ON OPTIMAL DECAY RATES FOR WEAK SOLUTIONS TO THE NAVIER-STOKES EQUATIONS IN $\mathbb{R}^n$

Tetsuro Miyakawa, Rokko, Maria Elena Schonbek, Santa Cruz

Dedicated to Professor Jindřich Nečas on his 70th birthday

Abstract. This paper is concerned with optimal lower bounds of decay rates for solutions to the Navier-Stokes equations in $\mathbb{R}^n$. Necessary and sufficient conditions are given such that the corresponding Navier-Stokes solutions are shown to satisfy the algebraic bound

$$\|u(t)\| \geqslant (t+1)^{-\frac{n+4}{2}}.$$

Keywords: decay rates, Navier-Stokes equations

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1. Introduction and the results

Consider the Navier-Stokes equations in $\mathbb{R}^n$, $n \geqslant 2$, which will be treated in this paper in the form of the integral equation

$$\text{(NS)} \qquad u(t) = \mathrm{e}^{-tA}a - \int_0^t \nabla \cdot \mathrm{e}^{-(t-s)A} P(u \otimes u)(s)\,\mathrm{d}s,$$

for prescribed initial velocity $a(x) = (a_1(x), \ldots, a_n(x))$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and unknown velocity $u(x,t) = (u_1(x,t), \ldots, u_n(x,t))$. Here, $A = -\Delta$ is the Laplacian on $\mathbb{R}^n$; $\{\mathrm{e}^{-tA}\}_{t \geqslant 0}$ is the heat semigroup; $P = (P_{jk})$ is the bounded projection onto divergence-free vector fields; $u \otimes v$ is the matrix with entries $(u \otimes v)_{jk} = u_j v_k$; $\nabla = (\partial_1, \ldots, \partial_n)$ with $\partial_j = \partial/\partial x_j$; and

$$(\nabla \cdot \mathrm{e}^{-tA} P(u \otimes u))_j = \sum_{k,\ell=1}^n \partial_\ell \mathrm{e}^{-tA} P_{jk}(u_\ell u_k), \quad j = 1, \ldots, n.$$

It is well known that for each $a \in \boldsymbol{L}^2$ with $\nabla \cdot a = 0$, (NS) has a weak solution u defined for all $t \geqslant 0$, satisfying the energy inequality

$$\|u(t)\|_2^2 + 2\int_0^t \|\nabla u\|_2^2 \, \mathrm{d}s \leqslant \|a\|_2^2 \quad \text{for all } t \geqslant 0.$$

Hereafter $\|\cdot\|_r$ denotes the L^r-norm.

As shown in [10], there exists a weak solution u such that

$$\|u(t)\|_2 \leqslant C(1+t)^{-\frac{n+2}{4}}, \tag{1.1}$$

whenever

$$a \in \boldsymbol{L}^2, \quad \nabla \cdot a = 0 \quad \text{and} \quad \int (1+|y|)|a(y)| \, \mathrm{d}y < \infty. \tag{1.2}$$

Assumption (1.2) implies $a \in \boldsymbol{L}^1$; so the divergence-free condition gives (see [4])

$$\int a(y) \, \mathrm{d}y = 0. \tag{1.3}$$

Furthermore, it is shown in [2] that in this case the solution u satisfies

$$\begin{aligned} \lim_{t\to\infty} t^{\frac{n+2}{4}} \Big\| u_j(t) + (\partial_k E_t)(\cdot) \int y_k a_j(y) \, \mathrm{d}y \\ + F_{\ell,jk}(\cdot,t) \int_0^\infty \int (u_\ell u_k)(y,s) \, \mathrm{d}y \, \mathrm{d}s \Big\|_2 = 0 \end{aligned} \tag{1.4}$$

for $j = 1, \ldots, n$, where

$$E_t(x) = (4\pi t)^{-n/2} \mathrm{e}^{-|x|^2/4t}, \quad F_{\ell,jk}(x,t) = \partial_\ell E_t(x)\delta_{jk} + \int_t^\infty \partial_\ell \partial_j \partial_k E_s(x) \, \mathrm{d}s.$$

(Hereafter, we use the summation convention). Equation (NS) is then written in the form

$$u_j(x,t) = \int E_t(x-y) a_j(y) \, \mathrm{d}y - \int_0^t \int F_{\ell,jk}(x-y,t-s)(u_\ell u_k)(y,s) \, \mathrm{d}y \, \mathrm{d}s, \ j = 1, \ldots, n,$$

as proved in [2]; and the integrals in (1.4) are finite, due to (1.1) and (1.2). Assertion (1.4) was first proved in [1] for smooth solutions when $n = 3$, and then extended in [2] to the case of weak solutions in all space dimensions by applying the spectral method as given in [3, 5].

