SIMPLIFIED MODELS OF QUANTUM FLUIDS IN NUCLEAR PHYSICS

B. Ducomet, Bruyères-le-Châtel

Dedicated to Prof. J. Nečas on the occasion of his 70th birthday

Abstract. We revisit a hydrodynamical model, derived by Wong from Time-Dependent-Hartree-Fock approximation, to obtain a simplified version of nuclear matter. We obtain well-posed problems of Navier-Stokes-Poisson-Yukawa type, with some unusual features due to quantum aspects, for which one can prove local existence. In the case of a one-dimensional nuclear slab, we can prove a result of global existence, by using a formal analogy with some model of nonlinear "viscoelastic" rods.

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1. Introduction

A large amount of work has been devoted in recent years to the study of nuclear motions based on hydrodynamics. In particular, the results of Wong et al. [13] show that a description of hydrodynamic type can be coherently built from the Time-Dependent-Hartree-Fock approximation, leading to non-trivial effects including, for example, a partial description of spin and isospin waves, which are specific quantum effects. If one considers that in a number of realistic physical situations like exotic nuclei, with currently several hundred of particles, one has to solve the corresponding quantum many body problem, one rapidly realizes that a tremendous amount of computation is necessary to solve it numerically. Practically, such large computations cannot be handled, even on the largest computers available.

In this context, one easily understands the benefit one can get by using an average framework such that hydrodynamics: one can hope to investigate at low cost some collective global properties of nuclei by using a restricted set of degrees of freedom, despite of some delicate questions about the relevance of such an approach in particular physical situations [2].

In this spirit, we considered in [5] a 3d simplified hydrodynamical model of nuclear matter, which leads to a consistent compressible Navier-Stokes system, with some special features due to quantum effects, leading to interesting phase transitions phenomena.

As usual, to get qualitative information, a simple attractive situation to begin with is the one dimension geometry.

In fact, a number of results have been achieved, concerning the dynamics of monodimensional "nuclear slabs". Although degenerate, this geometry has been used to test a more complicated model, in the Time-Dependent-Hartree-Fock framework [1], [13].

The plan of the paper is the following: in Section 2, we describe briefly the derivation of the complete model, then in Section 3 we consider a simplified temperature-dependent model by using a truncated version of the Skyrme nuclear interaction, and a barotropic model corresponding to the zero-temperature situation, which is meaningful in the quantum context.

Then we study (Section 4) the 1d lagrangian model, where we concentrate on the zero temperature case for which we give a global existence result.

2. TDHF theory and fluid equations

To be reasonably self-contained, we briefly derive hydrodynamics from the TDHF theory, without presenting full details, sending the reader back to the standard references [6], [8].

Let us begin with the N-body Schrödinger problem for two-body local interactions between particles (nucleons) of identical masses:

$$(1)\qquad \begin{cases} \imath\hbar \frac{\partial \Psi}{\partial t} = \mathbf{H}\Psi, \\ \Psi(0) = \Psi_0, \end{cases}$$

where the wave function $\Psi(x,t)$, with $x = (x_1, \dots, x_N) \in \mathbb{R}^{3N}$, is suitably antisymmetrized to take into account the fermionic character of the nucleons, and the hamiltonian of the system is $\mathbf{H} = \mathbf{T}+\mathbf{V}$, where $\mathbf{T} = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\Delta_i$ is the kinetic energy ($\hbar$ is Planck's constant, m is the common mass of the particles and Δ_i is the Laplace operator with respect to the x_i variable) and $\mathbf{V} = \frac{1}{2}\sum_{i,j=1}^{N} v(x_i - x_j)$ is the potential contribution.

As we have recalled in the introduction, it has been soon recognized that the problem (1) was intractable when N is large[1] and that the good framework to deal with N-body quantum problems was the "second quantization" which reformulates the problem into an equivalent one, with the advantage of incorporating ab initio the statistics of particles, and of concentrating on some global quantities (the matrix densities) carrying the relevant information, thus avoiding the need for dealing directly with the complete many-particle wave function. Moreover, the resulting formulation leads directly to tractable problems, by using various approximation schemes well known in atomic, nuclear, or solid state physics, under the general name of Hartree, or Hartree-Fock methods.

