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# SUPPORT PROPERTIES OF A FAMILY OF CONNECTED COMPACT SETS

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*Abstract.* A problem of finding a system of proportionally located parallel supporting hyperplanes of a family of connected compact sets is analyzed. A special attention is paid to finding a common supporting halfspace. An existence theorem is proved and a method of solution is proposed.

*Keywords*: set family, supporting hyperplane, lexicographic optimization, polyhedral approximation.

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## 1. Introduction

As is well-known, a three leg table stands stably and possesses a stable desk regardless of both the length an the shape of the legs. In mathematical terms, a system of three connected compact sets in the three-dimensional space possesses precisely two common supporting halfspaces provided an intuitive assumption concerning the location of these sets in the space is satisfied. The purpose of the paper is to analyze a generalized version of this problem and to make precise the intuitive assumption mentioned. Separation and support properties of convex sets have important consequences for optimization theory. They are contained immanently in all forms of duality theorems. Separation of convex set families plays an important role in the multicriteria decision making. Therefore, this topic has drawn attention of many authors (see [2], [8], [9], [19], [21]). Grygarová [6], [7] proposes several methods of constructing all supporting hyperplanes of two convex polytopes. In [13], an existence theorem concerning the solution of a special optimization problem is presented. This theorem has been formulated, above all, as a contribution to the theory of solving the so-called vague linear equation systems, where the columns of the system matrix can move in given compact convex sets. In this paper, we present a generalization of this theorem and show that it can be interpreted as a supporting theorem for a finite family of compact connected sets in $\mathbb{R}^m$. Furthermore, methods of constructing the supporting hyperplanes are proposed.

## 2. Problem of parallel supporting hyperplanes (PSH)

Let us consider the following problem: Given a family $F = \{\mathcal{A}^1, \ldots, \mathcal{A}^n\}$ of compact sets in $\mathbb{R}^m$, find halfspaces $H_1, \ldots, H_n$ such that

(i) $H^j$ is a supporting halfspace of $\mathcal{A}^j$;

(ii) the boundary hyperplanes $L_j = \mathrm{bd} H_j$ are parallel;

(iii) $0 \in H_j$ for $j \in N^-$, $0 \notin H_j$ for $j \in N^+$, where $N^+ \cup N^- = N = \{1, \ldots, n\}$;

(iv) the ratios $\alpha_j$ of the distances of the supporting hyperplanes $L_j$ from the origin are prescribed.

Let us consider $c = (c_j)$, where $c_j = -\alpha_j$ for $j \in N^-$, $c_j = \alpha_j$ for $j \in N^+$. Then the above mentioned task can be equivalently formulated as

Problem PSH. Find a vector $z^* \in \mathbb{R}^m$ such that

$$\min\{a^T z^*;\ a \in \mathcal{A}^j\} = c_j, \quad j \in N. \tag{1}$$

Then the halfspaces mentioned in (i) are defined as $H_j = \{x;\ (z^*)^T x \geqslant c_j\}$.

In order to be able to use an alternative formulation, let us introduce a few concepts (see also [13], [14], [15]).

**Definition 1.** Let $\mathcal{A}^1, \ldots, \mathcal{A}^n \subset \mathbb{R}^m$ be nonvoid compact sets. The set of matrices

$$\mathbf{A} = \{A;\ A = (a^1, \ldots, a^n),\ a^j \in \mathcal{A}^j,\ j = 1, \ldots, n\}$$

is called an $(m \times n)$-*vague matrix ($V$-matrix).*

Provided $X$ is a matrix of an adequate size, we have

$$\mathbf{A} + X = \{A + X;\ A \in \mathbf{A}\},\ \mathbf{A}X = \{AX;\ A \in \mathbf{A}\}.$$

The rank of a $V$-matrix $\mathbf{A}$ is defined as

$$\mathrm{rank}(\mathbf{A}) = \min\{\mathrm{rank}(A);\ A \in \mathbf{A}\}.$$

Recalling the close relation between a family $F$ and the corresponding $V$-matrix $\mathbf{A}$, we will use the notations $\mathbf{A}(F)$ and $F(\mathbf{A})$ respectively.