The argument of [10] suggests that the decay property (1.1) will be optimal in general. So we are interested in finding a class of weak solutions u satisfying the reverse estimate

$$\|u(t)\|_2 \geqslant Ct^{-\frac{n+2}{4}} \quad \text{at least for large } t.$$

In this paper we discuss this kind of *lower bound problem*.

Theorem A. *Under the assumption* (1.2), *let*

$$b_{k\ell} = \int y_\ell a_k(y)\,\mathrm{d}y, \qquad c_{k\ell} = \int_0^\infty \int (u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

(i) *We have*

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} \|u(t)\|_2 = 0 \tag{1.5}$$

if and only if $(b_{k\ell}) = 0$ *and* $(c_{k\ell}) = (c\delta_{k\ell})$ *for some constant* $c \geqslant 0$.

(ii) *There exists* $c' > 0$ *such that*

$$\|u(t)\|_2 \geqslant c' t^{-\frac{n+2}{4}} \quad \textit{for large } t > 0, \tag{1.6}$$

if and only if $(b_{k\ell}) \neq 0$ *or* $(c_{k\ell}) \neq (c\delta_{k\ell})$. *In particular,* u *satisfies* (1.6) *whenever* $(b_{k\ell}) \neq 0$.

R e m a r k. Theorem A (i) implies only that

$$\limsup_{t\to\infty} t^{\frac{n+2}{4}} \|u(t)\|_2 > 0 \tag{1.5′}$$

if and only if $(b_{k\ell}) \neq 0$ or $(c_{k\ell}) \neq (c\delta_{k\ell})$. Note, however, that our second assertion (1.6) is more stringent than (1.5′). Moreover, (1.6) holds for *all* large $t > 0$ and for *all* space dimensions, although $\|u(t)\|_2$ is only known to be lower semicontinuous when $n \geqslant 3$. We know nothing about the characterization of solutions satisfying $(c_{k\ell}) = (c\delta_{k\ell})$.

We next consider weak solutions u satisfying

$$\|u(t)\|_2 \leqslant C(1+t)^{-\frac{n}{4}}. \tag{1.7}$$

As shown in [3, 6, 10], such solutions exist for all $a \in \boldsymbol{L}^2$ satisfying

$$\nabla \cdot a = 0, \quad \|\mathrm{e}^{-tA} a\|_2 \leqslant C(1+t)^{-\frac{n}{4}}. \tag{1.8}$$

Theorem B. *Suppose a satisfies* (1.8) *and let u be a weak solution satisfying* (1.7). *Then*

$$\|u(t)\|_2 \geqslant ct^{-\frac{n}{4}} \quad \text{for large } t > 0, \tag{1.9}$$

if and only if

$$\|\mathrm{e}^{-tA}a\|_2 \geqslant ct^{-\frac{n}{4}} \quad \text{for large } t > 0. \tag{1.10}$$

The lemma below gives simple examples of a satisfying (1.10).

Lemma. *Let $a \in \boldsymbol{L}^2$, $\nabla \cdot a = 0$, and suppose that*

$$\int_{S^{n-1}} |\hat{a}(r,\omega)|^2 \,\mathrm{d}\omega \in L^\infty(\mathbb{R}_+), \quad \liminf_{r\to 0} \int_{S^{n-1}} |\hat{a}(r,\omega)|^2 \,\mathrm{d}\omega > 0, \tag{1.11}$$

where the Fourier transform $\hat{a}$ is defined by

$$\hat{a}(\xi) = \int \mathrm{e}^{-\mathrm{i}x\cdot\xi} a(x)\,\mathrm{d}x, \quad \mathrm{i} = \sqrt{-1},$$

S^{n-1} is the unit sphere of $\mathbb{R}^n$, and $\xi = (r,\omega)$ in polar coordinates. Then,

$$\|\mathrm{e}^{-tA}a\|_2 \leqslant C(1+t)^{-\frac{n}{4}} \text{ for all } t > 0; \quad \|\mathrm{e}^{-tA}a\|_2 \geqslant c't^{-\frac{n}{4}} \text{ for large } t > 0, \tag{1.12}$$

with constants $C > 0$ and $c' > 0$ independent of t.

