EXAMPLES FROM THE CALCULUS OF VARIATIONS IV. CONCLUDING REVIEW

Jan Chrastina, Brno

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Abstract. Variational integrals containing several functions of one independent variable subjected moreover to an underdetermined system of ordinary differential equations (the Lagrange problem) are investigated within a survey of examples. More systematical discussion of two crucial examples from Part I with help of the methods of Parts II and III is performed not excluding certain instructive subcases to manifest the significant role of generalized Poincaré-Cartan forms without undetermined multipliers. The classical Weierstrass-Hilbert theory is simulated to obtain sufficient extremality conditions. Unlike the previous parts, this article is adapted to the category of continuous objects and mappings without any substantial references to the general principles, which makes the exposition self-contained.

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Several rather profound mathematical theories can be best mastered only through a survey of examples: the classical moving frames, the topical solitons, and the recent derived categories may be (e.g.) stated in this connection. We follow the same strategy for analogous reasons.

In this eventually concluding Part IV, two of the simplest but typical Lagrange problems are recalled: first, the integral $\int f(x,u,v,\,\mathrm{d}u/\mathrm{d}x)\,\mathrm{d}x$ subjected to the constraint $\mathrm{d}v/\mathrm{d}x = g(x,u,v,\,\mathrm{d}u/\mathrm{d}x)$ and second, the integral $\int f(x,u,v,w,\,\mathrm{d}u/\mathrm{d}x,\,\mathrm{d}v/\mathrm{d}x)\,\mathrm{d}x$ constrained by $\mathrm{d}w/\mathrm{d}x = g(x,u,v,w,\,\mathrm{d}u/\mathrm{d}x,\,\mathrm{d}v/\mathrm{d}x)$. They were already discussed in Part I for the "generic case" and our aim is twofold: to involve some "exceptional subcases" and to rephrase the results in common terms of mathematical analysis. Concerning the first task, only quite elementary methods neglecting geometry (namely the equivalence theory) are applied but the results will not always follow by direct verification; the inevitable role of Poincaré-Cartan forms is transparently manifested. Concerning the second task, we will not already speak of smooth functions (hence of diffieties, which causes a certain lack of coherence) but employ the common continuous categories. (Further generalizations towards functional spaces and (e.g.) broken extremals are also possible but may be regarded as mere tedious technical impositions.) Our main task is to demonstrate a large diversity of the results and we do not aim for "universal recipes" of solution.

Three rather urgent and perspective topics are to be mentioned on this occasion. First, the investigation of "degenerate Lagrange problems" which should be defined by the property that the Lagrangian subspaces and the common Hamilton-Jacobi equations are insufficient to cope with the extremality properties. (It seems that appropriate generalizations, the "coisotropic subspaces", will be useful.) Second, the Mayer problem, which permits to involve the "extraordinary extremals" and to specify all "reasonable" boundary conditions and extremality functionals. (The Lagrange problem is included as a subcase with a certain specific structure.) Third, the generalization of our approach to the multiple variational integrals. (At the first place, introduction of generalized Poincaré-Cartan forms without uncertain multipliers for the multidimensional Lagrange problems.) We believe that essential achievements in all the directions mentioned are quite feasible and will be soon available.

First family of examples

1. Introduction. We recall the problem I 6 concerning the extremality properties of the integral $\int f(x, w_0^1, w_0^2, w_1^1)\,\mathrm{d}x$, where the variable functions $w_0^1 = w_0^1(x)$, $w_0^2 = w_0^2(x)$ are subjected to a differential constraint $w_1^2 = g(x, w_0^1, w_0^2, w_1^1)$. Recall that the lower indices denote the order of derivatives here (e.g., $w_1^i = \mathrm{d}w_0^i/\mathrm{d}x, w_2^i = \mathrm{d}^2 w_0^i/\mathrm{d}x^2$, and so on) which should not be confused with the abbreviation of partial derivatives of various functions (e.g., $f_0^i = \partial f/\partial w_0^i, f_{1x}^i = \partial^2 f/\partial w_1^i \partial x$, and so on). We suppose that f, g are real-valued functions in an open subdomain $\mathcal{D}(f, g) \subset \mathbb{R}^4$ and f, g, f_1^1, g_1^1 have the second order continuous derivatives therein. Let