**Definition 2.** A family $F$ is called *linearly independent* if $\mathrm{rank}(\mathbf{A}(F)) = n$.

**Definition 3.** A square $V$-matrix $\mathbf{A}$ is called *regular* if there exists no singular $A \in \mathbf{A}$.

Of course, a square $V$-matrix $\mathbf{A}$ is regular if and only if $F(\mathbf{A})$ is linearly independent. If $\mathbf{A}$ is regular, then the same holds for an arbitrary $\overline{\mathbf{A}} = (\overline{\mathcal{A}}^j) \subset \mathbf{A}$, where $\overline{\mathcal{A}}^j \subset \mathcal{A}^j \quad \forall j \in N$.

Problem PSH (alternative formulation): Given an $(m \times n)$-$V$-matrix $\mathbf{A}$ and a $c \in \mathbb{R}^m$, find a $z^* \in \mathbb{R}^m$ such that

(i) $A^T z^* \geqslant c \quad \forall A \in \mathbf{A}$;

(ii) $\exists A_* \in \mathbf{A}\colon\ A_*^T z^* = c$.

Let us denote by PSH$(\mathbf{A}, c)$ the PSH-problem applied to $\mathbf{A}$ and $c$.

Note. $z^*$ is a solution of PSH$(\mathbf{A}, c)$ if and only if $\beta z^*$ is a solution of PSH$(\mathbf{A}, \beta c)$ for a $\beta > 0$.

If $z^*$ is a solution of Problem PSH$(\mathbf{A}, c)$, then the pair $(z^*, A_*)$ that satisfies (ii) will be called an *extended solution.* Certainly, it is not surprising that Problem PSH is in general not solvable if $n > m$. Therefore, we will discuss, above all, the case $n = m$.

As has been shown in [13], [14], [15], [18], the solution of the PSH-problem is useful for solving the so-called vague linear equation systems.

**Theorem 1.** *If $\mathbf{A}$ is a connected regular square $V$-matrix, then there exists a unique solution of PSH-problem for an arbitrary $c \in \mathbb{R}^m$.*

We say that Theorem 1 is true for a $k, 0 \leqslant k \leqslant m$, if it holds for any

$$\mathbf{A}_k = (\mathcal{A}^1, \ldots, \mathcal{A}^{k-1}, a^k, \ldots, a^m), \tag{2}$$

where the columns $\mathcal{A}^1 \ldots, \mathcal{A}^{k-1}$ are vague and the remaining ones are definite vectors. Consider a connected regular $V$-matrix $\mathbf{A}_{k+1} = (\mathcal{A}^1, \ldots, \mathcal{A}^k, a^{k+1}, \ldots, a^m)$. For a vector $y \in \mathcal{A}^k$, let $g_k(y)$ be a solution of PSH$(\mathbf{A}_k(y), c)$, where $\mathbf{A}_k(y) = (\mathcal{A}^1, \ldots, \mathcal{A}^{k-1}, y, a^{k+1}, a^m)$. Provided Theorem 1 holds for a $k \in J = \{1, \ldots, m\}$, we have a mapping $g_k(y)\colon\ \mathcal{A}^k \to \mathbb{R}^m$.

**Lemma 1.** *Assume that Theorem 1 is true for a $k \in J$ and let $\mathbf{A}_{k+1}$ be a connected regular $V$-matrix. Further, let $z', z'' \in \mathbb{R}^m, y', y'' \in \mathcal{A}^k$ be such that*

$$z' = g_k(y', c),\ \ z'' = g_k(y'', c). \tag{3}$$

*Then the following implications hold for any $y \in \mathcal{A}^k$:*

$$(z')^T y'' < c_k, (z'')^T y \leqslant c_k \Rightarrow (z')^T y < c_k, \tag{4}$$

$$(z')^T y'' > c_k, (z'')^T y \geqslant c_k \Rightarrow (z')^T y > c_k. \tag{5}$$

Proof. Let us assume that there exists a $y \in \mathcal{A}^k$ that violates (4) and let $\varphi(t)\colon [0,1] \to \mathcal{A}^k$ be a curve connecting $y'', y$. Denoting $\mu'(t) = (z')^T\varphi(t)$ and $\mu''(t) = (z'')^T\varphi(t)$, we have $\mu'(0) < \mu''(0) = c_k, \mu'(1) \geqslant c_k \geqslant \mu''(1)$. Since $\mu', \mu''$ are continuous due to the regularity of $\mathbf{A}_{k+1}$, there exist a $\bar{t} \in [0,1]$ and a $\bar{\mu}$ such that $\mu'(\bar{t}) = \mu''(\bar{t}) = \bar{\mu}$. Evidently, $z', z''$ solve $\mathrm{PSH}_k(\varphi(\bar{t}), \bar{c})$ for $\bar{c} = (c_1, \ldots, c_{k-1}, \bar{\mu}, c_{k+1}, \ldots, c_m)$. Hence $z' = z''$ must hold due to the correctness of Theorem 1 for the $k$. This equality, however, contradicts $(z')^T y'' < c_k$, which proves (4). The other implication can be proved in the same way. □

