PURE STATES ON JORDAN ALGEBRAS

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Abstract. We prove that a pure state on a C^*-algebra or a JB algebra is a unique extension of some pure state on a singly generated subalgebra if and only if its left kernel has a countable approximative unit. In particular, any pure state on a separable JB algebra is uniquely determined by some singly generated subalgebra. By contrast, only normal pure states on JBW algebras are determined by singly generated subalgebras, which provides a new characterization of normal pure states. As an application we contribute to the extension problem and strengthen the hitherto known results on independence of operator algebras arising in the quantum field theory.

Keywords: JB algebras, C^*-algebras, pure states, state space independence of Jordan algebras

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1. Introduction and preliminaries

In mathematical foundations of the traditional Hilbert-space quantum mechanics the basic role is played by pure normal states on the algebra $B(H)$ of all bounded operators on a separable Hilbert space H. (Each normal state on $B(H)$ corresponds to a σ-additive measure on projections by Gleason's theorem [12].) It is well known that for each such a state ϱ there is a unique one-dimensional projection p which determines ϱ in the following sense: ϱ is the only pure state on $B(H)$ attaining value one at p. (We shall call p the determining element for ϱ.) This fact embodies the one-to-one correspondence between physical states and rays in the Hilbert space H. The aim of this note is to investigate the analogy of this duality in general operator

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algebraic quantum mechanics. Namely, in the first part of the paper we show that any pure state ϱ on a separable C^*-algebra and JB algebra admits a norm one, positive, determining element c. In this case c can be considered a 'generalized ray' representing the state ϱ. More generally, we prove that a pure state ϱ on a JB algebra (unital or non-unital) possesses a determining element if and only if its left kernel $\mathcal{L}_\varrho$ has a strictly positive element or, which is the same, if and only if $\mathcal{L}_\varrho$ is σ-unital. The proof is based on an analysis of approximate units on pure state spaces. As a consequence of our results we get that for any pure state on a separable C^*-algebra A there is a maximal abelian subalgebra B of A such that ϱ is the only extension of some pure state on B—a result proved in [3]. This contributes to the study of restrictions of pure states to real-valued homomorphisms on associative subalgebras which has been pursued intensively in the C^*-algebraic setting [1–3, 5–10, 21, 24].

In the second part of this note normal pure states on von Neumann algebras are characterized as those that admit determining elements. Therefore, any pure state ϱ on a von Neumann algebra with a determining element is a vector state, $\varrho(x) = (x\xi, \xi)$, the support of which is a one-dimensional projection generated by the unit vector ξ. (In the more general context of JBW algebras the pure states on M_3^8 and spin factors are also involved.) In other words, any pure state with a determining element has to be σ-additive on projections. This advocates on a more plausible physical ground the assumption of σ-additivity of states in Hilbert-space quantum mechanics adopted for solely technical reasons.

In the final part an application of the technique of determining elements to finding simultaneous extensions from infinitely many algebras is discussed. This situation often appears in the quantum field theory where extensions from local subalgebras corresponding to space-like separated regions in the space-time are considered. We generalize the hitherto known results on independence of two subalgebras [11, 14, 22, 25] to any collection of subalgebras so improving the classical characterization of C^*-independent commuting C^*-algebras due to H. Roos [22].

Besides, the results of this paper may be relevant to the discussion on hidden variables in quantum theory. Indeed, re-stating Theorem 2.5 we can say that a pure state with a σ-unital kernel is always uniquely determined by some pure state on a singly generated subalgebra, i.e. by a preparation of some minimal classical subsystem.

Let us recall basic facts on JB algebras and fix the notation. (For more details we refer the reader to the monograph [16].)

In the sequel A will always denote a JB algebra endowed with a product $\circ$. We write $A_1 = \{a \in A;\ \|a\| \leqslant 1\}$, $A^+ = \{a^2;\ a \in A\}$, $A_1^+ = A_1 \cap A^+$. For $a \in A$, mappings $T_a, U_a : A \to A$ are defined by putting $T_a(b) = a \circ b$, $U_a(b) = 2a \circ (a \circ b) - a^2 \circ b$. It is well known that $U_a(A^+) \subset A^+$.