$$\breve{\alpha} = f\,\mathrm{d}x + f_1^1\omega_0^1 + \frac{b}{a}\pi \qquad (\pi = \omega_0^2 - g_1^1\omega_0^1) \tag{1}$$

be the *Poincaré-Cartan* ($\mathcal{PC}$) *form* satisfying

$$\mathrm{d}\breve{\alpha} = \pi \wedge \xi + \Big(f_{11}^{11} - \frac{b}{a}g_{11}^{11}\Big)\omega_1^1 \wedge \omega_0^1 \quad \Big(\xi = e\,\mathrm{d}x + A\omega_0^1 - \Big(\frac{b}{a}\Big)_1^1\omega_1^1 - \Big(\frac{b}{a}\Big)_2^1\omega_2^1\Big) \tag{2}$$

where $\omega_r^i \equiv \mathrm{d}w_r^i - w_{r+1}^i\,\mathrm{d}x$ are the contact forms and

$$a = g_0^1 + g_0^2 g_1^1 - X g_1^1,\ b = f_0^1 + f_0^2 g_1^1 - X f_1^1, \tag{3}$$

$$e = f_0^2 - \frac{b}{a} g_0^2 - X\frac{b}{a},\ A = f_{10}^{12} - \frac{b}{a} g_{10}^{12} - \left(\frac{b}{a}\right)_0^1 - \left(\frac{b}{a}\right)_0^2 g_1^1, \tag{4}$$

$$X = \frac{\partial}{\partial x} + \sum w_{r+1}^1 \frac{\partial}{\partial w_r^1} + g\frac{\partial}{\partial w_0^2}.$$

We refer to Part I as concerns the true sense of these concepts, in particular $a \neq 0$ is supposed. Direct verification of (2–4) is quite easy. Recall that $e = 0$ stands for the *Euler-Lagrange* ($\mathcal{EL}$) equation.

We will see that the order of the $\mathcal{EL}$ equation determines the structure of our variational problem to a large extent. Clearly

$$e = -X\frac{b}{a} + \ldots = -\frac{Xb}{a} - bX\frac{1}{a} + \ldots = (af_{11}^{11} - bg_{11}^{11})w_3^1/a^2 + \ldots$$

as the top order terms are concerned. In principle, we may distinguish the subcases

(i) $e_3^1 \neq 0$ (hence $af_{11}^{11} \neq bg_{11}^{11}$), then the $\mathcal{EL}$ equation $e = 0$ can be equivalently represented by a condition $w_3^1 = E(x, w_0^1, w_0^2, w_1^1, w_2^1)$,

(ii) $e_3^1 = 0$ identically but $e_2^1 \neq 0$, then the $\mathcal{EL}$ equation can be rewritten as $w_2^1 = E(x, w_0^1, w_0^2, w_1^1)$,

(iii) $e_3^1 = e_2^1 = 0$ identically but $e_1^1 \neq 0$ with the $\mathcal{EL}$ equation rewritten as $w_1^1 = E(x, w_0^1, w_0^2)$,

(iv) $e = e(x, w_0^1, w_0^2)$ is free of all derivatives.

We shall discuss these possibilities separately.

Before passing to the main task, let us recall two kinds of curves appearing in this connection: the *admissible* ($\mathcal{A}$) *curves* satisfying the contact conditions $\omega_r^i \equiv 0$ (i.e., $w_{r+1}^i = \mathrm{d}w_r^i/\mathrm{d}x$) where the variables are related by the original differential constraints, e.g., $w_3^1 = g$, $w_4^1 = Xg, \ldots$ in our case) and the *critical* ($\mathcal{C}$) *curves* (alternatively: *extremals*) satisfying moreover the $\mathcal{EL}$ equations (e.g., $e = 0$, $Xe = 0, \ldots$, in our case). The order of derivatives under consideration will be specified only case by case and we do not try to obtain the best possible results.

