n=0}^{\infty} h_n < \infty \tag{5}$$

and

$$q_n \text{ and } \left(\frac{p_n}{n^\alpha}\right) \text{ are bounded for } n \in \mathbb{N} \tag{6}$$

where $\alpha \geqslant 1$. *If* $\{y_n\}$ *is any solution of equation* (1), *then* $y_n = O(n^\alpha)$, *that is,* $\lim_{n\to\infty} \sup\left(\frac{y_n}{n^\alpha}\right) < +\infty$.

P r o o f. From equation (1) we have

$$\frac{|y_n|}{n^\alpha} \leqslant \frac{|p_n|}{n^\alpha} + \sum_{s=0}^{n-1} q_n h_s \frac{f(s, y_s)}{|y_s|} \frac{|y_s|}{s^\alpha} \leqslant K + L \sum_{s=0}^{n-1} h_s \frac{|y_s|}{s^\alpha}$$

for some positive constants K and L. The result follows by the discrete Gronwall's inequality [1]. □

E x a m p l e 1. Consider the difference equation

$$y_n = n + \frac{1}{2^{n-1}} - \sum_{s=0}^{n-1} \frac{2^{-s}}{(s+1)(s+2)} y_s \left(1 - |y_s|\right). \tag{7}$$

All conditions of Theorem 1 are satisfied for $\alpha = 1$. If $\{y_n\}$ is any solution of equation (7), then $\{y_n\} = O(n)$. In fact $\{y_n\} = \{(n+2)\}$ is one such solution.

Theorem 2. *Suppose conditions* (3), (4) *and* (6) *hold. Further assume that*

$$\sum^{\infty} n^\alpha h_n < \infty, \tag{8}$$

$$\lim_{n\to\infty} \sup(p_n) = \infty, \quad \lim_{n\to\infty} \inf(p_n) = -\infty. \tag{9}$$

Then all solutions of equation (1) *are oscillatory.*

P r o o f. Since condition (5) of Theorem 1 is implied by (8) for $n \geqslant 1$, we can assume that the conclusion of Theorem 1 holds. Without loss of generality, suppose that $N \in \mathbb{N}$ is large enough so that $y_n > 0$ for $n \geqslant N$.
From equation (1),

$$y_n = p_n - \sum_{s=0}^{N-1} K(n,s) f(s, y_s) - \sum_{s=N}^{n-1} K(n,s) f(s, y_s) \tag{10}$$

$$\leqslant p_n + \sum_{s=0}^{N-1} q_n h_s f(s, y_s) + \sum_{s=N}^{n-1} q_n h_s s^\alpha \frac{f(s, y_s)}{y_s} \frac{y_s}{s^\alpha}. \tag{11}$$

Now $\{q_n\}$ is bounded, and by Theorem 1, $\{\frac{y_n}{n^\alpha}\}$ is bounded. In view of condition (8), the last two summations on the righthand side of (11) are finite. Since $y_n > 0$ and (9) holds, we obtain a contradiction. This completes the proof. $\square$

E x a m p l e 2. Consider the Volterra summation equation

$$y_n = (-1)^n(n+1) - \frac{2}{3}\left(\frac{(-1)^n}{2^n} + 1\right) - \sum_{s=0}^{n-1} \frac{2^{-s}}{(s+1)} y_s, \quad n \in \mathbb{N}. \tag{12}$$

Equation (12) satisfies all conditions of Theorem 2. Hence all solutions of equation (12) are oscillatory. In fact $\{y_n\} = \{(-1)^n(n+1)\}$ is one such solution.

Corollary 3.2 of Greaf and Thandapani [3] does not yield this conclusion.
Our next theorem improves condition (8) of Theorem 2.