In this way, one can obtain from (1) an evolution equation for global objects called *N-body density matrices*, and their associated kernels.

Let us consider

$$\mathcal{N}^{(N)}(x,x') = \Psi(x,t)\Psi^*(x',t), \tag{2}$$

where $*$ denotes the complex conjugate and $\Psi(x,t)$ is a solution of (1).

By differentiating this quantity with respect to time, one gets

$$\begin{aligned} &\imath\hbar\frac{\partial}{\partial t}\mathcal{N}^{(N)}(x,x') \\ &\quad = \Big\{ -\frac{\hbar^2}{2m}\sum_{i=1}^{N}(\Delta_i - \Delta_i') + \frac{1}{2}\sum_{i,j=1}^{N}(v(x_i - x_j) - v(x_i' - x_j'))\Big\}\mathcal{N}^{(N)}(x,x'), \end{aligned} \tag{3}$$

where $x = (x_1, \ldots, x_N)$ and $x' = (x_1', \ldots, x_N')$[2].

By integrating partially over the variables $(x_{s+1}, \ldots, x_N)$, one defines the *reduced density matrices* by

$$\begin{aligned} &\mathcal{N}^{(s)}(x_1 \ldots x_s; x_1' \ldots x_s') \\ &\quad = \frac{N!}{(N-s)!}\int \mathcal{N}^{(N)}(x_1 \ldots x_s y_{s+1} \ldots y_N; x_1' \ldots x_s' y_{s+1} \ldots y_N)\,\mathrm{d}y_{s+1} \ldots \mathrm{d}y_N, \end{aligned} \tag{4}$$

so, we get an infinite hierarchy of coupled equations for the family $\{\mathcal{N}^{(s)}(x_1 \ldots x_s; x_1' \ldots x_s')\}_{s=1\ldots N}$.

In particular, for $s = 1$, we obtain

$$\begin{aligned} \imath\hbar\frac{\partial}{\partial t}\mathcal{N}^{(1)}(x_1;x_1') = &-\frac{\hbar^2}{2m}(\Delta_1 - \Delta_1')\mathcal{N}^{(1)}(x_1;x_1') \\ &+ \int \mathrm{d}x_2 \lim_{x_2' \to x_2}(v(x_1,x_2) - v(x_1',x_2'))\mathcal{N}^{(2)}(x_1,x_2;x_1',x_2'). \end{aligned} \tag{5}$$

[1] There are frequently more than 200 particles in actual problems concerning exotic nuclei.
[2] We temporarily suppress the explicit time dependence.

In the TDHF theory, one assumes now that the many-body wave function Ψ is given, for each $t > 0$, by a Slater determinant built with the single-particle wave functions $\Phi_\alpha(x),\ \alpha = 1 \dots M$.

This strong assumption implies that $\mathcal{N}^{(1)}$ has the expression

$$\mathcal{N}^{(1)}(x; x') = \sum_{\alpha=1}^{M} \Phi_\alpha(x)\Phi_\alpha^*(x'),$$

where the summation holds on the above set of M occupied states.

The effect of this approximation is to produce an effective closure relation, by a factorization of the two-body density matrix

$$\mathcal{N}^{(2)}(x_1, x_2; x_1', x_2') = \mathcal{N}^{(1)}(x_1; x_1')\mathcal{N}^{(1)}(x_2; x_2') - \mathcal{N}^{(1)}(x_1; x_2')\mathcal{N}^{(1)}(x_2; x_1'),$$

and to limit the above hierarchy to one basic equation, for the one-body density matrix $N(x; x') := \mathcal{N}^{(1)}(x; x')$:

$$\begin{aligned} \imath\hbar\frac{\partial}{\partial t}N(x; x') &= -\frac{\hbar^2}{2m}(\Delta - \Delta')N(x; x') \\ &+ \int_{\mathbb{R}^3} [v(x-y) - v(x'-y)]\,[N(x; x')N(y; y) - N(x; y)N(y; x)]\ \mathrm{d}y, \end{aligned} \tag{6}$$

where we explicitly suppose the translation-invariance for the two-body interaction: $v(x, y) = v(x - y)$.