P r o o f. Parseval's relation gives

$$\|\mathrm{e}^{-tA}a\|_2^2 = (2\pi)^{-n} \int \mathrm{e}^{-2t|\xi|^2} |\hat{a}(\xi)|^2 \,\mathrm{d}\xi = (8\pi^2 t)^{-\frac{n}{2}} \int \mathrm{e}^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}})|^2 \,\mathrm{d}\eta$$

so that

$$(8\pi^2 t)^{\frac{n}{2}} \|\mathrm{e}^{-tA}a\|_2^2 = \int \mathrm{e}^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}})|^2 \,\mathrm{d}\eta.$$

The assumption and Fatou's lemma together imply

$$\begin{aligned}\liminf_{t\to\infty} (8\pi^2 t)^{\frac{n}{2}} \|\mathrm{e}^{-tA}a\|_2^2 &= \liminf_{t\to\infty} \int \mathrm{e}^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}})|^2 \,\mathrm{d}\eta \\ &\geqslant \int_0^\infty \mathrm{e}^{-r^2} \Big(\liminf_{t\to\infty} \int_{S^{n-1}} |\hat{a}(r(2t)^{-\frac{1}{2}},\omega)|^2 \,\mathrm{d}\omega\Big) r^{n-1} \,\mathrm{d}r > 0.\end{aligned}$$

This proves the second estimate of (1.12). The first estimate follows from $\|\mathrm{e}^{-tA}a\|_2 \leqslant \|a\|_2$ and

$$\begin{aligned}\|\mathrm{e}^{-tA}a\|_2^2 &= (8\pi^2 t)^{-\frac{n}{2}} \int \mathrm{e}^{-|\eta|^2} |\hat{a}(\eta(2t)^{-\frac{1}{2}})|^2 \,\mathrm{d}\eta \\ &\leqslant Ct^{-\frac{n}{2}} \Big\| \int_{S^{n-1}} |\hat{a}(\cdot,\omega)|^2 \,\mathrm{d}\omega \Big\|_\infty \int_0^\infty \mathrm{e}^{-r^2} r^{n-1} \,\mathrm{d}r.\end{aligned}$$

The proof is complete. $\square$

R e m a r k s. (i) Condition (1.11) implies that $\hat{a}$ is discontinuous at $\xi = 0$. Indeed, since $\nabla \cdot a = 0$, we have $\xi \cdot \hat{a}(\xi) = 0$; so if $\hat{a}$ is continuous at $\xi = 0$, we get $\omega \cdot \hat{a}(0) = 0$ for all unit vectors ω, and $\hat{a}(0) = 0$. (For this reason, $a \in \boldsymbol{L}^1$ implies (1.3)).

(ii) The assumption of Lemma is not vacuous. Indeed, suppose $\hat{a}$ is written in the form

$$\hat{a}(\xi) = f(|\xi|)g(\xi/|\xi|),$$

in terms of functions $f(r)$ and $g(\omega)$ such that

$$g \in \boldsymbol{L}^2(S^{n-1}), \quad g \not\equiv 0, \quad \omega \cdot g(\omega) \equiv 0 \quad (\omega \in S^{n-1})$$

and

$$f \in BC([0,\infty)), \quad \int_0^\infty |f(r)|^2 r^{n-1}\,\mathrm{d}r < \infty, \quad f(0) \neq 0.$$

Then, $\hat{a}$ satisfies condition (1.11).

(iii) In this connection, we note that under condition (1.2) we have

$$\|\mathrm{e}^{-tA}a\|_2 \geqslant ct^{-\frac{n+2}{4}} \quad \text{for large } t > 0 \tag{1.10$'$}$$

if and only if $(b_{k\ell}) \neq 0$. Indeed, using (1.2) and (1.3), we have (see Section 4)

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} \|\mathrm{e}^{-tA}a_k + \partial_\ell E_t b_{k\ell}\|_2 = 0, \quad k = 1,\dots,n. \tag{1.4$'$}$$

Suppose $(b_{k\ell}) \neq 0$. Then $(\sum\limits_k \|\partial_\ell E_t b_{k\ell}\|_2^2)^{1/2} = Ct^{-\frac{n+2}{4}}$ with $C > 0$; so we get

$$\|\mathrm{e}^{-tA}a\|_2 \geqslant \big(\sum_k \|\partial_\ell E_t b_{k\ell}\|_2^2\big)^{1/2} - \big(\sum_k \|\mathrm{e}^{-tA}a_k + \partial_\ell E_t b_{k\ell}\|_2^2\big)^{1/2} \geqslant ct^{-\frac{n+2}{4}}$$

for large $t > 0$. Conversely, if we assume (1.10′), then (1.4′) implies

$$\big(\sum_k \|\partial_\ell E_t b_{k\ell}\|_2^2\big)^{1/2} \geqslant \|\mathrm{e}^{-tA}a\|_2 - \big(\sum_k \|\mathrm{e}^{-tA}a_k + \partial_\ell E_t b_{k\ell}\|_2^2\big)^{1/2} \geqslant ct^{-\frac{n+2}{4}}$$

for large $t > 0$. Hence $\sum\limits_k \|\partial_\ell E_t b_{k\ell}\|_2^2 > 0$ for large $t > 0$, which implies $(b_{k\ell}) \neq 0$.