**Lemma 2.** *If the assumptions of Lemma* 1 *are satisfied, then*

$$(z')^T y'' < c_k \Leftrightarrow (z'')^T y' > c_k. \tag{6}$$

Proof. Setting $y = y'$ in (4) or (5), we obtain a false conclusion $(z')^T y' \neq c_k$. It means that the prediction must be false. □

Proof of Theorem 1 will be carried out by induction on $k$.

(i) If $k = 0$, then $\mathbf{A}_k \equiv A$ is an ordinary nonsingular matrix, so that $z^* = (A^{-1})^T c$.

(ii) Let us assume that the theorem holds for a $k - 1 \leqslant m - 1$ and let us form sequences $y^s$, $z^s$ recursively:

$$\begin{aligned} &y^0 \in \mathcal{A}^k \text{ arbitrarily chosen} \\ &z^s = g_k(y^s, c) && s = 0, 1, \ldots \\ (z^{s+1})^T y^s &= \min\{(z^s)^T y;\ y \in \mathcal{A}^k\} && s = 0, 1, \ldots \end{aligned} \tag{7}$$

Let us consider the following sequence of sets: $Q_s = \{y;\ (z^s)^T y \leqslant c_k, y \in \mathcal{A}^k\}$. Lemma 1 implies

$$Q_s \subset Q_{s+1} \quad \forall s = 0, 1, \ldots \tag{8}$$

and consequently there exists a nonvoid compact intersection $Q = \bigcap_0^\infty Q_s$. Let us choose a convergent subsequence $y^{s_i}$ such that $y^{s_i+1}$ is convergent as well and let us denote $z^* = g_k(y^*, c), z^{**} = g_k(y^{**}, c)$, where $y^{s_i} \to y^*, y^{s_i+1} \to y^{**}$. We have $z^* \in Q = \{y;\ (z^{**})^T y \leqslant c_k, y \in \mathcal{A}^k\}$ due to (8) and hence $(z^*)^T y^{**} \geqslant c_k$

follows by Lemma 2. For an arbitrary $y \in \mathcal{A}^k, (z^{s_i})^T y \geqslant (z^{s_i})^T y^{s_i+1}$ holds due to (7) and therefore

$$(z^*)^T y \geqslant (z^*)^T y^{**} \geqslant c_k. \tag{9}$$

Let us notice that

$$(z^*)^T y^* = c_k \tag{10}$$

holds by definition. Thus, $z^*$ solves PSH$(\mathbf{A}_{k+1}, c)$.

In order to prove uniqueness of $z^*$, let us suppose that there exist $z^1, z^2 \in \mathbb{R}^m$ and $y^1, y^2 \in A^k$ such that

$$z^i = g(y^i),\ \min\{(z^i)^T y;\ y \in \mathcal{A}^k\} = c_k,\ i = 1, 2.$$

Then we have $(z^1)^T y^2 \geqslant c_k$, $(z^2)^T y^1 \geqslant c_k$. Lemma 2 implies $(z^1)^T y^1 = (z^2) y^1 = c_k$ and hence $z^* = z^1 = z^2$ is determined uniquely by virtue of the inductive assumption. □

Under the assumptions of Theorem 1, we can write $z^* = \mathrm{PSH}(\mathbf{A}, c)$. Regularity of $\mathbf{A}$, of course, is not a necessary condition for the assertion of Theorem 1. In fact, the most important role is played by the local conditions in a neighbourhood of the system of the tangent points. Let us consider an extended solution $(z^*, A_*)$ of PSH$(\mathbf{A}, c)$ and denote $\Omega(A_*/\mathbf{A}) = \mathcal{O} \cap \mathbf{A}$, where $\mathcal{O}$ is an open set, $A_* \in \mathcal{O} \subset \mathbb{R}^{m \times m}$. Further, let conv$\mathbf{A}$ be the convex hull of $\mathbf{A}$.