We obtain a nonlinear integro-differential equation in 7 variables (x, x', t).

Actually, our aim is not to solve (6) but, by using a suitable averaging, to derive fluid approximations for it.

The starting point of the derivation of fluid equations is the decomposition [8] of $\Phi_\alpha(x)$ into an intrinsic motion of the nucleons described by $\varphi_\alpha(x)$, and a collective motion of the nucleus described by a real phase factor $S_\alpha(x)$ and an energy time-dependent factor $\Omega(t)$, giving the following form for the single-particle wave function:

$$\Phi_\alpha(x) = \varphi_\alpha(x)e^{\frac{\imath m}{\hbar}S_\alpha(x) - \imath\Omega(t)}, \tag{7}$$

where m is the nucleon mass.

So, the one-body density matrix $N(x; x')$ reads

$$N(x; x') = \sum_\alpha \varphi_\alpha(x)\varphi_\alpha(x')e^{\frac{\imath m}{\hbar}(S_\alpha(x) - S_\alpha(x'))}, \tag{8}$$

with summation on the occupied states.

The fluid limit corresponds to a local theory, giving conservation laws by passing formally to the limit $x' \to x$. Performing this limiting process first in (6), we have, by using (8):

$$\lim_{x' \to x} (\Delta - \Delta') N(x; x') = \frac{2\imath m}{\hbar} \nabla \cdot \sum_\alpha \varphi_\alpha^2(x) \nabla S_\alpha(x).$$

If we denote by $n(x)$ the diagonal part of $N(x; x')$: $n(x) = N(x; x) = \sum_\alpha \varphi_\alpha^2(x)$, we can introduce an average procedure for any one-particle quantity $A(x) = \{A_\alpha(x)\}$ as follows:

$$\langle A(x) \rangle := \frac{1}{n(x)} \sum_\alpha \varphi_\alpha^2(x) A_\alpha(x).$$

So, calling the quantity $\mathbf{u} = \langle \nabla S \rangle$ *collective velocity*, we get the equation

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}) = 0, \tag{9}$$

which is the familiar continuity equation for the density in ordinary hydrodynamics.

To obtain the dynamical equation, we apply the operator $\nabla - \nabla'$ to (6) and pass to the limit $x' \to x$. We find

$$\begin{aligned} \frac{\partial}{\partial t}(n\mathbf{u}) + \nabla \cdot (n\,\mathbf{u} \otimes \mathbf{u}) &= -\frac{1}{m} \nabla \cdot \pi - \frac{1}{m} \sum_\alpha \varphi_\alpha^2(x) \nabla \Big(\frac{\varepsilon_\alpha}{\varphi_\alpha^2(x)} \Big) \\ &- \frac{1}{m}\, n(x) \nabla \int_{\mathbb{R}^3} v(x - x')\, n(x')\, \mathrm{d}x' + \frac{1}{m} \int_{\mathbb{R}^3} \nabla v(x - x')\, |N(x, x')|^2\, \mathrm{d}x', \end{aligned} \tag{10}$$

where

$$\varepsilon_\alpha = \varphi_\alpha \Big(-\frac{\hbar^2}{2m} \Delta \Big) \varphi_\alpha$$

is the single-particle (intrinsic) kinetic energy density, and π is the *kinetic stress tensor*, defined by

$$\pi_{ij} = m \sum_\alpha \varphi_\alpha^2(x) \Big(\frac{\partial}{\partial x_i} S_\alpha - u_i \Big) \Big(\frac{\partial}{\partial x_j} S_\alpha - u_j \Big). \tag{11}$$

To identify macroscopic quantities on the right hand side of (10), we consider a specific form for the effective two-body nuclear interaction:

$$v(x - x') = v_S(n(x))\, \delta(x - x') + v_Y(x - x') + \Big(\frac{Z}{A} \Big)^2 v_C(x - x'),$$

where v_S is the short range density dependent contribution, v_Y is an intermediate range (Yukawa) term, and v_C is the proton-proton Coulomb contribution, in which we take a simplified $\frac{Z}{A}$ factor between the proton and nucleon densities.