The L^2 decay problem for weak solutions of the Navier-Stokes equations was successfully studied for the first time by [5] and the result was then systematically developed by [3, 6, 10]. Estimates (1.6) and (1.9) are studied in [6]–[9] in case $n = 2, 3$, and some sufficient conditions are obtained. Our Theorems A and B provide *necessary and sufficient conditions* for those estimates to hold. We further note that our lower bound estimates (1.6) and (1.9) hold in all space dimensions $n \geqslant 2$, although the function $\|u(t)\|_2$ is known only to be lower semicontinuous when $n \geqslant 3$. As will be seen in the proof below, this is due to (1.4) and the fact that the functions $\partial_\ell E_t(x)$ and $F_{\ell,jk}(x,t)$ are written in the form $t^{-\frac{n+1}{2}} K(xt^{-\frac{1}{2}})$ in terms of some bounded, integrable and uniformly continuous functions K.

We finally consider an example of two-dimensional flows u with $(b_{k\ell}) = 0$, $(c_{k\ell}) = (c\delta_{k\ell})$, which was first treated by [7].

Theorem C. *When $n = 2$, there is a smooth weak solution u such that $(b_{k\ell}) = 0$, $(c_{k\ell}) = (c\delta_{k\ell})$, and, with some constant $\gamma > 0$,*

$$\|u(t)\|_q \leqslant C_q \mathrm{e}^{-\gamma t} \quad \text{and} \quad |u(x,t)| \leqslant C_m \mathrm{e}^{-\gamma t}(1+|x|)^{-m} \tag{1.13}$$

for all $1 \leqslant q \leqslant \infty$ and all integers $m \geqslant 0$.

The above example was studied by [7, 8, 9], in which is given the exponential decay of $\|u(t)\|_q$ for $2 \leqslant q \leqslant \infty$. Our estimates (1.13) include the case $1 \leqslant q < 2$ as well as the decay estimates in the spatial direction. Theorem C is proved in [2].

In what follows we prove Theorems A and B, and conclude the paper with the proof of (1.4) which was given also in [2].

2. Proof of Theorem A

We begin with the following

Proposition 2.1. *Let $(b_{k\ell})$ and $(c_{k\ell})$ be real $n \times n$ matrices and let $(c_{k\ell})$ be symmetric. Then*

$$b_{k\ell}\partial_\ell E_t(x)\delta_{jk} + c_{k\ell}F_{\ell,jk}(x,t) = 0, \quad j = 1, \ldots, n, \tag{2.1}$$

for all $x \in \mathbb{R}^n$ and for some $t > 0$, if and only if

$$(b_{k\ell}) = 0 \quad \text{and} \quad (c_{k\ell}) = (c\delta_{k\ell}) \quad \text{for some } c \in \mathbb{R}. \tag{2.2}$$

Furthermore, (2.2) implies that (2.1) holds for all x and for all $t > 0$.

P r o o f. Assumption (2.1) implies, via the Fourier transformation,

$$\begin{aligned} b_{k\ell}\xi_\ell \mathrm{e}^{-t|\xi|^2}\delta_{jk} &= -c_{k\ell}\xi_\ell\Big(\mathrm{e}^{-t|\xi|^2}\delta_{jk} - \xi_j\xi_k\int_t^\infty \mathrm{e}^{-s|\xi|^2}\,\mathrm{d}s\Big) \\ &= -(c_{j\ell} - |\xi|^{-2}c_{k\ell}\xi_j\xi_k)\xi_\ell \mathrm{e}^{-t|\xi|^2} \end{aligned}$$

for some $t > 0$, and we get $|\xi|^2(b_{j\ell} + c_{j\ell})\xi_\ell = \xi_j c_{k\ell}\xi_k\xi_\ell$. Taking $\xi_j = 0$ for any fixed j, $\xi_\ell = 1$ for any fixed $\ell \neq j$, and $\xi_k = 0$ for all k such that $k \neq j$ and $k \neq \ell$, we easily obtain $b_{j\ell} + c_{j\ell} = 0$ whenever $j \neq \ell$, and so

$$|\xi|^2(b_{jj} + c_{jj})\xi_j = \xi_j c_{k\ell}\xi_k\xi_\ell, \quad j = 1, \ldots, n.$$