Theorem 3. *Suppose all conditions of Theorem* 1 *hold. Further suppose that* (9) *holds and*

$$\lim_{n\to\infty} \sup(q_n) \sum_{s=N}^{n-1} s^\alpha h_s < \infty. \tag{13}$$

Then all solutions of equation (1) *are oscillatory.*

P r o o f. Without any loss of generality, let $y_n > 0$ for $n \geqslant N \in \mathbb{N}$ be a solution of equation (1). As in the proof of Theorem 1 , we see that

$$y_n = O(n^\alpha). \tag{14}$$

From equation (1),

$$\begin{aligned} y_n &\leqslant p_n - \sum_{s=0}^{N-1} K(n,s) f(s, y_s) + \sum_{s=N}^{n-1} q_n s^\alpha h_s \frac{f(s,y_s)}{y_s} \frac{y_s}{s^\alpha} \\ &\leqslant p_n + \sum_{s=0}^{N-1} K(n,s) f(s,y_s) + M q_n \sum_{s=N}^{n-1} s^\alpha h_s \frac{y_s}{s^\alpha}. \end{aligned} \tag{15}$$

From (13), (14) and the boundedness of $\{p_n\}$, we see that the last two summations in inequality (15) are bounded. From (9) we have

$$\lim_{n\to\infty} \sup(p_n) = \infty$$

and

$$\lim_{n\to\infty} \inf(p_n) = -\infty$$

which gives a contradiction. The proof is now complete. $\square$

E x a m p l e 3. Consider the difference equation

$$
\begin{aligned}
y_n = (-1)^{n+1}(2n+1) &- \frac{\mathrm{e}}{(\mathrm{e}+1)(n+1)(n+2)}\left(\frac{(-1)^n}{\mathrm{e}^n}-1\right)\\
&- \sum_{s=0}^{n-1}\frac{(4s^2+4s+2)\mathrm{e}^{-s}}{(n+1)(n+2)(2s+1)}\left(\frac{y_s^2}{1+y_s^2}\right),\ n\in\mathbb{N}.
\end{aligned}
\tag{16}
$$

Here, if we choose $\alpha = 1$ and

$$K(n,s)=\frac{(4s^2+4s+2)\mathrm{e}^{-s}}{(n+1)(n+2)(2s+1)},\ n\geqslant s=0,\ s>n,$$

then all conditions of Theorem 3 are satisfied. Therefore, all solutions of equations (16) are oscillatory. In fact $\{y_n\} = \{(-1)^{(n+1)}(2n+1)\}$ is one such solution.

In our next theorem we do not require that $\{p_n\}$ be unbounded.

Theorem 4. *Suppose conditions* (3)–(5) *hold,* $\{q_n\}$ *and* $\{p_n\}$ *are bounded and*

$$\lim_{n\to\infty}\inf\sum_{s=0}^{n-1}p_s=-\infty,\qquad \lim_{n\to\infty}\sup\sum_{s=0}^{n-1}p_s=+\infty. \tag{17}$$

Further suppose that

$$\sum_{n=0}^{\infty}q_n<\infty. \tag{18}$$

Then all solutions of equation (1) *are bounded and oscillatory.*

P r o o f. Let $\{y_n\}$ be any solution of equation (1). Then boundedness of $\{y_n\}$ follows by the discrete Gronwall's inequality since $\{p_n\}$ is bounded. Now suppose that $\{y_n\}$ is nonoscillatory, without loss of generality suppose there exists a large $N\in\mathbb{N}$ such that $y_n>0$ for $n\geqslant N$.
From equation (1),

$$
\begin{aligned}
\sum_{s=1}^{n-1}y_s &= \sum_{s=N}^{n-1}p_s-\sum_{s=N}^{n-1}\sum_{t=0}^{s-1}K(s,t)f(t,y_t)\\
&= \sum_{s=N}^{n-1}p_s-\sum_{s=N}^{n-1}\sum_{t=0}^{N-1}K(s,t)f(t,y_t)-\sum_{s=N}^{n-1}\sum_{t=N}^{s-1}K(s,t)f(t,y_t)\\
&\leqslant \sum_{s=N}^{n-1}p_s+\sum_{s=N}^{n-1}q_s\sum_{t=0}^{N-1}h_t\frac{f(t,y_t)}{y_t}y_t+\sum_{s=N}^{n-1}q_s\sum_{t=N}^{s-1}h_t\frac{f(t,y_t)}{y_t}y_t.
\end{aligned}
\tag{19}
$$

Since $\{y_n\}$ and $\frac{f(n,y_n)}{y_n}$ are bounded and conditions (5) and (18) hold, the last two summations on the righthand side of (19) are finite. Since

$$\sum_{s=N}^{n-1} y_s > 0$$

for $n \geqslant N$, we have a contradiction to condition (17). This completes the proof of the theorem. □