The first contribution will be included in the equation of state of the medium, the other two will contribute to the self-interacting force field.

Now [13], one assumes that the kinetic stress tensor π can be split into two pieces: a thermal contribution $\pi^{(t)}$ of perfect gas type, and a term $\pi^{(v)}$ modelling dissipative processes of viscous nature.

By gathering all of these contributions, one defines a global stress tensor σ, with

$$\sigma = \pi^{(q)} + \pi^{(i)} + \pi^{(t)} + \pi^{(v)}. \tag{12}$$

The explicit values considered for these terms depend on the physical context.

For the quantum term, one can take a Thomas-Fermi-Weisäcker approximation [8]

$$\pi^{(q)} = \Big[\frac{1}{5}\frac{\hbar^2}{2m}\Big(\frac{3\pi^2}{2}\Big)^{2/3} n^{5/3} + \frac{1}{6}\frac{\hbar^2}{18m}\Big(\frac{|\nabla n|}{n}\Big)^2 - \frac{2}{3}\frac{\hbar^2}{8m}\Delta n\Big]\mathbf{I}.$$

To simplify the exposition, we consider only the low order approximation $\pi^{(q)} = \frac{1}{5}\frac{\hbar^2}{2m}(\frac{3\pi^2}{2})^{2/3} n^{5/3}\mathbf{I}$.

We suppose that the time scale of the macroscopic motion is much greater than the microscopic relaxation time leading to a local equilibrium, then we can introduce a local temperature θ.

Moreover, at high temperature, nuclear matter is a Fermi gas and behaves like a perfect gas, so we take the simple law $\pi^{(t)} = nk\theta\mathbf{I}$, where k is the Boltzmann constant.

We have now to specify the kind of effective nuclear interaction we consider.

In fact, nuclear interaction in itself is not exactly known. It is actually a challenging problem in high energy physics, as its complete solution amounts to the understanding of Quantum Chromodynamics (QCD), considered to be a good theory of strong interactions. Moreover, the interaction between two nucleons is strongly affected by the environment of nuclear matter, which increases the difficulty of the problem.

To perform realistic computations, nuclear physicists have been led to introduce various kinds of effective interactions, of phenomenological character, taking into account the surrounding nuclear medium with slightly different properties, which has been compared favorably with experiments (see [8], [3]). In order to simplify the exposition we limit ourself in the following, to a "toy" interaction, used to describe infinite nuclear matter [9]: the Skyrme interaction for a spin-isospin saturated system

$$\pi^{(i)} = \Big(\frac{3}{8}t_0 n^2 + \frac{1}{8}t_3 n^3\Big)\mathbf{I},$$

where the first term is repulsive ($t_0 < 0$), while the second term is attractive ($t_3 > 0$).

The viscous stress contribution is taken as $\pi^{(v)} = 2\eta\mathbf{d} + \zeta\, Tr(\mathbf{d})\,\mathbf{I}$, where $\mathbf{d}$ is the strain tensor with entries $\mathbf{d}_{ij} = \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i}\right)$, where $Tr(\mathbf{d}) = \nabla\cdot\mathbf{u}$ is the trace of $\mathbf{d}$, and η and ζ are two viscosity coefficients, which ought to be fitted with experiments.