We let $\xi_j = 1$ and $\xi_k = 0$ for $k \neq j$, to get $b_{jj} + c_{jj} = c_{jj}$; so $b_{jj} = 0$. This implies

$$|\xi|^2 c_{jj}\xi_j = \xi_j c_{k\ell}\xi_k\xi_\ell, \quad j = 1, \ldots, n. \tag{2.3}$$

Hence, $c_{11} = \ldots = c_{nn} = c_{k\ell}\xi_k\xi_\ell|\xi|^{-2}$. We then set $j = 1$, $\xi_1 = \xi_2 = 1$ and $\xi_k = 0$ for $k \geqslant 3$ in (2.3), to get $2c_{11} = c_{11} + c_{22} + c_{12} + c_{21} = 2(c_{11} + c_{12})$ since $c_{k\ell} = c_{\ell k}$ by assumption. Therefore, $c_{12} = 0$. We thus obtain $c_{j\ell} = 0 = -b_{j\ell}$ whenever $j \neq \ell$; so $(b_{k\ell}) = 0$ and $(c_{k\ell}) = (c\delta_{k\ell})$. That (2.2) implies (2.1) for *all* $t > 0$ is easily seen from

$$F_{k,jk} = \partial_j E_t + \int_t^\infty \partial_j \Delta E_s \, \mathrm{d}s = \partial_j E_t + \int_t^\infty \partial_j \partial_s E_s \, \mathrm{d}s = \partial_j E_t - \partial_j E_t = 0,$$

where $\partial_s = \partial/\partial s$. The proof of Proposition 2.1 is complete. $\square$

To establish Theorem A, it suffices in view of (1.4) to prove the following

Proposition 2.2. *Let a satisfy* (1.2) *and define*

$$b_{k\ell} = \int y_\ell a_k(y) \, \mathrm{d}y, \quad c_{k\ell} = \int_0^\infty \int (u_\ell u_k)(y, s) \, \mathrm{d}y \, \mathrm{d}s.$$

Then we have

$$\textit{either} \quad (b_{k\ell}) \neq 0 \quad \textit{or} \quad (c_{k\ell}) \neq (c\delta_{k\ell}), \tag{2.4}$$

if and only if a corresponding weak solution u satisfies

$$\|u(t)\|_2 \geqslant c' t^{-\frac{n+2}{4}} \quad \textit{for large } t > 0 \tag{2.5}$$

with a constant $c' > 0$ indenpendent of t.

Proof. In what follows we write

$$\boldsymbol{b}_\ell = (b_{1\ell}, \ldots, b_{n\ell}), \quad \boldsymbol{F}_{\ell,k} = (F_{\ell,1k}, \ldots, F_{\ell,nk}).$$

Assume first (2.4). By Proposition 2.1, we have $\|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 = Ct^{-\frac{n+2}{4}}$ for all $t > 0$ with some $C > 0$, and so (1.4) implies

$$\begin{aligned}\|u(t)\|_2 &\geqslant \|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 - \|u(t) + \partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \\ &= Ct^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}) \geqslant c' t^{-\frac{n+2}{4}}\end{aligned}$$

for large $t > 0$. Assume next (2.5). By (1.4) we have

$$\|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \geqslant \|u(t)\|_2 - \|u(t) + \partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \geqslant c' t^{-\frac{n+2}{4}} - o(t^{-\frac{n+2}{4}}),$$

and so

$$\|\partial_\ell E_t \boldsymbol{b}_\ell + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 > 0 \quad \text{for large } t > 0.$$

We thus obtain (2.4) by Proposition 2.1. This proves Proposition 2.2. □

3. Proof of Theorem B

Suppose that $n \geqslant 3$. We have

$$c_{k\ell} = \int_0^\infty \int (u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s < \infty;$$

so the argument given in [2, Sect. 5] applies to our present situation, implying

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} \|u(t) - \mathrm{e}^{-tA} a + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 = 0. \tag{3.1}$$

Suppose (1.9) holds. Since $\|\boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 = C t^{-\frac{n+2}{4}}$, it follows from (3.1) that

$$\begin{aligned}
\|\mathrm{e}^{-tA} a\|_2 &\geqslant \|u(t)\|_2 - \|-u(t) + \mathrm{e}^{-tA} a - \boldsymbol{F}_{\ell,k} c_{k\ell} + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \\
&\geqslant \|u(t)\|_2 - \|u(t) - \mathrm{e}^{-tA} a + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 - \|\boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \\
&\geqslant c t^{-\frac{n}{4}} - C t^{-\frac{n+2}{4}} \geqslant c' t^{-\frac{n}{4}}
\end{aligned}$$

for large $t > 0$. This proves (1.10). Conversely, if (1.10) holds, then (3.1) implies

$$\begin{aligned}
\|u(t)\|_2 &\geqslant \|\mathrm{e}^{-tA} a\|_2 - \|\boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 - \|u(t) - \mathrm{e}^{-tA} a + \boldsymbol{F}_{\ell,k} c_{k\ell}\|_2 \\
&\geqslant c t^{-\frac{n}{4}} - C t^{-\frac{n+2}{4}} \geqslant c' t^{-\frac{n}{4}}
\end{aligned}$$

for large $t > 0$. This proves (1.9) in case $n \geqslant 3$.