A closed subspace I of A is called a *Jordan ideal* if $T_a(A) \subset I$ for all $a \in I$. Similarly, a closed subspace U of A is said to be a *quadratic ideal* if $U_a(A) \subset U$ for all $a \in U$. Both the Jordan and the quadratic ideal is a subalgebra. We will use the symbol $C[a_1, \dots, a_n]$ to denote the JB subalgebra of A generated by elements $a_1, \dots, a_n$. The elements $a, b \in A$ are said to be *operator commuting* if $T_a T_b = T_b T_a$. The algebra A is called associative if it consists of operator commuting elements. The associative subalgebra $C[a]$ is said to be *singly generated*.

Denote by A^* the dual space of A. The *state space* $S(A)$ and the *quasi-state space* $Q(A)$ of A is defined as the set of all positive, norm one elements in A^* and the set of all positive elements in the unit ball of A^*, respectively.

The second dual A^{**} of A is a JBW algebra whose product is separately weak* continuous and extends the original product in A. In the sequel A will always be viewed as a weak* dense subalgebra of its second dual. Under this identification all functionals on A can be viewed as normal functionals on A^{**}. The range projection $r(a)$ of $a \in A$ is a projection in A^{**} defined as the smallest projection p with $p \circ a = a$. The symbol $\overline{S}$ will always denote the weak* closure of the set S in a given JBW algebra.

In this paper we will be mainly concerned with pure states (extreme points) in $S(A)$. For every pure state ϱ on A there exists a unique minimal non-zero projection $s(\varrho)$ in A^{**} such that $U_{s(\varrho)}(a) = \varrho(a)s(\varrho)$ for every $a \in A$ [16]. We shall call $s(\varrho)$ the *support projection* of a pure state ϱ. The symbol z_{at} will denote the supremum of all atomic projections in A^{**}. Denote by $\mathcal{L}_f = \{a \in A;\ f(a^2) = 0\}$ the *left kernel* of a positive functional f on A. The space $\mathcal{L}_f$ is a quadratic ideal contained in the kernel $\operatorname{Ker} f$. Moreover, $(\operatorname{Ker} f)^+ = \mathcal{L}_f^+$. With any pure state ϱ we associate a representation $\pi_\varrho\colon A \to c(\varrho)A^{**}\colon a \to c(\varrho)a$, where $c(\varrho)$ is the smallest central projection in A^{**} covering the support projection $s(\varrho)$.

The symbol M_3^8 will denote a matrix Jordan algebra of all 3×3 hermitean matrices over Cayley numbers. (It is known that M_3^8 is not isomorphic to any JC algebra.)

2. Approximate units and pure states

An *approximate unit* in a JB algebra A is a family $(u_\lambda)_{\lambda \in J}$ of elements in A_1^+ indexed by an upwards directed set J such that

(i) $0 \leqslant u_\lambda \leqslant u_\mu$ whenever $\lambda \leqslant \mu$ in J,

(ii) $\|u_\lambda \circ a - a\| \to 0$ $(\lambda \in J)$ for all $a \in A$.

Note that a functional $f \in A^*$ is positive exactly when $\|f\| = \lim_{\lambda \in J} \|f(u_\lambda)\|$ whenever (u_λ) is an approximate unit of A (see e.g. [16]). Conversely, it has been proved by C. A. Akemann [4] that an increasing net $(u_\lambda)_{\lambda \in J}$ of non-negative elements in the unit ball of a C^*-algebra A is an approximate unit if and only $\|f\| = \lim f(u_\lambda)$ for all pure states f on A. The equivalence of conditions (ii) and (iii) in the following theorem is a generalization of this result to the context of JB algebras. The proof is, except one step, the same as for C^*-algebras. Nevertheless, we state the full argument here for the sake of completeness.