So, we finally get the expression

$$\sigma = -p\mathbf{I} + 2\eta\mathbf{d} + \zeta\, Tr(\mathbf{d})\,\mathbf{I},$$

where the (effective) pressure is defined by

$$p(n,\theta) = \frac{3}{8}t_0 n^2 + \frac{1}{8}t_3 n^3 + \frac{\hbar^2}{m}\frac{1}{5}\Big(\frac{3\pi^2}{2}\Big)^{2/3} n^{5/3} + nk\theta,$$

and we obtain the dynamical equation of motion

$$\frac{\partial}{\partial t}(n\mathbf{u}) + \nabla\cdot(n\,\mathbf{u}\otimes\mathbf{u}) = \frac{1}{m}\nabla\cdot\sigma - \frac{1}{m}\,n\nabla V_{YC}. \tag{13}$$

We notice that, apart from the pressure forces, there are supplementary terms coming from the intrinsic kinetic energy term together with the contribution from the nuclear interaction (δ density type and exchange).

To get a macroscopic equation for the energy, one performs a similar limit process, by applying the operator $\nabla\cdot\nabla'$ to (6) and then passing to the limit $x' \to x$.

If we define the specific internal energy e, we get finally after some algebra [13] the familiar equation

$$n\Big(\frac{\partial e}{\partial t} + \mathbf{u}\nabla e\Big) = \sigma\colon\, \mathbf{d} - \nabla\cdot\mathbf{Q}, \tag{14}$$

where $\sigma\colon\, \mathbf{d} = \sum_{ij}\sigma_{ij}\mathbf{d}_{ij}$ and $\mathbf{Q}$ is a phenomenonogical heat flux, a simple expression of which is $\mathbf{Q} = -\kappa(n,\theta)\nabla\theta$.

At this point we emphasize that, as in classical statistical physics, phenomenology appears here at three different places: the effective interaction, the viscous dissipation $\sigma^{(v)}$, and the thermal flux $\mathbf{Q}$. Unfortunately, the last two types are the less known, and probably the more difficult to get from collision experiments.

By a simple numerical renormalization, we can suppress the dependence in mass m and charge Z, and we consider the following set of equations for our model[3]:

$$\begin{cases} \frac{\partial n}{\partial t} + \nabla \cdot (n\,\mathbf{u}) = 0, \\ n\frac{D\mathbf{u}}{Dt} = \nabla \cdot \tau - \ n\ \nabla\Psi, \\ n\frac{De}{Dt} = \tau \colon\ \mathbf{d} + \nabla \cdot (\kappa \nabla T) \end{cases} \tag{15}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$ is the derivative along the vector field $\mathbf{u}$.

The stress tensor τ is the renormalized expression of σ given by $\tau = -p\mathbf{I} + \mathbf{D}$, where the (effective) pressure is defined by

$$p(n,T) = \frac{3}{8}t_0 n^2 + \frac{1}{8}t_3 n^3 + \frac{\hbar^2}{m}\frac{1}{5}\Big(\frac{3\pi^2}{2}\Big)^{2/3}\ n^{5/3} + nkT.$$

The total interaction potential is

$$\Psi(n) = \Psi^Y(n) + \Psi^C(n),$$

where the Yukawa term is

$$\Psi^Y(n) = \int_{\mathbb{R}^3} v_Y(x - x')\ n(x')\,\mathrm{d}x$$

with $v_Y(x,x') = g_Y \frac{e^{-\mu|x-x'|}}{|x-x'|}$, with the coupling constant g_Y and the mass scale μ, and the Coulomb term between protons is

$$\Psi^C(n) = \int_{\mathbb{R}^3} v_C(x - x')\ n(x')\,\mathrm{d}x,$$

with $v_C(x,x') = g_C \frac{1}{|x-x'|}$, and the coupling constant $g_C = \frac{e^2}{2}\left(\frac{Z}{A}\right)^2$ (e is the electron charge).