**Theorem 2.** *Let us assume that*

(i) $\overline{\mathbf{A}} \subset \mathrm{conv}\mathbf{A}$ *is a connected regular* $V$*-matrix;*

(ii) $(z^*, A_*)$ *is an extended solution of PSH*$(\overline{\mathbf{A}}, c)$*;*

(iii) $\exists \Omega(A_*/\mathrm{conv}\mathbf{A}) \subset \overline{\mathbf{A}}$.

*Then* $z^*$ *is the unique solution of PSH*$(\mathbf{A}, c)$.

Proof. Assumptions (ii), (iii) imply that $z^* = \mathrm{PSH}(\mathbf{A}, c)$. Indeed,

$$H_j = \{y;\ (z^*)^T y \geqslant c_j\} \tag{11}$$

is a supporting halfspace of $\overline{\mathcal{A}}^j$ at $a_*^j$ and, due to (iii), it is a supporting halfspace of conv$\mathcal{A}^j$ as well. □

## 3. Common supporting halfspace

Besides the original idea, outlined in the previous section, a solution of the PSH-problem can be interpreted in several other ways:

1° Assume that $|c| > 0$ and denote $\overline{\mathcal{A}}^j = (1/c_j)\mathcal{A}^j$. Then $L = \{y;\ (z^*)^T y = 1\}$ is a common supporting hyperplane of all $\overline{\mathcal{A}}^j$'s. Furthermore, the sets with positive multipliers lie in the same halfspace determined by $L$ while those having negative multipliers lie in the opposite one.

2° Let us assume that $c_j = \alpha$ for $j \in \widetilde{J} \subset J$. Then

$$H = \{y;\ (z^*)^T y \geqslant \alpha\} \tag{12}$$

is a common supporting halfspace of the sets $\mathcal{A}^j, j \in \widetilde{J}$. The remaining $c_j$ specify the position of $H$ with respect to the sets $\mathcal{A}^j$ for $j \in J \setminus \widetilde{J}$.

**Theorem 3.** *If $F = \{\mathcal{A}^1, \ldots, \mathcal{A}^m\}$ is a linearly independent family of connected compact sets, then there exist precisely two common supporting halfspaces of all sets $\mathcal{A}^j \in F$.*

P r o o f. Let us consider $\sigma \in \{+1, -1\}, e = (1)^m$ and $z^* = \mathrm{PSH}(\mathbf{A}, \sigma e)$. Since the common supporting halfspace $H = \{y;\ (z^*)^T y \geqslant \sigma\}$ is invariant with respect to a positive multiplier of $(z^*, \sigma e)$, this theorem is an immediate consequence of Theorem 1. Apparently, $0 \in H$ for $\sigma = -1$ and $0 \notin H$ in the opposite case. □

The supporting halfspace is a geometric concept while the linear independence is an algebraic one. It is evident that the existence of a common supporting halfspace does not depend on the location of the set with respect to the origin, i.e., it is invariant with respect to a translation of the whole family. Therefore, a more transparent interpretation of this problem can be expressed in terms of the affine space.

**Definition 4.** A family $\widetilde{F} = \{\mathcal{P}^1, \ldots, \mathcal{P}^n\}$ is called *affinely independent* if the following implication holds: If $\mathcal{P} = \sum\limits_{j=1}^{n} \lambda_j \mathcal{P}^j$, where $\sum\limits_{j=1}^{n} \lambda_j = 0$, $\sum\limits_{j=1}^{n} |\lambda_j| \neq 0$, then $0 \notin \mathcal{P}$.

**Lemma 3.** *A family $\widetilde{F}$ is affinely independent if and only if there exists no $(n-2)$-dimensional affine subspace that intersects all sets of $\widetilde{F}$.*

P r o o f. Let us consider $X = \bigl\{x = \sum\limits_{j=1}^{n-1} \lambda_j x^j;\ \sum\limits_{j=1}^{n-1} \lambda_j = 1, x^j \in \mathcal{P}^j\bigr\}$. The dimension of $X$ is at most equal to $n-2$. If $X$ intersects $\mathcal{P}^n$, there is an $x^n \in \mathcal{P}^n \cap X$. Denoting $\lambda_n = -1$, we obtain $\sum \lambda_j x^j = 0, \sum \lambda_j = 0$. This reasoning holds conversely as well. Thus, $X \cap \mathcal{P}^n \neq 0$ if and only if $0 \notin \mathcal{P}$. □

Consider $\mathcal{A}^j = \{a^j = (p^j, 1);\ p^j \in \mathcal{P}_j\} \in \mathbb{R}^{m+1}$, $j = 1, \ldots, n$ and $\mathcal{A} = \sum_j \lambda_j \mathcal{A}^j$, where $\sum_{\mathrm{j}}^{\mathrm{n}} |\lambda_{\mathrm{j}}| \neq 0$. Apparently, $0 \in \mathcal{A}$ if and only if $0 \in \mathcal{P}$. Thus, $F'$ is affinely independent if and only if $F = \{\mathcal{A}^1, \ldots, \mathcal{A}^n\}$ is linearly independent.