2. Subcase (i). In accordance with the order of the function e, we will suppose the existence of continuous third order derivatives. Consequently, the $\mathcal{C}$-curves satisfy the system

$$\mathrm{d}w_0^1 - w_1^1\,\mathrm{d}x = \mathrm{d}w_1^1 - w_2^1\,\mathrm{d}x = \mathrm{d}w_2^1 - E\,\mathrm{d}x = \mathrm{d}w_1^2 - g\,\mathrm{d}x = 0. \tag{5}$$

Because of the additional variable w_2^1, we introduce an open subdomain $\mathcal{D}(E) \subset \mathcal{D}(f,g) \times \mathbb{R}$, the definition domain of E, with coordinates $x, w_0^1, w_0^2, w_1^1, w_2^1$. By a lucky accident, $\breve{\alpha}$ can be regarded as a differential form on $\mathcal{D}(E)$ and formula (2) can be accordingly adapted: (2_1) remains valid if ξ is replaced by

$$\bar{\xi} = A\omega_0^1 - \left(\frac{b}{a}\right)_1^1 \omega_1^1 - \left(\frac{b}{a}\right)_2^1 \bar{\omega}_2^1 \qquad (\bar{\omega}_2^1 = \mathrm{d}w_2^1 - E\,\mathrm{d}x). \tag{6}$$

We have introduced the restriction $e = 0$ (hence $w_3^1 = E$) which affects neither $\breve{\alpha}$ nor $\mathrm{d}\breve{\alpha}$. (In terms of Part I, $\breve{\alpha} = \mathbf{e}^*\breve{\alpha}$ is identified.)

We are interested in *Lagrangian subspaces* $\mathbf{l}\colon \mathbf{L} \subset \mathcal{D}(E)$, i.e., in subspaces of the maximal possible dimension satisfying the *Hamilton-Jacobi* ($\mathcal{HJ}$) *condition* $\mathbf{l}^*\,\mathrm{d}\breve{\alpha} = 0$. In order to determine the dimension of $\mathbf{L}$, recall that the module

$$\mathrm{Adj}\ \mathrm{d}\breve{\alpha} = \{Z\rfloor\,\mathrm{d}\alpha\} = \{\omega_0^1, \omega_1^1, \pi, \bar{\xi}\} = \{\omega_0^1, \omega_0^2, \omega_1^1, \bar{\omega}_2^1\}$$

has a local basis such that

$$\mathrm{Adj}\ \mathrm{d}\breve{\alpha} = \{\,\mathrm{d}u^1, \mathrm{d}v^1, \mathrm{d}u^2, \mathrm{d}v^2\}, \ \mathrm{d}\breve{\alpha} = \mathrm{d}u^1 \wedge \mathrm{d}v^1 + \mathrm{d}u^2 \wedge \mathrm{d}v^2,$$

consequently $\dim \mathbf{L} = 3$ since two interrelations between $u^1, \dots, v^2$ are needed to kill $\mathrm{d}\breve{\alpha}$ (cf. also I (4)). Passing to more details, let us deal with the case when x, w_0^1, w_0^2 may be chosen for coordinates on $\mathbf{L}$. Then the $\mathcal{HJ}$ condition locally reads $\mathbf{l}^*\breve{\alpha} = \mathrm{d}W$, where $W = W(x, w_0^1, w_0^2)$ is a certain function. In quite explicit terms

$$\mathbf{l}^*\Big(f - f_1^1 w_1^1 - \frac{b}{a}(w_1^2 - g_1^1 w_1^1)\Big) = W_x, \mathbf{l}^*\Big(f_1^1 - \frac{b}{a}g_1^1\Big) = W_0^1, \mathbf{l}^*\tfrac{b}{a} = W_0^2, \tag{7}$$

by using (1_1). One can then observe that

$$x, w_0^1, w_0^2, p = f_1^1 - \frac{b}{a}g_1^1, q = \frac{b}{a}$$

can be locally chosen for alternative coordinates on $\mathcal{D}(E)$. In terms of the *Hamilton function* H defined by

$$H(x, w_0^1, w_0^2, p, q) = -f + f_1^1 + w_1^1 + \frac{b}{a}(w_1^2 - g_1^1 w_1^1), \tag{8}$$

equations (7) lead to the *Hamilton-Jacobi equation*

$$W_x + H(x, w_0^1, w_0^2, W_0^1, W_0^2) = 0 \tag{9}$$

for the unknown function W. This is formally the equation (7_1), while the other equations $(7_{2,3})$ determine the embedding $\mathbf{L} \subset \mathcal{D}(E)$.