2.1. Theorem. *Let $(u_\lambda)_{\lambda \in J} \subset A_1^+$ be an increasing net of elements in a JB algebra A. The following statements are equivalent:*
(i) $u_\lambda \nearrow 1$ *in* A^{**},
(ii) $f(u_\lambda) \nearrow 1$ *for every pure state f of A,*
(iii) $(u_\lambda)_{\lambda \in J}$ *is an approximate unit of A.*

Proof. The implication (i)$\Longrightarrow$ (ii) is trivial because every state on A is normal when considered as a state on A^{**}.

(ii)$\Longrightarrow$ (iii) Observe that for each fixed $x \in A$ the net $(x^2 - U_x(u_\lambda))_\lambda$ is a decreasing net of non-negative elements in A. Since (u_λ) is an increasing net in A^{**}, there is an element $0 \leqslant u \leqslant 1$ in A^{**} with $u_\lambda \nearrow u$. Therefore, $f(u) = 1$ whenever f is a pure state on A. Hence, $u \circ z_{at} = z_{at}$. Every pure state f being concentrated on $z_{at} A^{**}$, simple calculations give

$$\begin{aligned} f\big(x^2 - U_x(u_\lambda)\big) &= f\big(z_{at}x^2 - z_{at}U_x(u_\lambda)\big) \to f\big(z_{at}x^2 - z_{at}U_x(u)\big) \\ &= f(z_{at}x^2 - z_{at}x^2) = 0. \end{aligned}$$

As $\big(x^2 - U_x(u_\lambda)\big)$ is decreasing, $\|x^2 - U_x(u_\lambda)\| \searrow \varepsilon \geqslant 0$. Assume $\varepsilon > 0$ and try to reach a contradiction. Putting $K_\lambda = \big\{f \in Q(A);\ f\big(x^2 - U_x(u_\lambda)\big) \geqslant \varepsilon\big\}$ we get a system of weak* closed non-empty faces in the quasi-state space which enjoys the finite intersection property. Employing the compactness of $Q(A)$, we have that $K = \bigcap K_\lambda \neq \emptyset$. By the Krein-Milman theorem there is an extreme point f of the weak* compact convex set K. Since K_λ is a face f has to be a non-zero extreme point of $Q(A)$, i.e. a pure state on A. But this is absurd because of $f(x^2 - U_x(u_\lambda)) \to 0$. Therefore $\|x^2 - U_x(u_\lambda)\| \to 0$ for all $x \in A$. Evoking the inequality

$$\|a \circ b\| \leqslant \|a\| \cdot \|U_b(a)\| \text{ for all } a \in A^+ \text{ and } b \in A$$

[16, Lemma 3.5.2. (ii), p. 86] we can write

$$\begin{aligned} \|x - x \circ u_\lambda\|^2 = \|x \circ (1 - u_\lambda)\|^2 &\leqslant \|1 - u_\lambda\| \ \|U_x(1 - u_\lambda)\| \\ &= \|1 - u_\lambda\| \ \|x^2 - U_x(u_\lambda)\| \to 0. \end{aligned}$$

(iii)$\Longrightarrow$(i) Let $(u_\lambda)_{\lambda \in J}$ be an approximate unit for A. By separate weak* continuity of multiplication in A^{**} we have $u_\lambda \nearrow p$ for some projection $p \in A^{**}$. Suppose that $p < 1$. Then there is a normal state ϱ on A^{**} with $\varrho(p) = 0$. Hence,

$$0 \leqslant \varrho(u_\lambda^2) \leqslant \varrho(u_\lambda) \leqslant \varrho(p) = 0,$$

and so $\varrho(x \circ u_\lambda)^2 \leqslant \varrho(x^2)\varrho(u_\lambda^2) = 0$ by the Schwarz inequality. On the other hand, since $\|x \circ u_\lambda - x\| \to 0$, we obtain that $\varrho(x) = 0$ for every $x \in A$, which is impossible. Therefore $p = 1$ and the result follows. □

An element x of a JB algebra A is said to be *strictly positive* if $f(x) > 0$ for every non-zero $f \in A^*_+$. Employing now Theorem 2.1 and arguments analogous to the well known case of C^*-algebras we see that A admits a strictly positive element exactly when there is $h \in A^+$ with $r(h) = 1$, or equivalently, when A admits (an operator commuting) countable approximate unit.