**Theorem 4.** *Let $\widetilde{F} = \{\mathcal{P}^1, \ldots, \mathcal{P}^{m+1}\}$ be an affinely independent family of connected compact sets. Then, for each $i \in \{1, \ldots, m+1\}$, there exist precisely two common supporting halfspaces $H_i^k = \{x;\ (v^{ki})^T y \geqslant \alpha_k\}$, $k = 1, 2$ of the sets $\mathcal{P}^j, j \neq i$ such that $\mathcal{P}^i \subset \mathrm{int} H_i^1, \mathcal{P}^i \cap H_i^2 = \emptyset$.*

Proof. Let $\sigma$ be equal to either $+1$ or $-1$. Consider $\mathbf{A} = \{(a^j);\ a^j \in \mathcal{P}^j \times \{1\}\}$ and choose $c = \sigma e^j$, where $e^j \in \mathbb{R}^{m+1}$ is the $j$-th unit vector. According to Theorem 1, there exist $A_* = (a_*^j) = (p_*^j, 1) \in \mathbf{A}$ and a unique $z^* = (v^*, -\alpha)$ such that

$$
\begin{aligned}
(v^*)^T p \geqslant \alpha; &\quad \forall p \in \mathcal{P}^j,\ j \neq i, \\
(v^*)^T p_*^j = \alpha; &\quad \forall j \neq i, \\
(v^*)^T p_*^i = \alpha + \sigma.&
\end{aligned}
$$

The hyperplane $\{y;\ (v^*)^T y = \alpha\}$ cannot intersect $\mathcal{P}^i$ due to the affine independence of $F'$. Hence $\sigma (v^*)^T y > \sigma\alpha$ holds for all $y$'s belonging to $P^i$. Thus, the halfspace $\{y;\ (v^*)^T y \geqslant \alpha\}$ possesses the properties of $H^1$ or $H^2$ for $\sigma = \pm 1$, respectively. □

Let us denote by aff $F$ the *affine hull* of $F$, i.e. $\mathrm{aff}\, F = \bigcup \{\sum \lambda_j \mathcal{A}^j;\ \sum \lambda_j = 1\}$.

**Definition 5.** $F$ is called *completely affinely independent* (CAI) if it is affinely independent and

$$\mathrm{aff}\, F \neq \mathbb{R}^m. \tag{13}$$

**Proposition 1.** *If $F = \{\mathcal{A}^j\}_1^m$ is a CAI family of connected compact sets, then there exist precisely two common supporting halfspaces of all $\mathcal{A}^j \in F$.*

Proof. For an arbitrary $h \notin \mathrm{aff}\, F$, the family $F' = \{\mathcal{A}^1, \ldots, \mathcal{A}^m, \{h\}\}$ is affinely independent. Thus, Theorem 4 can be applied. □

The condition (13) cannot be omitted even if all the sets are simply connected. Let us consider a trivial example of $F$ for $m = 2$: $\mathcal{A}^1 = \{0\}$, $\mathcal{A}^2 = \{a;\ a^T a = 1, a_1 \leqslant \frac{1}{2}\}$. Even though $\mathcal{A}^1, \mathcal{A}^2$ are affinely independent, there exists no common supporting halfspace as $\mathrm{aff}\, F = \mathbb{R}^m$.

**Lemma 4.** *$F = \{\mathcal{A}^j\}_1^n$ is CAI if and only if there exists an $h \in \mathbb{R}^m$ such that $F' = \{\mathcal{A}^j - h\}_1^n$ is linearly independent.*

P r o o f. Let us consider a nontrivial combination $x = \sum \lambda_j (x^j - h)$, $h \in \mathbb{R}^m$, where $x^j \in \mathcal{A}^j$. The following implications evidently hold:

$$\sum \lambda_j = 0 \Rightarrow \sum \lambda_j x^j = x, \tag{14}$$

$$\sum \lambda_j \neq 0 \Rightarrow \sum \lambda'_j x^j = h + \beta x, \text{ where } \beta = \Big(\sum \lambda_j\Big)^{-1}, \ \sum \lambda'_j = 1. \tag{15}$$

(i) If there are $\lambda_j$ such that $x = 0$, then either $F$ is not affinely independent due to (14) or $h \in \operatorname{aff} F$ due to (15). The latter assertion implies $\operatorname{aff} F = \mathbb{R}^m$ since $h$ has been chosen arbitrarily.