The equations (5) can also be written as $\mathrm{d}u^1 = \mathrm{d}u^2 = \mathrm{d}v^1 = \mathrm{d}v^2 = 0$ and it follows that every Lagrangian subspace is fibered by $\mathcal{C}$-curves. (Alternatively: $\mathcal{C}$-curves are Cauchy characteristics of the $\mathcal{HJ}$ equation.) In terms of the coordinates x, w_0^1, w_0^2 on $\mathbf{L}$, we have a generalization of the *Mayer fields* of extremals. At this stage, all ingredients of the Weierstrass-Hilbert method are at hand.

Let $W(x, w_0^1, w_0^2)$ be a solution of (9) on a simply connected open subdomain $\mathcal{D}(W) \subset \mathbb{R}^3$. Let $P(t) \in \mathbf{L}$ $(0 \leqslant t \leqslant 1)$ be a $\mathcal{C}$-curve embedded into the relevant Lagrangian subspace. (Alternatively: $P(t) \in \mathbf{L}$ is a segment of a Cauchy characteristic of the $\mathcal{HJ}$ equation.) Let moreover

$$Q(t) = (x(t), w_0^1(t), w_0^2(t), w_1^1(t), w_2^1(t)) \in \mathcal{D}(E), \quad 0 \leqslant t \leqslant 1$$

be an $\mathcal{A}$-curve such that the "projection into $\mathbf{L}$"

$$R(t) = (x(t), w_0^1(t), w_0^2(t), r_1^1(t), r_2^1(t)) \in \mathbf{L}, \quad 0 \leqslant t \leqslant 1$$

is defined. (Recall that the components r_1^1, r_2^1 are determined by $(7_{2,3})$. This is the case when $(x(t), w_0^1(t), w_0^2(t)) \in \mathcal{D}(W)$, $0 \leqslant t \leqslant 1$.) Assuming moreover the "fixed ends" conditions

$$x(P(t)) = x(Q(t)), w_0^i(P(t)) = w_0^i(Q(t)) \qquad (t = 0, 1;\ i = 1, 2)$$

for the simplicity of exposition, we have

$$\int_Q \alpha - \int_P \alpha = \Bigl(\int_Q \alpha - \int_R \breve{\alpha}\Bigr) + \Bigl(\int_R \breve{\alpha} - \int_P \breve{\alpha}\Bigr). \tag{10}$$

The last summand vanishes due to Green's theorem (R and P make a loop in $\mathbf{L}$ where $\mathrm{d}\breve{\alpha} = 0$) and the middle term is equal to the integral $\int_0^1 \mathcal{E} x'(t)\,\mathrm{d}t$ where

$$\begin{aligned}\mathcal{E} = {}& f(\ldots, w_1^1) - f(\ldots, r_1^1) - f_1^1(\ldots, r_1^1)(w_1^1 - r_1^1)\\ &- \frac{b(\ldots, r_1^1, r_2^1)}{a(\ldots, r_1^1, r_2^1)}(g(\ldots, w_1^1) - g(\ldots, r_1^1) - g_1^1(\ldots, r_1^1)(w_1^1 - r_1^1))\end{aligned}$$

$(\ldots = x, w_0^1, w_0^2$ are parameters) is the *Weierstrass function*.

O b s e r v a t i o n. The inequalities $x' \geqslant 0$, $\mathcal{E} \geqslant 0$ ensure the minimum. The introduction of a continuous category does not cause many formal changes of the previous "smooth approach" of Part I but the sense of the constructions becomes a little obscure.