When $n = 2$, we introduce

$$c_{k\ell}(t) = \int_0^{t/2} \int (u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

instead of $c_{k\ell}$. The argument of [2, Sect. 5] is then modified to yield

$$\|u(t) - \mathrm{e}^{-tA} a + \boldsymbol{F}_{\ell,k} c_{k\ell}(t)\|_2 \leqslant C t^{-1} \log(1+t). \tag{3.1$'$}$$

See also Section 4 below. Since

$$\|\boldsymbol{F}_{\ell,k}c_{k\ell}(t)\|_2 \leqslant Ct^{-1}\int_0^{t/2}\|u(s)\|_2^2\,\mathrm{d}s \leqslant Ct^{-1}\log(1+t),$$

this implies $\|u(t)-\mathrm{e}^{-tA}a\|_2 \leqslant Ct^{-1}\log(1+t)$. Now we can prove the result in the same way as in the case $n \geqslant 3$. Indeed, (1.10) implies

$$\|u(t)\|_2 \geqslant \|\mathrm{e}^{-tA}a\|_2 - \|u(t)-\mathrm{e}^{-tA}a\|_2 \geqslant ct^{-\frac{1}{2}} - Ct^{-1}\log(1+t) \geqslant c't^{-\frac{1}{2}}$$

for large $t>0$, while (1.9) yields

$$\|\mathrm{e}^{-tA}a\|_2 \geqslant \|u(t)\|_2 - \|u(t)\mathrm{e}^{-tA}\|_2 \geqslant ct^{-\frac{1}{2}} - Ct^{-1}\log(1+t) \geqslant c't^{-\frac{1}{2}}$$

for large $t>0$. The proof of Theorem B is complete.

4. Proof of (1.4)

Here we present the proof of (1.4) given in [2]. The same method can be applied to the proof of (3.1) and (3.1′) with no essential change. Let a satisfy (1.2) and so (1.3). We first prove

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\left\|\mathrm{e}^{-tA}a + (\partial_k E_t)(\cdot)\int y_k a(y)\,\mathrm{d}y\right\|_2 = 0. \tag{4.1}$$

Direct calculation gives

$$\begin{aligned}\mathrm{e}^{-tA}a &= \int [E_t(x-y)-E_t(x)]a(y)\,\mathrm{d}y = -\int\int_0^1 (\partial_k E_t)(x-y\theta)y_k a(y)\,\mathrm{d}\theta\,\mathrm{d}y \\ &= -(\partial_k E_t)(x)\int y_k a(y)\,\mathrm{d}y - \int\int_0^1 [(\partial_k E_t)(x-y\theta)-(\partial_k E_t)(x)]y_k a(y)\,\mathrm{d}\theta\,\mathrm{d}y,\end{aligned}$$

so

$$\mathrm{e}^{-tA}a + (\partial_k E_t)(x)\int y_k a(y)\,\mathrm{d}y = -\int\int_0^1 [(\partial_k E_t)(x-y\theta)-(\partial_k E_t)(x)]y_k a(y)\,\mathrm{d}\theta\,\mathrm{d}y.$$

We can write $(\partial_k E_t)(x) = t^{-\frac{n+1}{2}}(\partial_k E_1)(xt^{-\frac{1}{2}})$, to obtain

$$\left\|\mathrm{e}^{-tA}a + (\partial_k E_t)(\cdot)\int y_k a(y)\,\mathrm{d}y\right\|_2 \leqslant Ct^{-\frac{n+2}{4}}\int\int_0^1 \varphi_t(y,\theta)|y||a(y)|\,\mathrm{d}\theta\,\mathrm{d}y.$$

Here $\varphi_t(y,\theta) = \|(\nabla E_1)(\cdot - y\theta t^{-\frac{1}{2}}) - (\nabla E_1)(\cdot)\|_2$ is bounded and $\lim_{t\to\infty} \varphi_t(y,\theta) = 0$ for any fixed (y,θ). Since $|y||a(y)|$ is integrable by (1.2), the dominated convergence theorem yields

$$\lim_{t\to\infty} \iint_0^1 \varphi_t(y,\theta)|y||a(y)| \,\mathrm{d}\theta \,\mathrm{d}y = 0.$$