(ii) If $x \neq 0$ for each nontrivial combination of $\lambda_j$, then (14), (15) imply that $F$ is affinely independent and, at the same time, $h \notin \operatorname{aff} F$. □

**Corollary 1.** *A family $F = \{\mathcal{A}^j\}_1^m$ is CAI if and only if there exists an $h$ such that $\mathbf{A}(F) - he^T$ is regular.*

**Corollary 2.** *If an $m$-member family $F$ is CAI, then* $\operatorname{rank}(\mathbf{A}(F)) \geqslant m - 1$.

Let us formulate these results in the framework of the linear space. Provided a square $V$-matrix $\mathbf{A}$ is given, the problem of finding a common supporting halfspace of a family $F(\mathbf{A})$ can be formulated as follows:

P r o b l e m CSH$(\mathbf{A}, \sigma)$. Find a $z^*$ such that $\exists h : z^* = \mathrm{PSH}(\mathbf{A} - he^T, \sigma e)$. Corollary 1 indicates that the regularity assumption of Theorem 1 can be weakened.

**Definition 6.** A square $V$-matrix $\mathbf{A}$ is called *nearly regular* (NR) if the following implication holds:

$$(A \in \mathbf{A}, A^T v = 0) \ \Rightarrow \ \mathbf{A} + ve^T \quad \text{is regular.}$$

Notice that a regular $V$-matrix is NR.

Immediately, we obtain the following results:

**Proposition 2.** *If $\mathbf{A}$ is NR, then $F(\mathbf{A})$ is CAI.*

**Corollary 3.** *If a square connected $V$-matrix $\mathbf{A}$ is NR, then $F(\mathbf{A})$ possesses precisely two common supporting halfspaces.*

As we have already mentioned, local conditions at a matrix $A \in \mathbf{A}$ are worth studying. We will say that $\mathbf{A}$ is locally NR at $A \in \mathbf{A}$ if it is NR in a neighbourhood $\mathcal{O}$ of $A$.

**Proposition 3.** *Let the following assumptions be satisfied for an $A \in \mathbf{A}$:*

(i) $\operatorname{rank}(A) = m - 1$;
(ii) $A^T w = e$ *has no solution.*
*Then* $\mathbf{A}$ *is locally NR at* $A$.

P r o o f. Consider an $A \in \mathbf{A}$ and let $A^T v = 0$, $(A+ve^T)^T w = A^T w + (v^T w)e = 0$ hold for $v, w \neq 0$. If $\alpha = v^T w \neq 0$, then we have $A^T(-\alpha^{-1}w) = e$, which contradicts the assumption (ii). Thus $v^T w = 0$ and, consequently, $A^T w = 0$ must hold. Since $\operatorname{rank}(A) = m-1$, we have $w = \beta v$ and hence $\beta v^T v = 0$. This contradiction with the assumptions $v \neq 0$, $\beta \neq 0$ implies nonsingularity of $A + v^T e$ and the same holds in a sufficiently small neighbourhood of $A$. $\square$

**Corollary 4.** *Let* $\mathbf{A}$ *be a connected compact* $V$*-matrix and let an* $A \in \mathbf{A}$ *satisfy the conditions of Proposition* 3. *Then there exists a neighbourhood* $\mathcal{O}$ *of* $A$ *such that* $F(\overline{\mathbf{A}})$ *possesses precisely two common supporting halfspaces for any compact* $\overline{\mathbf{A}} \in \mathcal{O} \cap \mathbf{A}$.

## 4. Solving PSH-problem

The PSH-problem recalls the semi-infinite programming [10], [11]. However, there is no objective function here. We find a special feasible vector of a semi-infinite system of linear inequalities, the existence of which has been discussed in the previous sections.

A method for solving the PSH-problem can be derived immediately from the proof of Theorem 1. First of all, let us consider the case of a polyhedral $V$-matrix.