3. Subcase (ii). Let us suppose the (equivalent) identities

$$af^{11}_{11} = bg^{11}_{11},\ \left(\frac{b}{a}\right)^1_2 = 0,\ e^1_3 = 0 \tag{11}$$

are true, hence (2) simplifies to

$$\mathrm{d}\breve{\alpha} = \pi \wedge \xi \qquad \left(\xi = e\,\mathrm{d}x + A\omega^1_0 - \left(\frac{b}{a}\right)^1_1 \omega^1_1\right). \tag{12}$$

Then the congruence

$$0 = \mathrm{d}^2\breve{\alpha} \cong \pi \wedge \left(e^1_2\omega^1_2 \wedge \mathrm{d}x - \left(\frac{b}{a}\right)^1_1 \mathrm{d}x \wedge \omega^1_2\right) \qquad (\mathrm{mod}\,\omega^1_0, \omega^1_1)$$

implies the identity

$$e^1_2 = -\left(\frac{b}{a}\right)^1_1.$$

Let us assume $e^1_2 \neq 0$, hence $(b/a)^1_1 \neq 0$ in this section.

By lucky accident, $\breve{\alpha}$ can be expressed in terms of variables x, w^1_0, w^2_0, w^1_1. The restriction $e = 0$ (hence $w^1_2 = E$) does not affect $\breve{\alpha}$, therefore (12) remains true if ξ is replaced by

$$\bar{\xi} = A\omega^1_0 - (b/a)^1_1\bar{\omega}^1_1 \qquad (\bar{\omega}^1_1 = \mathrm{d}w^1_1 - E\,\mathrm{d}x).$$

Clearly Adj $\mathrm{d}\breve{\alpha} = \{\pi, \bar{\xi}\}$. It follows that $\mathrm{d}\pi \cong 0$ (mod $\pi, \bar{\xi}$), however,

$$\begin{aligned}\mathrm{d}\pi &= \mathrm{d}(\,\mathrm{d}w^2_0 - g\,\mathrm{d}x - g^1_1\omega^1_0) = -\,\mathrm{d}g \wedge \mathrm{d}x - g^1_1\,\mathrm{d}x \wedge \bar{\omega}^1_1 - \mathrm{d}g^1_1 \wedge \omega^1_0 \\ &\cong -(g^1_0 + g^2_0 g^1_1)\omega^1_0 \wedge \mathrm{d}x - Xg^1_1\,\mathrm{d}x \wedge \omega^1_0 = a\,\mathrm{d}x \wedge \omega^1_0 \quad (\mathrm{mod}\,\pi, \bar{\xi})\end{aligned}$$

by direct calculation. So we have the contradiction $a = 0$.

Observation. The subcase (ii) never occurs since (11) and $e = 0$ together imply $a = 0$. Some formally quite reasonable possibilities cannot be in fact realized.

4. Subcase (iii). Let us suppose (11) and $e^1_2 = 0$ hold true, hence $(b/a)^1_1 = 0$. Then $\mathrm{d}\breve{\alpha} = \pi \wedge (e\,\mathrm{d}x + a\omega^1_0)$ and the identity $\mathrm{d}^2\breve{\alpha} = 0$ implies

$$A = e^1_1, eg^{11}_{11} = 0 \tag{13}$$

by simple calculation. We will assume $e^1_1 \neq 0$, therefore $g^{11}_{11} = f^{11}_{11} = 0$ in virtue of $(13_2, 11_1)$. It follows that

$$f = Mw^1_1 + N, g = Kw^1_1 + L \tag{14}$$

with certain coefficients depending on x, w_0^1, w_0^2. Conversely, assuming (14), one can find

$$a = L_0^1 + L_0^2 K - K_x - K_0^2 L,\; b = N_0^1 + N_0^2 K - M_x - M_0^2 L \tag{15}$$

independent of w_1^1, hence $(b/a)_1^1 = 0$ is realized. One can observe that the inequalities

$$a \neq 0,\; e_1^1 = A = M_0^2 - b/aK_0^2 - (b/a)_0^1 - (b/a)_0^2 \neq 0 \tag{16}$$

can be realized, too, even with $K = 0$ (which includes the "peculiar" Lagrange problem I 7 where moreover $M = 0$ identically).