This proves (4.1). Now let u satisfy (1.1). We next show that the function

$$w(t) = u(t) - \mathrm{e}^{-tA}a = -\int_0^t\!\!\int \boldsymbol{F}_{\ell,k}(x-y,t-s)(u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s$$

satisfies

$$\lim_{t\to\infty} t^{\frac{n+2}{4}} \left\| w(t) + \boldsymbol{F}_{\ell,k}(\cdot,t)\int_0^\infty\!\!\int (u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \right\|_2 = 0. \tag{4.2}$$

Indeed, we have

$$\begin{aligned}
w(t) &+ \boldsymbol{F}_{\ell,k}(x,t)\int_0^\infty\!\!\int (u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&= \boldsymbol{F}_{\ell,k}(x,t)\int_{t/2}^\infty\!\!\int (u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad - \int_0^{t/2}\!\!\int [\boldsymbol{F}_{\ell,k}(x-y,t-s) - \boldsymbol{F}_{\ell,k}(x,t-s)](u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad - \int_0^{t/2}\!\!\int [\boldsymbol{F}_{\ell,k}(x,t-s) - \boldsymbol{F}_{\ell,k}(x,t)](u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\quad - \int_{t/2}^{t}\!\!\int \boldsymbol{F}_{\ell,k}(x-y,t-s)(u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s \\
&\equiv I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

It is easy to see that

$$t^{\frac{n+2}{4}}\|I_1\|_2 \leqslant C\int_{t/2}^\infty (1+s)^{-1-\frac{n}{2}}\,\mathrm{d}s \to 0 \quad \text{as } t\to\infty. \tag{4.3}$$

We write I_3 in the form

$$I_3 = \int_0^{t/2}\!\!\iint_0^1 s(\partial_t \boldsymbol{F}_{\ell,k})(x,t-s\theta)(u_\ell u_k)(y,s)\,\mathrm{d}\theta\,\mathrm{d}y\,\mathrm{d}s$$

to get

$$\begin{aligned}
\|I_3\|_2 &\leqslant C\int_0^{t/2}\!\!\iint_0^1 s(t-s\theta)^{-1-\frac{n+2}{4}}|u(y,s)|^2\,\mathrm{d}\theta\,\mathrm{d}y\,\mathrm{d}s \\
&\leqslant Ct^{-1-\frac{n+2}{4}}\int_0^{t/2} s\|u(s)\|_2^2\,\mathrm{d}s
\end{aligned}$$

and so

$$t^{\frac{n+2}{4}}\|I_3\|_2 \leqslant Ct^{-1}\int_0^t (1+s)^{-\frac{n}{2}}\,\mathrm{d}s \to 0 \quad \text{as } t\to\infty. \tag{4.4}$$

To estimate I_2, note that we can write $\boldsymbol{F}_{\ell,k}(x,t) = t^{-\frac{n+1}{2}}K(xt^{-\frac{1}{2}})$, to get

$$\begin{aligned}\|I_2\|_2 &\leqslant Ct^{-\frac{n+2}{4}}\int_0^{t/2}\int \|K(\cdot - y(t-s)^{-\frac{1}{2}}) - K(\cdot)\|_2|u(y,s)|^2\,\mathrm{d}y\,\mathrm{d}s\\ &\equiv Ct^{-\frac{n+2}{4}}\int_0^{t/2}\int \varphi_t(y,s)|u(y,s)|^2\,\mathrm{d}y\,\mathrm{d}s \equiv Ct^{-\frac{n+2}{4}}\int_0^{t/2}\psi_t(s)\,\mathrm{d}s.\end{aligned}$$

Since $\psi_t(s) \leqslant C\|u(s)\|_2^2$, the dominated convergence theorem implies

$$\lim_{t\to\infty}\int_0^M \psi_t(s)\,\mathrm{d}s = 0 \quad \text{for any fixed } M>0.$$

Given $\varepsilon > 0$, choose $M > 0$ so that $\int_M^\infty \|u(s)\|_2^2\,\mathrm{d}s < \varepsilon$. Then for $t > 2M$,

$$\int_0^{t/2}\psi_t(s)\,\mathrm{d}s \leqslant \int_0^M \psi_t(s)\,\mathrm{d}s + C\int_M^\infty \|u(s)\|_2^2\,\mathrm{d}s \leqslant \int_0^M \psi_t(s)\,\mathrm{d}s + C\varepsilon.$$