A l g o r i t h m o f l e x i c o g r a p h i c o p t i m i z a t i o n ( L O )

Set $s := 0 : f = false$ and choose an $A_0 \in \mathbf{A}$ arbitrarily
**repeat** Find the solution $z^s$ of the system $A_s^T z = c$
    $j := 1$
    **repeat** Find $y^j \in \mathcal{A}^j$ such that $r_j := (z^s)^T y^j = \min\{(z^k)^T y | y \in \mathcal{A}^j\}$
        **if** $r_j < c_j$ **then** $k := j, f = true$
        $j := j + 1$
    **until** $j > m$ **or** $f = true$
    **if** $f = true$ **then** replace the $k$-th column of $A_s$ by $y^k$ and
        denote the modified matrix by $A_{s+1}$
    $s := s + 1$
**until** $f = false$
$z^* := z^s$ is the solution of the PSH-problem.

**Theorem 5.** *If $\mathbf{A}$ is a polyhedral regular $V$-matrix, then Algorithm LO is finite.*

Proof. Since the solutions of the problem of finding $y^j$ can be chosen among the vertices of the polyhedron $\mathcal{A}^j$, an accumulation point $A_* = (a_*^1, \ldots, a_*^m)$ of $A_s$ is achieved in a finite number of steps. Let us consider an arbitrary $A \in \mathbf{A}$ and suppose that

$$A^T z^* \ngeqslant c_k. \tag{16}$$

Let us denote $k = \min\{j;\ (a^j)^T z^* < c_j, j \in J\}$. According to the construction of $(A_s, z^s)$, $z^*$ solves $\mathrm{CSH}_k(\mathbf{A}_k(a_*^k), c)$. With regard to the proof of Theorem 1, we can assert that $(z^*)^T y \geqslant (z^*)^T a_*^k = c_k \quad \forall y \in \mathcal{A}^k$, which contradicts the assumption (16). Thus, $\min\{(z^*)^T y;\ y \in \mathcal{A}^j\} = c_j \quad \forall j \in J$. □

Provided $\mathbf{A}$ is an interval matrix, Algorithm LO is identical with the so-called sign-accord algorithm proposed by Rohn [20].

Let us notice that $z^*$ is the solution of $\mathrm{CSH}(\mathbf{A}, c)$ if and only if it solves $\mathrm{CSH}(\overline{\mathbf{A}}, c)$, where $\overline{\mathbf{A}} = (\mathrm{conv}\mathcal{A}^j)$. Now, we consider a convex vague matrix $\mathbf{A}$. The following algorithm realizes the idea of sequentially approximating $\mathcal{A}^j$, in a neighbourhood of a tangent point, by the convex hull of a growing finite set $\mathcal{B}^j$. The sets $\mathcal{B}^j$ define a polyhedral $V$-matrix $\mathbf{B} = (\mathrm{conv}\mathcal{B}^1, \ldots, \mathrm{conv}\mathcal{B}^m)$. Then, in each step, $\mathrm{PSH}(\mathbf{B}, c)$ is solved instead of $\mathrm{PSH}(\mathbf{A}, c)$.

Algorithm of sequential polyhedral approximation (SPA).

Set t:=0;
Choose $A_* := (a_*^1, \ldots, a_*^m), \mathcal{B}^j := \{a_*^j\}, a_*^j \in \mathcal{A}^j (j \in J)$ arbitrarily
**repeat** Starting with $A_0 = A_*$, solve $\mathrm{CSH}(B^V, c)$ by using LO-Algorithm;
    Set $z^t := z^*$, $t := t + 1$;
    Find $\varepsilon(z^*) := (z^*)^T y^k - c_k = \min_{j=1}^{m} \min\{(z^*)^T y - c_j;\ y \in \mathcal{A}^j\}$;
    Add $y^k$ to $\mathcal{B}^k$
**until** $\varepsilon(z^*) = 0$
$z^*$ is the solution of $\mathrm{PCH}(\mathbf{A}, c)$.