Instead of the domain $\mathcal{D}(f, g)$, we may employ the common definition domain $\mathcal{D}(M, N, K, L) \subset \mathbb{R}^3$ of our coefficients. The $\mathcal{A}$-curves will have first order continuous derivatives and satisfy $\mathrm{d}w_0^2 - g\,\mathrm{d}x = 0$, the $\mathcal{C}$-curves satisfy moreover $\mathrm{d}w_0^1 = E\,\mathrm{d}x$, by definition.

By using (14), the $\mathcal{PC}$ form (1) simplifies to

$$\breve{\alpha} = N\,\mathrm{d}x + M\,\mathrm{d}w_0^1 + \frac{b}{a}\pi \qquad (\pi = \mathrm{d}w_0^2 - K\,\mathrm{d}w_0^1 - L\,\mathrm{d}x). \tag{17}$$

This is a form on $\mathcal{D}(M, \dots, L)$. Inserting $e = 0$ (hence $w_1^1 = E$) into (12), we obtain

$$\mathrm{d}\breve{\alpha} = A\varpi_0^2 \wedge \varpi_0^1 \qquad (\varpi_0^1 = \mathrm{d}w_0^1 - E\,\mathrm{d}x, \varpi_0^2 = \mathrm{d}w_0^2 - (KE + L)\,\mathrm{d}x).$$

It follows that there exist two-dimensional Lagrangian subspaces $\mathbf{l}\colon \mathbf{L} \subset \mathcal{D}(M, \dots, L)$, however, one can observe that the Weierstrass-Hilbert method cannot be directly applied.

In more detail: the decomposition (10) leads to a reasonable Weierstrass function after certain adaptations. First, there exist (local) coordinates $\tilde{x}, \tilde{w}_0^1, \tilde{w}_1^1$ such that π is a multiple of the form $\mathrm{d}\tilde{w}_0^1 - \tilde{w}_1^1\,\mathrm{d}\tilde{x}$, second, then $\breve{\alpha} \cong \tilde{f}\,\mathrm{d}\tilde{x} + \mathrm{d}\tilde{g} \pmod{\pi}$ for appropriate functions $\tilde{f}$ and $\tilde{g}$, third, the form $\tilde{f}\,\mathrm{d}\tilde{x}$ may replace the original form $f\,\mathrm{d}x$ as the extremality properties are concerned. Altogether, we have the variational integral $\int \tilde{f}(\tilde{x}, \tilde{w}_0^1, \tilde{w}_1^1)\,\mathrm{d}\tilde{x}$ constrained by the mere contact form $\mathrm{d}\tilde{w}_0^1 - \tilde{w}_1^1\,\mathrm{d}\tilde{x} = 0$. This is the simplest classical problem II (1) with $m = 1$. Unfortunately, the above adaptations can be explicitly performed only in some favourable cases. For instance, if $K = 0$ identically, then the choice $\tilde{x} = x$, $\tilde{w}_0^1 = w_0^2$, $\tilde{w}_1^1 = L$ is possible.

To obtain more explicit results, we may apply the coordinates-free reformulation of the Legendre and Jacobi criteria, the "rotation principle" of Part III. The vector fields

$$X = \frac{\partial}{\partial x} + E\frac{\partial}{\partial w_0^1} + (KE + L)\frac{\partial}{\partial w_0^2},\; Y = \frac{\partial}{\partial w_0^1} + K\frac{\partial}{\partial w_0^2}$$

can be employed, see III (9_1) and II ($9_{2,3}$) with $\Theta = \{\pi\}$. Then we obtain the Legendre condition $\mathrm{d}\breve{\alpha}([X,Y],Y) = -Aa > 0$ for the minimum. Concerning the Jacobi condition, the variations $Z = v\partial w_0^1 + w\partial/\partial w_0^2$ of a given $\mathcal{C}$-curve are determined by the linear system

$$\frac{\mathrm{d}v}{\mathrm{d}x} = E_0^1 v + E_0^2 w,\ \frac{\mathrm{d}w}{\mathrm{d}x} - K\frac{\mathrm{d}v}{\mathrm{d}x} = (K_0^1 E + L_0^1)v + (K_0^2 E + L_0^2)w.$$

There should be a solution satisfying $\pi(Z) = w + Kv \neq 0$.