This implies that

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\|I_2\|_2 = 0. \tag{4.5}$$

It remains to prove

$$\lim_{t\to\infty} t^{\frac{n+2}{4}}\|I_4\|_2 = 0. \tag{4.6}$$

To do so, we follow the arguments of [3, 5]. The function

$$v(t) = -\int_\tau^t\int \boldsymbol{F}_{\ell,k}(x-y,t-s)(u_\ell u_k)(y,s)\,\mathrm{d}y\,\mathrm{d}s = u(t) - \mathrm{e}^{-(t-\tau)A}u(\tau)$$

defined for $t \geqslant \tau > 0$ satisfies

$$\partial_t v + Av = -P(u\cdot\nabla u) \quad (t>\tau), \quad v(\tau) = 0.$$

(We may assume v is smooth, replacing u by the approximate solutions u_N given in [3]). Since $(P(u\cdot\nabla v), v) = (u\cdot\nabla v, v) = 0$, the standard energy integral method gives

$$\partial_t\|v\|_2^2 + 2\|A^{1/2}v\|_2^2 = -2(u\cdot\nabla u, v) = 2(u\cdot\nabla v, u) = 2(u\cdot\nabla v, u_0)$$

and

$$
\begin{aligned}
2|(u\cdot\nabla v, u_0)| &\leqslant 2\|u\|_2\|A^{1/2}v\|_2\|u_0\|_\infty \leqslant C\|u\|_2\|A^{1/2}v\|_2(t-\tau)^{-\frac{n}{4}}\tau^{-\frac{n+2}{4}}\\
&\leqslant C\|A^{1/2}v\|_2(t-\tau)^{-\frac{n+1}{2}}\tau^{-\frac{n+2}{4}} \leqslant \|A^{1/2}v\|_2^2 + C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}},
\end{aligned}
$$

where $u_0(t) = \mathrm{e}^{-(t-\tau)A}u(\tau)$. We thus obtain

$$\partial_t\|v\|_2^2 + \|A^{1/2}v\|_2^2 \leqslant C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}}.$$

Let $\{E_\lambda\}_{\lambda\geqslant 0}$ be the spectral measure associated to A. Since $\|A^{1/2}v\|_2^2 \geqslant \varrho(\|v\|_2^2 - \|E_\varrho v\|_2^2)$ for any $\varrho > 0$, the above estimate yields

$$\partial_t\|v\|_2^2 + \varrho\|v\|_2^2 \leqslant \varrho\|E_\varrho v\|_2^2 + C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}}.$$

But, $\|E_\varrho v\|_2^2 \leqslant C\varrho^{\frac{n+2}{2}}\Big(\int_\tau^t \|u\|_2^2\,\mathrm{d}s\Big)^2$ as shown in [3, 5]; so

$$\partial_t\|v\|_2^2 + \varrho\|v\|_2^2 \leqslant C\varrho^{\frac{n+4}{2}}\Big(\int_\tau^t \|u\|_2^2\,\mathrm{d}s\Big)^2 + C(t-\tau)^{-n-1}\tau^{-1-\frac{n}{2}}.$$

Here we set $\varrho = m/(t-\tau)$, $m > 0$, and multiply both sides by $(t-\tau)^m$, to obtain

$$\partial_t((t-\tau)^m\|v\|_2^2) \leqslant C_m(t-\tau)^{m-\frac{n}{2}-2}\Big(\int_\tau^t \|u\|_2^2\,\mathrm{d}s\Big)^2 + C(t-\tau)^{m-n-1}\tau^{-1-\frac{n}{2}}.$$

Now fix m so that $m > n/2+2$ and $m > n+1$, and integrate the above inequality, to get

$$\|v(t)\|_2^2 \leqslant C(t-\tau)^{-2-\frac{n}{2}}\int_\tau^t\Big(\int_\tau^s \|u\|_2^2\,\mathrm{d}\sigma\Big)^2\mathrm{d}s + C(t-\tau)^{-n}\tau^{-1-\frac{n}{2}}.$$

Inserting $\tau = t/2$ yields $v(t) = I_4$, so

$$t^{n+\frac{n}{2}}\|I_4\|_2^2 \leqslant Ct^{n-1}\Big(\int_{t/2}^\infty \|u\|_2^2\,\mathrm{d}s\Big)^2 + Ct^{-1} \leqslant Ct^{-1} \to 0$$

as $t\to\infty$. This proves (4.6).

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Authors' addresses: *Tetsuro Miyakawa*, Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan, e-mail: `miyakawa@math.kobe-u.ac.jp`; *Maria Elena Schonbek*, Department of Mathematics, University of California, Santa Cruz, CA 95064, USA, e-mail: `schonbek@math.ucsc.edu`.