**Theorem 6.** *If $\mathbf{A}$ is a regular convex $V$-matrix, then the sequence $z^t$ produced by Algorithm SPA converges to the solution.*

Proof. Independently of the step of the SPA-Algorithm, $\mathbf{B}$ is a regular polyhedral matrix and therefore the corresponding application of the LO-Algorithm is finite. Consider an accumulation point $\widetilde{z}$ of $z^p$ and suppose that $\varepsilon(\widetilde{z}) < 0$. It means that there exists a $\widetilde{y} \in \mathcal{A}^k$ and an $\widetilde{\varepsilon} < 0$ such that $(\widetilde{z})^T \widetilde{y} - c_k \leqslant \widetilde{\varepsilon}$ for infinitely many $p$'s. On the other hand, starting with such a $p$, $\widetilde{y}$ can be joined to $\mathcal{B}^k$ and consequently $(z^p)^T\widetilde{y} - c_k \geqslant 0$ will hold for all sufficiently large $p$'s. Hence $\varepsilon(\widetilde{z}) = 0$ must be satisfied. □

**Corollary 5.** *Let $(z^*, A_*)$ be an extended solution of the PSH-problem and let there exist an $\Omega = \Omega(A_*/\mathbf{A})$ such that any $\overline{\mathbf{A}} \subset \Omega$ is a connected regular $V$-matrix. Then Algorithm SPA is locally convergent in $\Omega$.*

Numerical experiments support the hypothesis that the method described converges fairly quickly so that the cardinality of the sets $\mathcal{B}^j$ can be usually expressed in one-digit numbers.

Now, let us investigate the CSH-problem of finding the two supporting hafspaces of an $F = \{\mathcal{A}^j\}_1^m$. A possible singularity of $\mathbf{A}(F)$ can cause difficulties if the supporting hyperplane passes through the zero point. Since the supporting properties are invariant with respect to translation, we can modify $\mathbf{A}(F)$ by adding a vector to each of its columns. The existence of a suitable vector is guaranteed provided $F$ is CAI. Of course, solving the problem we usually do not know the vector $h$ mentioned in Lemma 4. However, if $\mathbf{A}(F)$ is nearly regular, such a vector can be easily found, if necessary. This fact is utilized in the following algorithm. Moreover, the two phases of the solution process are integrated.

Algorithm CSH

Given $\mathbf{A}$ and $\sigma \in \{\pm 1\}$.
Set $v := 0$;
Choose a nonsingular $A_0 = (a_0^1, \dots, a^m) \in \mathbf{A}$ and define $\mathcal{B}^j := \{a_0^j\}$.
**repeat** Set $s := 0, f = false$
  **repeat if** $A_s$ is singular **then** find a nontrivial solution $v$ of $A_s^T v = 0$.
    Find the solution $z^s$ of the system $(A_s + ve^T)^T z = \sigma e$
    $j := 1$
    **repeat** Find $y^j \in \mathcal{B}^j$ such that $r_j := (z^s)^T y^j = \min\{(z^k)^T y;\ y \in \mathcal{B}^j\}$
      **if** $r_j < 0$ **then** $k := j, f = true$
      $j := j + 1$
    **until** $j > m$ **or** $f = true$
    **if** $f = true$ **then** replace the $k$-th column of $A_s$ by $y^k$ and denote the modified matrix by $A_{s+1}$
    $s := s + 1$
  **until** $f = false$
  Set $z^t := z^*, t := t + 1$;
  Find $\varepsilon(z^*) := (z^*)^T y^k = \min_{j=1}^{m} \min\{(z^*)^T y;\ y \in \mathcal{A}^j\}$;

Add $y^k$ to $\mathcal{B}^k$

**until** $\varepsilon(z^*) = 0$

$z^*$ is the solution of CSH($\mathbf{A}, \sigma$).

The following theorem follows from Theorems 5, 6 and Corollary 3.

**Theorem 7.** *If* $\mathbf{A}$ *is a connected NR* $V$*-matrix, then Algorithm CSH is convergent.*

**Corollary 6.** *Assume that the following conditions are satisfied:*

(i) $\mathbf{A}$ *is a square connected* $V$*-matrix;*

(ii) $(z^*, A_*)$ *is an extended solution of CSH(*$\mathbf{A}, \sigma$*);*

(iii) $\operatorname{rank}(A_*) \geqslant m - 1$*;*

(iv) *If* $A_*$ *is singular, then* $A_*^T w = e$ *has no solution.*

*Then, Algorithm CSH is locally convergent in a neigbourhood of* $A_*$*.*

P r o o f. According to Proposition 3, $\mathbf{A}$ is locally NR in a neighbourhood of $A_*$. Thus, Theorem 7 can be applied. □

Of course, an effective application of the CSH-method can be hardly carried out without assuming convexity of the sets $\mathcal{A}^j$. For some important special types of these sets, the problems of finding $y^s$ can be solved by using simple explicit formulae ([15]).

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