O b s e r v a t i o n. Geometrical reformulations of the classical analytical results and the equivalence theory of variational problems deserve more attention even from the point of view of very earthy applications.

5. Subcase (iv). If $e = 0$ identically, then $\breve{\alpha}$ is a total differential (see I 5 (vi)) and we omit comments. Therefore, let us assume $e \neq$ const., $\mathrm{d}e \neq 0$ (but $e_1^1 = e_2^1 = e_3^1 = 0$). We may again suppose (14) with $a \neq 0$ but $A = e_1^1 = 0$ identically, hence $\mathrm{d}\breve{\alpha} = e\pi \wedge \mathrm{d}x$, see either (2, 12) and (16). We find ourselves in the space $\mathcal{D}(M,\dots,L)$ with $\mathcal{A}$-curves $Q(t)$ satisfying $Q^*\pi = 0$ (cf. (17_2) for the form π) and $\mathcal{C}$-curves $P(t)$ such that moreover $P^*e = 0$, hence $P^*\,\mathrm{d}e = 0$. It follows that π is a multiple of the form $\mathrm{d}e$ at every point $P(t)$, explicitly $\pi = \mathcal{E}(t)\,\mathrm{d}e$. If the $\mathcal{C}$-curve $P(t)$, $0 \leqslant t \leqslant 1$, and an $\mathcal{A}$-curve $Q(t)$, $0 \leqslant t \leqslant 1$, constitute a loop, then

$$\int_Q \alpha - \int_P \alpha = \int_Q \breve{\alpha} - \int_P \breve{\alpha} = \iint \mathrm{d}\breve{\alpha} = \iint e\pi \wedge \mathrm{d}x.$$

If moreover $Q(t)$ is near enough to $P(t)$, the factor π can be approximated by $\mathcal{E}(t)\,\mathrm{d}e$. Then, assuming (e.g.) $\mathcal{E}(t) > 0$ ($0 \leqslant t \leqslant 1$), the last double integral is nonnegative.

O b s e r v a t i o n. We have a “Weierstrass-like” function $\mathcal{E}(t)$ along the $\mathcal{C}$-curve $P(t)$, $0 \leqslant t \leqslant 1$, ensuring only a local extremum. A somewhat paradoxically, the seemingly simplest problems may cause many difficulties.

Second family of examples

6. Introduction. Let us return to the integral $\int f(x, w_0^1, w_0^2, w_0^3, w_1^1, w_1^2)\,\mathrm{d}x$ constrained by $w_1^3 = g(x, w_0^1, w_0^2, w_0^3, w_1^1, w_1^2)$, see the first half of Section I 8. We suppose that f, g are real-valued functions in an open subdomain $\mathcal{D}(f,g) \subset \mathbb{R}^6$ and $f, g, f_1^1, f_1^2, g_1^1, g_1^2$ have continuous second order derivatives therein. Then the $\mathcal{PC}$ form

$$\breve{\alpha} = f\,\mathrm{d}x + f_1^1\omega_0^1 + f_1^2\omega_0^2 + c\pi \qquad (\pi = \omega_0^3 - g_1^1\omega_0^1 - g_1^2\omega_0^2) \tag{18}$$

satisfies the congruence

$$d\breve{\alpha} \cong (e^1\pi + e^2\omega_0^1) \wedge dx + \sum c_r^i\omega_r^i \wedge \pi + (f_{11}^{ij} - cg_{11}^{ij})\omega_1^i \wedge \omega_0^j \tag{19}$$

$(i, j, r = 1, 2)$ modulo all products $\omega_0^i \wedge \omega_0^j, \omega_0^i \wedge \pi$, where

$$c = b^2/a^2, a^j \equiv g_0^j + g_1^j g_0^3 - Xg_1^j, b_j \equiv f_0^j + g_1^j f_0^3 - Xf_1^j \tag{20}$$

$$e^1 = f_0^3 - cg_0^3 - Xc, e^2 = b^1 - ca^1 \tag{21}$$

(we suppose $a^2 \neq 0$) with the operator

$$X = \frac{\partial}{\partial x} + \sum w_{s+1}^i \frac{\partial}{\partial w_s^i} + g\frac{\partial}{\partial w_0^3} \quad (i = 1, 2).$$