Now we find, by using standard thermodynamical identities, that the internal energy e is given by

$$e(n,\theta) = \frac{3}{8}t_0 n + \frac{1}{16}t_3 n^2 + \frac{3\hbar^2}{10m}\Big(\frac{3\pi^2}{2}\Big)^{2/3}\ n^{2/3} + k\theta.$$

In real problems, the parameters of the model are adjusted to reproduce some properties of nuclear matter[4], defining the Skyrme-Yukawa model [1].

[3] We renormalize the nucleon mass and charge ($m = 1$).

[4] In suitable units, one has $t_0 = -498\ MeV\ fm^3$, $t_3 = 17270\ MeV\ fm^6$, $g\mu = -167\ MeV\ fm$, $\mu = 0.46\ fm$, $\frac{\hbar^2}{m} = 41.5\ MeV\ fm^2$.

We consider, for a given initial configuration Ω_0 and for each x in Ω_0, the initial conditions

$$(n, \mathbf{u}, \theta)(x, 0) = (n_0, \mathbf{u}_0, \theta_0)(x). \tag{16}$$

Until this point, we have considered infinite nuclear matter. However, to take into account some finite size effects, we take, for each $t \geqslant 0$, the following dynamical (free) boundary conditions:

$$\begin{cases} (\tau + P_e \mathbf{I}) \cdot \mathbf{n} = 0 \\ \frac{D\psi}{Dt} = 0, \end{cases} \tag{17}$$

where P_e is an extra phenomenological pressure, modelling the surface effects, $\psi(x, t) = 0$ is the equation of the boundary $S_t = \partial\Omega_t$, and $\mathbf{n}$ is the exterior normal.

Finally, we put a thermal Neumann condition on S_t:

$$\nabla\theta \cdot \mathbf{n} = 0. \tag{18}$$

At this point, two points deserve to be emphasized:

1. As the presence of the Planck's constant $\hbar$ shows, some quantum effects are taken into account in the model, which can pretend to capture some features of the physical problem.
2. Due to the particular equation of state (including a negative term ($t_0 < 0$)), some unusual behaviour is expected, especially the asymptotic convergence toward a non trivial static state: in fact, the physics described by our simplified model shares some analogy with the Van der Waals liquid-gas transition.

In nuclear physics, the zero-temperature model is not trivial, as quantum fluctuations are present, and is meaningful to describe the ground state of nuclei and their low energy excitations. In this case, we discard the energy equation and we can also, in the first approach, neglect the (subdominant) Coulomb term.

So, we get the following system in the domain $\Omega_t = \mathbb{R}^3$:

$$\begin{cases} \frac{\partial n}{\partial t} + \nabla \cdot (n\, \mathbf{u}) = 0, \\ n\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) = \nabla \cdot \mathbf{T} - n\nabla\Psi, \end{cases} \tag{19}$$

where the stress tensor is $\mathbf{T}(n, s) = -P\mathbf{I} + \mathbf{D}$, the pressure is given by $P(n) = \frac{3}{8}t_0 n^2 + \frac{1}{8}t_3 n^3 + \frac{1}{5}\left(\frac{3\pi^2}{2}\right)^{2/3} \frac{\hbar^2}{m} n^{5/3}$, and the "self-interacting" potential is $\Psi^Y(n) = g_Y \int_{\Omega_t} \frac{e^{-\mu|x-x'|}}{|x-x'|}\, n(x')\, \mathrm{d}x$.

We also consider, for a given initial configuration Ω_0 and for each x in Ω_0, the initial conditions

$$(n, \mathbf{u})(x, 0) = (n_0, \mathbf{u}_0)(x). \tag{20}$$

We take, for each $t \geqslant 0$, the free dynamical boundary conditions

$$\begin{cases} (\mathbf{T} + P_e \mathbf{I}) \cdot \mathbf{n} = 0, \\ \dfrac{D\psi}{Dt} = 0, \end{cases} \tag{21}$$

where, as before, P_e is a positive pressure, $\psi(x, t) = 0$ is the equation of the boundary $S_t = \partial\Omega_t$, and $\mathbf{n}$ is the exterior normal.

