A NOTE ON THE DOMINATION NUMBER OF A GRAPH AND ITS COMPLEMENT

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Abstract. If G is a simple graph of size n without isolated vertices and $\overline{G}$ is its complement, we show that the domination numbers of G and $\overline{G}$ satisfy

$$\gamma(G)+\gamma(\overline{G}) \leqslant \begin{cases} n-\delta+2 & \text{if} \quad \gamma(G)>3, \\ \delta+3 & \text{if} \quad \gamma(\overline{G})>3, \end{cases}$$

where δ is the minimum degree of vertices in G.

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Introduction

Graphs, considered here, are *finite* and *simple* (without loops or multiple edges), and [1,2] are followed for terminology and notation.

Let $G=(V,E)$ be an *undirected graph* with the set of *vertices* V and the set of edges E. The *complement* $\overline{G}$ of G is the graph with vertex set V, two vertices being adjacent in $\overline{G}$ if and only if they are not adjacent in G.

For any vertex v of G, the *neighbour set* of v is the set of all vertices adjacent to v; this set is denoted by $N(v)$. A vertex is said to be *isolated* if its neighbour is empty. Suppose that W is a nonempty subset of V. The subgraph of G, whose vertex set is W and whose edge set is the set of those edges of G that have both ends in W, is called the subgraph of G *induced* by W and is denoted by $G[W]$. A set of vertices in a graph is said to be *dominating* if every vertex not in the set is adjacent to one or more vertices in the set. A *minimal dominating set* is a dominating set such that no proper subset of it is also a dominating set.

The *domination number* $\gamma(G)$ of G is the size of the smallest minimal dominating set.

The Main results

In the sequel, we will denote $n = |V|$ and $\delta = \min\limits_{v \in V} |N(v)|$.

Theorem 1. *If $G = (V, E)$ is a graph without isolated vertices and $\gamma(G) > 3$, then $\gamma(G) + \gamma(\overline{G}) \leqslant n - \delta + 2$.*

P r o o f. Let $v \in V$ be such that $\delta = |N(v)|$ (obviously, since G has no isolated vertices, we have $\delta \geqslant 1$) and $W = V - (N(v) \cup \{v\})$. If W is empty, then $\gamma(G) = 1$, contradicting the hypothesis. Thus $|W| \geqslant 1$ and, by the choice of v, it follows that $|N(w)| \geqslant \delta$ for each $w \in W$.

Consequently, if all vertices of W are isolated in $G[W]$, then $(w, u) \in E$ for every $w \in W$ and $u \in N(v)$, that is, $\{v, u\}$ is a dominating set in G for each $u \in N(v)$. Thus, $\gamma(G) = 2$, contradicting the hypothesis. Let now $Z \subset W (Z \neq W)$ be the set of isolated vertices in $G[W]$ (Z can be empty or nonempty), and $Z^* = W - Z$. Let also $D \subseteq Z^*$ be a minimal dominating set in $G[Z^*]$.

If Z is empty, then $D \cup \{v\}$ is a dominating set of G, and we have $\gamma(G) \leqslant |D \cup \{v\}| = 1 + |D|$. Hence, $|D| \geqslant \gamma(G) - 1$.

If Z is nonempty, then, since $\delta \leqslant |N(z)|$ for each $z \in Z$, we have $(z, u) \in E$ for every $z \in Z$ and $u \in N(v)$. Consequently, for each $u \in N(v)$, $D \cup \{v\} \cup \{u\}$ is a dominating set of G and, therefore, we have $\gamma(G) \leqslant |D \cup \{v\} \cup \{u\}| = 2 + |D|$. Hence, $|D| \geqslant \gamma(G) - 2$. Thus we always have

$$|D| \geqslant \gamma(G) - 2. \tag{1}$$

By (1), since $\gamma(G) > 3$, we can choose $B \subseteq D$ such that $|B| = \gamma(G) - 3$.

Let $C \subseteq Z^*$ be the set of vertices in $G[Z^*]$ dominated by B, and $C^* = Z^* - C$. Suppose Z to be empty. If there exists $c \in C$ such that $(c, c^*) \in E$ for each $c^* \in C^*$, then $B \cup \{v\} \cup \{c\}$ is a dominating set in G, that is, $\gamma(G) \leqslant |B \cup \{v\} \cup \{c\}| = 2 + |B| = \gamma(G) - 1$; a contradiction. Thus for every $c \in C$ there exists $c^* \in C^*$ such that $(c, c^*) \notin E$. If there exists $u \in N(v)$ such that $(u, c^*) \in E$ for each $c^* \in C^*$, then $B \cup \{v\} \cup \{u\}$ is a dominating set in G, that is, $\gamma(G) \leqslant |B \cup \{v\} \cup \{u\}| = 2 + |B| = \gamma(G) - 1$; a contradiction. Thus for every $u \in N(v)$ there exists $c^* \in C^*$ such that $(u, c^*) \notin E$. On the other hand, by the choice of v, for each $c^* \in C^*$ we have $(v, c^*) \notin E$. Consequently, $C^* = C^* \cup Z$ is a dominating set in $\overline{G}$.