Example 4. The difference equation

$$y_n = \frac{1}{n+1} + \frac{2}{(n+1)(n+2)}\left(1 - \frac{1}{2^n}\right) - \sum_{s=0}^{n-1} \frac{(s+1)2^{-s}}{(n+1)(n+2)} y_s \tag{20}$$

satisfies all conditions of Theorem 4 except condition (17). The equation (20) has a bounded nonoscillatory solution $\{y_n\} = \{\frac{1}{n+1}\}$.

Theorem 5. *In addition to conditions of Theorem* 4, *assume*

$$\Delta p_n \to 0, \quad \Delta q_n \to 0, \quad h_n q_{n+1} \to 0 \tag{21}$$

as $n \to \infty$. *Further assume*

$$|\Delta_n K(n,s)| \leqslant |h_n \Delta q_n|, \quad s \leqslant n, \quad n \in N. \tag{22}$$

Let $\{y_n\}$ *be any solution of equation* (1). *Then* $\{y_n\}$ *is bounded and* $\Delta y_n \to 0$ *as* $n \to \infty$.

Proof. From equation (1),

$$\Delta y_n = \Delta p_n - K(n+1,n) f(n,y_n) - \sum_{s=0}^{n-1} \Delta_n K(n,s) f(s,y_s)$$

$$|\Delta y_n| \leqslant |\Delta p_n| + M q_{n+1} h_n |y_n| + |\Delta q_n| \sum_{s=0}^{n-1} |h_s| f(s,y_s). \tag{23}$$

Since the conditions of Theorem 4 hold, $\{y_n\}$ is bounded. From conditions (21) and (22), we see that (23) implies $\Delta y_n \to 0$ as $n \to \infty$. This completes the proof of the theorem. □

In the following theorem we prove a partial converse of Theorem 2.

Theorem 6. *Suppose conditions* (3), (4), (6) *and* (8) *hold. Let* $\{y_n\}$ *be a properly unbounded oscillatory solution of equation* (1). *Then*

$$\lim_{n\to\infty} \sup(p_n) = +\infty$$

and

$$\lim_{n\to\infty} \inf(p_n) = -\infty.$$

P r o o f. From equation (1),

$$y_n = p_n - \sum_{s=0}^{N-1} K(n,s) f(s,y_s) - \sum_{s=N}^{n-1} K(n,s) f(s,y_s). \tag{24}$$

Since the conditions of Theorem 1 are satisfied, we see that $\{\frac{y_n}{n^\alpha}\}$ is bounded. Now

$$\Bigl|\sum_{s=N}^{n-1} K(n,s) f(s,y_s)\Bigr| \leqslant \sum_{s=N}^{n-1} q_n s^\alpha h_s \frac{f(s,y_s)}{y_s} \frac{y_s}{s^\alpha}. \tag{25}$$

In view of condition (8) of Theorem 2, we find that the lefthand side of (25) is bounded. Thus the last two summations in (24) are finite. The conclusion now follows from the fact that $\{y_n\}$ is a properly unbounded oscillatory solution of equation (1). □

Corollary 7. *Suppose conditions* (3), (4), (6) *and* (8) *hold. Then a necessary and sufficient condition for all properly unbounded solutions of equation* (1) *to be oscillatory is that*

$$\lim_{n\to\infty} \sup(p_n) = +\infty$$

and

$$\lim_{n\to\infty} \inf(p_n) = -\infty.$$

R e m a r k 1. From the conditions given in Theorems 2–6 and Corollary 7 one can see that the oscillatory behavior of solutions of equation (1) may depend on the behavior of the sequence $\{p_n\}$.

Next we study the asymptotic behavior of solutions of equation (2). To prove our results we need the following lemma which is a discrete analogue of Lemma 2 given in [9].