The question of when a pure state is determined by its values on some singly generated subalgebra gives impetus to the following definition.

2.2. Definition. Let ϱ be a pure state on a JB algebra A. We say that an element $a \in A_1$ is *determining* for ϱ if ϱ is the only pure state on A with $\varrho(a) = 1$.

It is easily verified that if a is a determining element for ϱ then its positive part is determining for ϱ as well.

2.3. Remark. Note that if f is a state on A, $a \in A_1^+$, and $f(a) = 1$, then f is multiplicative on a in the sense of the equality $f(a^2) = f(a)^2$. In that case even $f(a \circ b) = f(a)f(b)$ for any $b \in A$. (These observations can be derived by using the Schwarz inequality and the technique of the approximate unit in the same way as in the case of C^*-algebras considered e.g. in [5, 21].)

In particular, if a is a determining element for a pure state ϱ, then ϱ restricts to a pure state on $C[a]$ and is uniquely determined by this restriction. By the Hahn-Banach and Krein-Milman theorem ϱ is then uniquely determined by its (necessarily pure) restriction to any maximal associative subalgebra containing a. On the other hand, whenever a pure state ϱ is the only extension of some pure state on a singly generated subalgebra B of A, then $\varrho|B$ has a determining element $a \in B$ (use Example 2.5 below and the spectral theorem) and so a must be a determining element for ϱ as well.

Therefore, the presence of a determining element implies the restriction property of ϱ studied in [1, 2, 5, 7, 9, 10, 21].

2.4. Example. Let us examine the case of A being associative. Then A is representable as an algebra $C_0^R(X)$ of all real-valued continuous functions defined on a locally compact Hausdorff space X vanishing at infinity. Given a pure state ϱ, we can find a point $x \in X$ such that $\varrho(f) = f(x)$ for all $f \in C_0^R(X)$. A state ϱ has a determining element if and only if there is a countable system (U_n) of neighbourhoods of x such that $\bigcap_n U_n = \{x\}$. Indeed, suppose f is a non-negative determining element for ϱ. So $0 \leqslant f \leqslant 1$, and $f(y) = 1$ exactly when $y = x$. Putting $U_n = f^{-1}(1-1/n, \infty)$ we get the desired system of neighbourhoods of x. For the converse, let (U_n) be a countable system of neighbourhoods of $\{x\}$ satisfying $\bigcap U_n = \{x\}$. Assume that the system (U_n) is decreasing. We can always find $f_n \in C_0^R(X)$ such that $0 \leqslant f_n \leqslant 1$, $f_n(x) = 1$ and $f_n|(X \setminus U_n) = 0$. Set $f = \sum_{n=1}^{\infty} (1/2^n) f_n$. Then $0 \leqslant f \leqslant 1$, and $f(y) = 1$ exactly when $f_n(y) = 1$ for all n; or alternatively, if and only if $y \in \bigcap U_n = \{x\}$.

This example shows that only those pure states on associative algebras which can be, as points in the spectrum of A, separated from the other points by a countable system of neighbourhoods admit determining elements. Thus, if we consider e.g. $X = [0,1]^c$, where c is the continuum, then no pure state on the corresponding function algebra has a determining element. The following theorem is a generalization of this facts to general JB algebras.

2.5. Theorem. *Let ϱ be a pure state on a JB algebra A. If the left kernel $\mathcal{L}_\varrho$ has a strictly positive element, then ϱ admits a determining element.*

The converse implication is true provided A is unital.