Suppose Z to be nonempty. By the choice of v, we have $(v,z) \notin E$ for each $z \in Z$. Also, $(z,c) \notin E$ for every $z \in Z$ and $c \in C$. Suppose that there exists $u \in N(v)$ such that $(u,c^*) \in E$ for each $c^* \in C^*$. Since $\delta \leqslant |N(z)|$ for each $z \in Z$, we have $(t,z) \in E$ for every $z \in Z$ and $t \in N(v)$. Hence $B \cup \{u\} \cup \{v\}$ is a dominating set in G, that is, $\gamma(G) \leqslant |B \cup \{u\} \cup \{v\}| = 2 + |B| = \gamma(G) - 1$; a contradiction. Thus for every $u \in N(v)$ there exists $c^* \in C^*$ such that $(u,c^*) \notin E$. Consequently, $C^* \cup Z$ is a dominating set in $\overline{G}$.

So we have

$$(2)\quad \gamma(\overline{G}) \leqslant |C^* \cup Z| = |C^*| + |Z| = |Z^*| - |C| + |Z| = |W| - |C| = n - \delta - 1 - |C|.$$

However, because $G[Z^*]$ does not contain isolated vertices, it follows that $|B| \leqslant |C|$ and, by (2), since $|B| = \gamma(G) - 3$, we obtain $\gamma(G) + \gamma(\overline{G}) \leqslant n - \delta + 2$. □

Theorem 2. *If $G = (V,E)$ is a graph without isolated vertices and $\gamma(\overline{G}) > 3$, then $\gamma(G) + \gamma(\overline{G}) \leqslant \delta + 3$.*

Proof. Let $v \in V$ be such that $\delta = |N(v)|$ (obviously, since G has no isolated vertices, we have $\delta \geqslant 1$). Obviously, $N(v) \cup \{v\}$ is a dominating set in $\overline{G}$, that is, $\gamma(\overline{G}) \leqslant |N(v) \cup \{v\}| = 1 + \delta$. Thus $\delta \geqslant \gamma(\overline{G}) - 1$ and, since $\gamma(\overline{G}) > 3$, we can choose $B \subseteq N(v)$ such that $|B| = \gamma(\overline{G}) - 3$. Let $B^* = N(v) - B$ and $W = V - (N(v) \cup \{v\})$. If W is empty, then the minimum degree of vertices in G is less than δ, contradicting the choice of v. Hence $|W| \geqslant 1$. Let $w \in W$. We have $|B \cup \{v\} \cup \{w\}| = 2 + |B| = \gamma(\overline{G}) - 1$, that is, $B \cup \{v\} \cup \{w\}$ is not a dominating set in $\overline{G}$. Consequently, there exists $x \in V$ such that $(x,v) \in E$, $(x,w) \in E$ and $(x,b) \in E$ for each $b \in B$. Obviously, since G does not contain loops, $x \in B^*$. So for every $w \in W$ there exists $b^*_w \in B^*$ such that $(b^*_w, w) \in E$, $(b^*_w, v) \in E$ and $(b^*_w, b) \in E$ for each $b \in B$. Hence B^* is a dominating set in G, that is, $\gamma(G) \leqslant |B^*| = |N(v)| - |B| = \delta - \gamma(\overline{G}) + 3$. Therefore, $\gamma(G) + \gamma(\overline{G}) \leqslant \delta + 3$. □

Corollary. *If G is a graph without isolated vertices such that $\gamma(G) > 3$ and $\gamma(\overline{G}) > 3$, then $\gamma(G) + \gamma(\overline{G}) \leqslant \lfloor (n+5)/2 \rfloor$ (we use $\lfloor x \rfloor$ to denote the integer less than or equal to x).*

Proof. It follows from the above theorems. □

References

[1] *C. Berge*: Graphes et Hypergraphes. Dunod, Paris, 1970.

[2] *J. A. Bondy, U. S. R. Murty*: Graph Theory with Applications. Macmillan Press, 1976.

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