Lemma 8. *Let $\{w_n\}$ be a positive nondecreasing real sequence such that $\lim_{n\to\infty} w_n = \infty$. If*

$$\sum_{n=N}^{\infty} \frac{F_n}{w_n} < \infty$$

for all $F_n \geqslant 0$, $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} \frac{1}{w_n} \sum_{s=N}^{n-1} F_s = 0.$$

Theorem 9. *Assume that*

$$|f(n,u)| \leqslant a_n \varphi(|u|) \tag{26}$$

where $\{a_n\}$ is a positive real sequence and $\varphi\colon [0,\infty) \to (0,\infty)$ is a positive, non-decreasing continuous function such that

$$\frac{1}{v}\varphi(u) \leqslant \varphi\left(\frac{u}{v}\right) \tag{27}$$

for $u \geqslant 0$, $v \geqslant 1$ and φ is sub-multiplicative for $u \geqslant 0$. Further assume that there exists a non-negative function $Q(n,s)$ such that

$$|K(n,s)| \leqslant Q(n,s) \tag{28}$$

for $n \geqslant s \in \mathbb{N}$. If $w_n \geqslant 1$ satisfies the hypothesis of Lemma 8 and $Q(n,s)$ is non-increasing in n for each s and

$$\lim_{n\to\infty} \frac{p_n}{w_n} < \infty, \tag{29}$$

$$\sum^{\infty} Q(s,s) a_s < \infty, \tag{30}$$

and

$$\sum^{\infty} \frac{Q(s,s)}{w_s} a_s \varphi(w_s) < \infty, \tag{31}$$

then for any solution $\{y_n\}$ *of equation* (2) *we have*

$$\lim_{n\to\infty} \frac{y_n}{w_n}$$

exists and

$$\lim_{n\to\infty} \frac{y_n}{w_n} = \lim_{n\to\infty} \frac{p_n}{w_n}.$$

Proof. Since

$$\lim_{n\to\infty} \frac{p_n}{w_n} < \infty,$$

there exists a constant c such that $\left|\frac{p_n}{w_n}\right| \leqslant c$ for all $n \in \mathbb{N}$. Dividing the equation (2) by w_n, we have

$$\frac{y_n}{w_n} = \frac{p_n}{w_n} + \frac{1}{w_n} \sum_{s=0}^{n-1} K(n,s) f(s,y_s) \tag{32}$$

and by the assumptions

$$\left|\frac{y_n}{w_n}\right| \leqslant \left|\frac{p_n}{w_n}\right| + \frac{1}{w_n} \sum_{s=0}^{n-1} Q(n,s) a_s \varphi\left(|y_s|\right) \leqslant c + \sum_{s=0}^{n-1} Q(s,s) a_s \varphi\Big(\left|\frac{y_s}{w_s}\right|\Big).$$

Using Theorem 1.6.4 of [8], we have

$$\left|\frac{y_n}{w_n}\right| \leqslant c + G^{-1}\Big[\sum_{s=0}^{n-1} Q(s,s) a_s\Big]$$

where

$$G(r) = \sum_{s=0}^{r-1} \frac{1}{\varphi(s)}.$$

Hence by (30) there exists a constant $c_1 > 0$ such that

$$c_1 = c + G^{-1}\Big[\sum_{s=0}^{\infty} Q(s,s) a_s\Big],$$

so that

$$|y_n| \leqslant c_1 w_n$$

for $n \in \mathbb{N}$. Thus $\lim\limits_{n\to\infty} \left|\frac{y_n}{w_n}\right|$ exists. Now

$$\begin{aligned} \left|\frac{1}{w_n} \sum_{s=0}^{n-1} K(n,s) f(s,y_s)\right| &\leqslant \frac{1}{w_n} \sum_{s=0}^{n-1} Q(n,s) a_s \varphi(|y_s|) \\ &\leqslant \frac{1}{w_n} \sum_{s=0}^{n-1} Q(s,s) a_s \varphi(c_1) \varphi(w_s). \end{aligned} \tag{33}$$

Set $F_s = Q(s,s)a_s\varphi(w_s)$ in Lemma 8, then

$$\sum_{s=0}^{\infty} \frac{F_s}{w_s} < \infty$$

by (32), so Lemma 8 implies that

$$\lim_{n\to\infty} \frac{1}{w_n} \sum_{s=0}^{n-1} F_s = 0$$

and taking the limit as $n \to \infty$ in (33), we conclude

$$\lim_{n\to\infty} \left| \frac{1}{w_n} \sum_{s=0}^{n-1} K(n,s) f(s, y_s) \right| = 0.$$

As $n \to \infty$ in (32), we have

$$\lim_{n\to\infty} \frac{y_n}{w_n} = \lim_{n\to\infty} \frac{p_n}{w_n}$$

and the proof is complete. □

We conclude this paper with the following theorem.