Proof. We first prove the existence of an element $a \in A_1^+$ with $\varrho(a) = 1$. This follows from [15, Proposition].

Let now $0 \leqslant x \leqslant 1$ be a strictly positive element of $\mathcal{L}_\varrho$. Set $c = a - x$ and take a pure state φ of A such that $\varphi(c) = 1$. Then $\varphi(a), \varphi(x) \in [0,1]$ immediately implies that $\varphi(a) = 1$ and $\varphi(x) = 0$. An element x being a strictly positive element of $\mathcal{L}_\varrho$, we have $\varphi|\mathcal{L}_\varrho = 0$. The projection $1 - s(\varrho)$ is open in A^{**} and so $1 - s(\varrho)$ is in the weak* closure of $\mathcal{L}_\varrho$. Hence $\varphi(1 - s(\varrho)) = 0$ by normalcy of φ. Therefore $\varphi = \varrho$.

Conversely, assume that A is unital and $0 \leqslant c \leqslant 1$ is a determining element for ϱ. Letting $x = 1 - c$ we get a strictly positive element of $\mathcal{L}_\varrho$. For this let us take an arbitrary pure state φ of $\mathcal{L}_\varrho$ with $\varphi(x) = 0$. On extending φ canonically to a normal pure state on A^{**}, we have that $\varphi(c) = 1$, while $\varphi \neq \varrho$, contradicting the assumption. The proof is completed. $\square$

2.6. Remark. The assumption of unitality of A is essential in Theorem 2.5. For a counterexample take a JB algebra $A = R \oplus M$, where R is one-dimensional and M has no strictly positive element. Then a pure state ϱ on M concentrated at the one-dimensional direct summand R has a determining element 1_R. On the other hand $\mathcal{L}_\varrho = M$.

Let us remark that any separable algebra has a strictly positive element because it has a countable approximate unit. Therefore Theorem 2.5 improves results in [1, 3, 14] concerning separable algebras.

3.1. Lemma. *Let M be an associative JBW algebra with an atomless projection lattice. For any $f \in M^+$ of norm one there are two pure states ϱ_1, ϱ_2 of M such that $\varrho_1(f) = \varrho_2(f) = 1$.*

Proof. We can assume that $M = L^\infty(X, \mathcal{M}, m)$, where m is an atomless Radon measure on an algebra $\mathcal{M}$ of all Borel sets of a locally compact Hausdorff space X. Set $M_n = \left\{x \in X;\ f(x) > 1 - \frac{1}{n}\right\}$. Then (M_n) is a decreasing sequence of Borel sets with non-zero measures. Employing atomlessness we can find disjoint Borel sets A and B such that both A and B have intersections of non-zero measures with any member of the sequence (M_n). Hence, $\|f|A\|_\infty = \|f|B\|_\infty = 1$. In other words, if we take $f_1 = fp, f_2 = f(1-p)$, where p is a characteristic function of A, then

$$f_1, f_2 \leqslant f, \quad f_1 f_2 = 0 \text{ and } \|f_1\|_\infty = \|f_2\|_\infty = 1.$$

By the Hahn-Banach and Krein-Milman theorems there are pure states ϱ_1, ϱ_2 of M such that $\varrho_1(f_1) = \varrho_2(f_2) = 1$. Since any pure state is multiplicative on M we have

$$0 = \varrho_1(f_1 f_2) = \varrho_1(f_1)\varrho_1(f_2).$$

Hence, $\varrho_1(f_2) = 0$ and so $\varrho_1 \neq \varrho_2$. The proof is completed. □

Unlike separable JB algebras for which every pure state has a determining element, on JBW algebras only normal pure states have this property.