By using the methods of [10], [11], [12], we can study [5] the global existence and some qualitative properties of the solution of the barotropic system (19), (20), (21).

4. The one-dimensional zero-temperature model

Let us consider the mixed "fixed-free" problem

$$\begin{cases} u_t - v_x = 0, \\ v_t = \sigma_x + \beta x, \end{cases} \tag{22}$$

with mixed boundary conditions

$$\begin{cases} v(0, t) = 0, \\ \sigma(M, t) = -P. \end{cases} \tag{23}$$

We are going to solve this problem by using a suitable regularization method, introduced by Kuttler and Hicks [7] to study some models of viscoelastic rods, allowing very mild hypotheses on the regularity of the data (see also [4] for related stability results).

By using the lagrangian expression of the spatial position $r(x, t)$ defined by

$$\begin{cases} r_x = u, \\ r_t = v, \end{cases} \tag{24}$$

one can transform the problem (22) into the following quasi-linear parabolic problem for the unknown $r(x, t)$, with nonlinear boundary condition:

$$\begin{cases} r_{tt} = \sigma(r_x)_x + g(x), \\ r(x, 0) = r_0(x), \\ r_t(x, 0) = r_1(x), \\ r(0, t) = 0, \\ \sigma(r_x)(M, t) = -P, \end{cases} \tag{25}$$

with $\sigma(r_x) = -p(r_x) + \nu_0 \frac{r_{tx}}{r_x}$, $g(x) = \beta x$, $r_0(x) = \int_0^x u_0(y)\,\mathrm{d}y$ and $r_1(x) = v_0(x)$.

One first checks easily the following

Lemma 1. *Let r be a solution of* (25) *of "positive density"* ($r_x > 0$).
If P is large enough, one has

$$\|r_t\|_{L^2(0,M)} \leqslant C,$$

where C does not depend on T.

As there is a possibility for the specific volume $u = r_x$ to be large, the quasilinear equation (25) can degenerate, so our first step is the investigation of the following regularized problem:

$$\begin{cases} \tilde{r}_{tt} = \tilde{\sigma}(\tilde{r}_x)_x + g(x), \\ (\tilde{r}, \tilde{r}_t)(x,0) = (u_0(x), u_1(x)), \\ \tilde{r}(0,t) = 0, \\ \tilde{\sigma}(\tilde{r}_x)(M,t) = -P \end{cases} \tag{26}$$

with $\tilde{\sigma}(w) = -\tilde{p}(w) + \tilde{\nu}(w) w_t$ and $g(x) = \beta x$.

We suppose here that $s \to \tilde{p}(s)$ is a bounded, non negative Lipschitz continuous function such that

$$\int_1^\infty \tilde{p}(s)\,\mathrm{d}s \leqslant C_1, \tag{27}$$

and that $s \to \tilde{\nu}(s)$ is a continuous function such that

$$0 < \delta \leqslant \tilde{\nu}(s) \leqslant C_2 \tag{28}$$

for some positive δ, C_1, C_2.

One checks the following elementary facts:

1. If we put
$$\widetilde{\mathcal{E}}(w) \equiv \int_1^w \tilde{p}(s)\,\mathrm{d}s,$$
by multiplying (25) with $r := \tilde{r}$ by $\tilde{r}_t$, integrating by parts, and using a Gronwall argument, one has
$$\|\tilde{r}_t(t,\cdot)\|_{L^2(0,M)} \leqslant C_2,$$
where
$$C_2 = \Big(2C_1 - 2\int_0^M \mathcal{E}\left((\tilde{r}_0)_x\right)\,\mathrm{d}x + \|r_1\|_H^2 + \frac{1}{3}\beta^2 M^3 T\Big)e^T.$$