Theorem 10. *Assume conditions* (26)–(28) *hold and let* $w_n \geqslant 1$ *be as in Lemma* 8. *Let* $Q(n,s) = h_n q_s$ *where* $h_n \geqslant 1$ *and non-decreasing with* $q_s \geqslant 0$ *for all* $s \in \mathbb{N}$. *Assume*

$$\lim_{n\to\infty} \frac{p_n}{w_n h_n} < \infty, \tag{34}$$

$$\sum^{\infty} h_s q_s a_s < \infty, \tag{35}$$

and

$$\sum^{\infty} \frac{q_s}{w_s} a_s \varphi(w_s) < \infty. \tag{36}$$

Then for any solution $\{y_n\}$ *of equation* (2) *we have*

$$\lim_{n\to\infty} \frac{y_n}{w_n h_n}$$

exists and

$$\lim_{n\to\infty}\frac{y_n}{w_nh_n}=\lim_{n\to\infty}\frac{p_n}{w_nh_n}.$$

P r o o f. Since

$$\lim_{n\to\infty}\frac{p_n}{w_nh_n}<\infty,$$

there exists a constant $c_2<\infty$ such that

$$\left|\frac{p_n}{w_nh_n}\right|\leqslant c_2,$$

for all $n\in\mathbb{N}$. Dividing the equation (2) by w_nh_n, we have

$$\frac{y_n}{w_nh_n}=\frac{p_n}{w_nh_n}+\frac{1}{w_nh_n}\sum_{s=0}^{n-1}K(n,s)f(s,y_s),\ \ n\in\mathbb{N}. \tag{37}$$

Using the assumptions of the theorem, we have from (37)

$$\left|\frac{y_n}{w_nh_n}\right|\leqslant c_2+\sum_{s=0}^{n-1}h_sq_sa_s\varphi\Big(\left|\frac{y_s}{w_sh_s}\right|\Big).$$

Again using Theorem 1.6.4 of [8], we have

$$\left|\frac{y_n}{w_nh_n}\right|\leqslant c_2+G^{-1}\Big(\sum_{s=0}^{n-1}h_sq_sa_s\Big)$$

and as in the last theorem, by virtue of the condition (35) there exists a positive constant c_3 such that

$$c_3=c_2+G^{-1}\Big(\sum_{s=0}^{\infty}h_sq_sa_s\Big)$$

and

$$\left|\frac{y_n}{w_nh_n}\right|\leqslant c_3$$

for $n\in\mathbb{N}$. Further $|y_n|\leqslant c_3w_nh_n$ for $n\in\mathbb{N}$, and hence

$$\begin{aligned}\left|\frac{1}{w_nh_n}\sum_{s=0}^{n-1}K(n,s)f(s,y_s)\right|&\leqslant\frac{1}{w_nh_n}\sum_{s=0}^{n-1}h_nq_sa_s\varphi\left(c_3w_sh_s\right)\\&\leqslant\frac{\varphi(c_3)}{w_n}\sum_{s=0}^{n-1}q_sa_s\varphi(h_s)\varphi(w_s).\end{aligned}$$

Setting $F_s = q_s a_s \varphi(h_s)\varphi(w_s)$ and applying Lemma 8, we obtain using (35) that

$$\lim_{n\to\infty} \frac{1}{w_n} \sum_{s=0}^{n-1} F_s = 0.$$

Hence

$$\lim_{n\to\infty} \frac{1}{w_n h_n} \sum_{s=0}^{n-1} K(n,s) f(s, y_s) = 0$$

and the result follows from (37). $\square$

R e m a r k 2. Once again, from the conditions given in Theorems 9 and 10 we see that the asymptotic behavior of solutions $\{y_n\}$ of equation (2) may depends on the behavior of the sequence $\{p_n\}$.

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