3.2. Theorem. *Let ϱ be a pure state on a JBW algebra M. Then ϱ has a determining element, if and only if ϱ is normal.*

Proof. Let us suppose that ϱ is a pure state on M admitting a determining element c. Fix a maximal associative subalgebra of M containing c. Decompose A into the discrete and continuous parts A_d and A_c, respectively. So A_d is either zero or has an atomic projection lattice and A_c has no non-zero minimal projection. Suppose that ϱ is concentrated on A_c. It means that $\varrho|A_c$ has a determining element $f \in A_c^+, \|f\| = 1$. But this is impossible according to Lemma 3.1. Therefore ϱ has to be concentrated on A_d and has a determining element in this algebra. Since A_d is discrete it is isomorphic to the algebra $C^R(\beta\kappa)$ of all continuous real-valued functions on the Stone-Čech compactification of a cardinal κ endowed with the discrete topology. Therefore $\varrho|A_d$ is a Dirac measure concentrated at a point $x_\varrho \in \beta\kappa$. Now we can use Example 2.4 to deduce that $\{x_\varrho\}$ has a countable basis of neighbourhoods. This can occur exactly when $x_\varrho \in \kappa$. So ϱ has to be concentrated at a point $x_\varrho \in \kappa$ which corresponds to an atomic projection p in A. Since A is maximal p has to be an atomic projection in M. This implies that ϱ is normal, which concludes the proof. $\square$

4. Extension problem

In the concluding part of this paper we show one application of the technique of determining elements to the problem of the simultaneous extensions of states. Suppose we are given a system (A_α) of JB subalgebras of a JB algebra A. We shall consider the question of when there is a simultaneous extension for any system of states defined on local subalgebras A_α's. According to the Hahn-Banach theorem an obvious necessary condition for the existence of such extension is the following one: for any n-tuple $(a_{\alpha_1}, \ldots, a_{\alpha_n}) \in (A_{\alpha_1}, \ldots, A_{\alpha_n})$ of positive, norm one elements there is a state φ of A with $\varphi(a_{\alpha_1}) = \ldots = \varphi(a_{\alpha_n}) = 1$. In the following theorem we show that this condition is also sufficient.

4.1. Theorem. *Let $(A_\alpha)_{\alpha \in G}$ be a system of JB subalgebras of a JB algebra A. Suppose that for any finite system of positive, norm one elements $a_{\alpha_1} \in A_{\alpha_1}, \ldots, a_{\alpha_n} \in A_{\alpha_n}$ there is a state ϱ of A with*

$$\varrho(a_{\alpha_1}) = \ldots = \varrho(a_{\alpha_n}) = 1.$$

Assume that φ_α is a state on A_α. Then there is a state φ on A extending all states of φ_α's.

Proof. First we prove the statement under the condition that A is separable. Let us consider finitely many subalgebras $A_{\alpha_1}, \ldots, A_{\alpha_n}$. Let ψ_{α_i} $(i = 1, \ldots, n)$ be a pure state on A_{α_i}. Our previous results guarantee the existence of a positive, norm one determining element $c_{\alpha_i} \in A_{\alpha_i}$ for each ψ_{α_i}. By assumption there is a state ψ of A with $\psi(c_{\alpha_i}) = 1$ for all $i = 1, \ldots, n$. The state ψ is automatically an extension of ψ_{α_i} for all $i = 1, \ldots, n$. Let us suppose first that

$$\varphi_{\alpha_1} = \sum_{i=1}^{m} \lambda_i \varrho_i$$

is a convex combination of pure states $\varrho_1, \ldots, \varrho_m$ on A_{α_1}, the states $\varphi_{\alpha_2}, \ldots, \varphi_{\alpha_n}$ being pure. The above reasoning implies that there is a state φ of A extending $\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_n}$. Finally, employing the Krein-Milman theorem and the compactness of the quasi state space we can find the simultaneous extension of the states $\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_n}$, where φ_{α_1} is arbitrary and $\varphi_{\alpha_2}, \ldots, \varphi_{\alpha_n}$ are pure. Proceeding in the same way we can show that there is a common extension for arbitrary states $\varphi_{\alpha_1}, \ldots, \varphi_{\alpha_n}$. Let $\mathcal{K}$ denote the system of all finite subsets of the index set G. For each $K \in \mathcal{K}$ the set $S_K = \{\varphi \in Q(A);\ \varphi|A_i = \varphi_i|A_i \text{ for all } i \in K\}$ is a non-empty and closed subset of $Q(A)$. Moreover, the system $(S_K)_{K\in\mathcal{K}}$ has the finite intersection property. By compactness there is a state in the intersection of all sets S_K that is a common extension of the family $(\varphi_\alpha)_{\alpha\in G}$.