2. Let us denote by E the space $E = \{u \in H^1(0,M) : \; u(0) = 0\}$.

Then the following weak formulation of (25) holds for each test function $\varphi \in C_0^\infty(0,T;E)$

$$
\begin{aligned}
\int_0^T \int_0^M \varphi_t \; \tilde r_t \, \mathrm{d}x \, \mathrm{d}t &= \int_0^T \int_0^M \varphi_x \; \widetilde{\mathcal V}(\tilde r_x) \, \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^T \int_0^M \varphi \; g \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_0^M \varphi_x \; \tilde p(\tilde r_x) \, \mathrm{d}x \, \mathrm{d}t + P \int_0^T \varphi(M,t) \, \mathrm{d}t.
\end{aligned} \tag{29}
$$

Finally, we put

$$\widetilde{\mathcal V}(w) \equiv \int_1^w \tilde \nu(s) \, \mathrm{d}s,$$

and assume that $\tilde p$ and $\tilde \nu$ “look like” the actual quantities p and ν, by satisfying the condition

$$\exists C_0 > 0, \; \exists P_0 > 0 : \; |P_0 - \tilde p(w)| \; \widetilde{\mathcal V}(s) > 0, \tag{30}$$

provided that $|\widetilde{\mathcal V}(w)| \geqslant C_0$.

Then we have the following direct adaptation of a result of [7]:

Theorem 1. *Let us denote by T an arbitrary positive number.*

1. *If*

$$\tilde r(\cdot,0) \in E, \; \tilde r_x(\cdot,0) \in L^\infty(0,M), \; \tilde r_t(\cdot,0) \in L^2(0,M),$$

and if $\tilde p$ and $\tilde \nu$ satisfy the condition (30), *then there exists a unique solution to* (26) *satisfying $\tilde r, \tilde r_t \in L^2(0,T;E)$.*

2. *If we put*

$$C_3 = \max\Big\{ \frac{\beta}{\sqrt 3} M^{3/2} T + P_0 T + C_2, \|r_1\|_H + \|\widetilde{\mathcal V}\left((\tilde r_0)_x\right)\|_\infty, C_0 \Big\},$$

then

$$\widetilde{\mathcal V}(\tilde r_x(t,x)) \leqslant 3C_3 \tag{31}$$

for each $t \in [0,T]$ and a.e. $x \in [0,M]$.

Of course, the actual state functions p and ν do not satisfy (27) and (28), so, in order to apply the above result to the problem (25), one first performs an “ε”-regularization on the actual problem; then one has just to prove uniform bounds for the specific volume, and finally pass to the limit $\varepsilon \to 0$. We skip the details.

Finally, by using the same regularization process, we can prove the global existence of a weak solution:

Theorem 2. *Let us denote by T an arbitrary positive number.*
Let us suppose now that r_0 and r_1 satisfy the conditions

$$\begin{cases} r_0 \in W^{1,\infty}(0,M) \cap E, \ 0 < \underline{r} \leqslant r_{0x} \leqslant \overline{r}, \ \text{a.e.}, \\ r_1 \in L^2(0,M). \end{cases} \tag{32}$$

Then there exists a unique weak solution $r \in L^2((0,T) \times (0,M))$, with $r_x, \ r_t, \ r_{tx} \in L^2((0,T) \times (0,M))$, such that for each test function $\varphi \in C_0^\infty(0,T;E)$ we have

$$\begin{aligned} \int_0^T \int_0^M \varphi_t \ r_t \, \mathrm{d}x \, \mathrm{d}t &= \int_0^T \int_0^M \varphi_x \ \nu_0 \log r_x \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^T \int_0^M \varphi \ g \, \mathrm{d}x \, \mathrm{d}t - \int_0^T \int_0^M \varphi_x \ p(r_x) \, \mathrm{d}x \, \mathrm{d}t + P \int_0^T \varphi(M,t) \, \mathrm{d}t. \end{aligned} \tag{33}$$

Moreover, there exist two constants u_m and u_M such that $0 < u_m \leqslant r_x(x,t) \leqslant u_M$.

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Author's address: *B. Ducomet*, CEA-Département de Physique Théorique et Appliquée, BP 12, 91680 Bruyères-le-Châtel, France, e-mail: `ducomet@bruyeres.cea fr`.