Let A be now arbitrary. Denote by $\mathcal{F}$ the set of all non-empty finite subsets of the set $\bigcup\limits_{\alpha\in G} A_\alpha$. For $F \in \mathcal{F}$ set

$$S_F = \{\varrho \in Q(A);\ \varrho|F \cap A_\alpha = \varphi_\alpha \text{ for all } \alpha \in G \text{ with } F \cap A_\alpha \neq \emptyset\}.$$

Every set S_F is non-empty. Indeed, we can take a separable JB algebra $A(F)$ generated by a set F. By the previous part of the proof there is a state on $A(F)$ extending all states $\varphi_\alpha|A(F) \cap A_\alpha$. By extending this state to the whole algebra we get an element of S_F. Moreover, it can be easily checked that the system $(S_F)_{F\in\mathcal{F}}$ is a system of closed subsets of $Q(A)$ enjoying the finite intersection property. Therefore, $\bigcap\limits_{F\in\mathcal{F}} S_F \neq \emptyset$ and any element of this intersection is the desired common extension of the states $(\varphi_\alpha)_{\alpha\in G}$. The proof is completed. □

It has been proved by H. Roos [22] that any pair of states φ_1 and φ_2 of mutually commuting C^*-algebras A_1 and A_2 has a common extension to some larger algebra if and only if $ab \neq 0$ whenever a and b are non-zero elements of A_1 and A_2, respectively. As a consequence of Theorem 4.1 we can generalize this result to infinite families of operator commuting JB algebras.

4.2. Corollary. *Let $(A_\alpha)_{\alpha\in G}$ be a system of mutually operator commuting JB subalgebras of a JB algebra A. Suppose that every finite collection of non-zero elements $a_{\alpha_1} \in A_{\alpha_1}, \ldots a_{\alpha_n} \in A_{\alpha_n}$ has a non-zero product $a_{\alpha_1} \circ \ldots \circ a_{\alpha_n}$. Then there is a common extension of any family $(\varphi_\alpha)_{\alpha\in G}$ of states on A_α's.*

Proof. Let us take the positive, norm one elements $a_1 \in A_{\alpha_1}, \ldots, a_n \in A_{\alpha_n}$. Then $\|a_1 \circ \ldots \circ a_n\| = 1$. Indeed, if $\|a_1 \circ \ldots \circ a_n\| < 1$ held then the homomorphism

$$\Phi : C[a_1] \otimes C[a_2] \otimes \ldots \otimes C[a_n] \to C[a_1, a_2, \ldots, a_n]$$

defined on the tensor product of associative algebras which is uniquely determined by the condition

$$\Phi(f_1 \otimes \ldots \otimes f_n) = f_1 \ldots f_n, \text{ for all } f_1 \in C[a_1], \ldots, f_n \in C[a_n]$$

would not be an isometry. Therefore $\operatorname{Ker}\Phi$ would be non-trivial and we would be able to find non-zero elements $f_1 \in C[a_1], \ldots, f_n \in C[a_n]$ with $\Phi(f_1 \otimes \ldots \otimes f_n) = f_1 \ldots f_n = 0$—contradicting the assumption of Corollary.

Hence, there is a pure state φ of A such that $\varphi(a_1 \circ \ldots \circ a_n) = 1$ and so $\varphi(a_1) = \ldots = \varphi(a_n) = 1$ since $a_1 \circ \ldots \circ a_n \leqslant a_i$ for each i by commutativity. The assertion now follows from Theorem 4.1. $\square$

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