Formulae (19–21) can be directly verified. Recall that $e^1 = e^2 = 0$ stand for the $\mathcal{EL}$ system. We will again distinguish between $\mathcal{A}$-curves satisfying the contact conditions and $\mathcal{C}$-curves (or: *extremals*) that moreover fulfil the $\mathcal{EL}$ system.

Unlike the previous family of examples, the order of derivatives effectively occurring in the functions e^1, e^2 is insufficient to determine the structure of the problem. For lack of space, we cannot state a complete classification but restrict ourselves only to a few instructive subcases.

7. The generic subcase. Expansions of the kind

$$dF = XF\,dx + \sum(F_0^i + F_0^3 g_1^i)\omega_0^i + F_0^3\pi + \sum F_r^j\omega_r^j \tag{22}$$

$(i = 1, 2;\ j = 1, 2, 3;\ r = 0, 1, \ldots)$ for the functions $F = e^1, e^2, c$ together with the congruence

$$d\pi \cong dx \wedge (a^1\omega_0^1 + a^2\omega_0^2 + g_0^3\pi) - \sum g_{11}^{ij}\omega_1^i \wedge \omega_0^j \tag{23}$$

inserted into the identity $d^2\breve{\alpha} = 0$ yield useful formulae

$$e^{1i}{}_3 + c_2^i \equiv 0,\ e^{1i}{}_2 + c_1^i \equiv c_2^i g_0^3 - Xc_2^i \tag{24}$$

$$e^{2i}{}_2 \equiv c_2^i a^1 - f_{11}^{i1} + cg_{11}^{i1}, c_2^i a^2 - f_{11}^{i2} + cg_{11}^{i2} \equiv 0 \tag{25}$$

by comparison of the coefficients of summands $\omega_3^i \wedge \pi \wedge dx, \omega_2^i \wedge \pi\,dx, \omega_2^i \wedge \omega_0^j \wedge dx$ $(i, j = 1, 2)$. It follows that e^2 is of the second order if and only if the couple of functions

$$(e^{21}{}_2, e^{22}{}_2) = (c_2^1 a^1 - f_{11}^{11} + cg_{11}^{11}, c_2^2 a^1 - f_{11}^{12} + cg_{11}^{12}) \tag{26}$$

is nonvanishing. Then the third order summands in e^1 and Xe^2 are linearly independent if and only if the couple

$$(27) \qquad (e^{11}_{\ 3}, e^{12}_{\ 3}) = (-c^1_2, -c^2_2)$$

is not a multiple of the couple $((Xe^2)^1_3, (Xe^2)^2_3) = (e^{21}_{\ 2}, e^{22}_{\ 2})$, i.e., of the couple (26). Using (25_2), one can verify that both conditions are satisfied if and only if

$$(28) \qquad \det(f^{ij}_{11} - cg^{ij}_{11}) \neq 0.$$

Assuming moreover $e^{22}_{\ 2} \neq 0$ (which is a technical provision) for better clarity, the equation $e^2 = 0$ can be adapted as $w^2_2 = \dots$ (the variable w^2_2 is separated on the left), hence the system $e^2 = Xe^2 = e^1 = 0$ can be equivalently rewritten as

$$w^2_2 = E, w^2_3 = F, w^1_3 = G \qquad (F = XE),$$

where E, F, G are certain functions of x, w^1_0, w^2_0, w^3_0, w^1_1, w^2_1, w^1_2. Our reasoning will be carried out in the definition domain $\mathcal{D}(E,G) \subset \mathbb{R}^7$ of these functions E, G.

In accordance with such assumptions, we shall suppose the existence of the third order continuous derivatives for all $\mathcal{A}$-curves. By definition, they satisfy the system

$$\mathrm{d}w^1_0 - w^1_1\,\mathrm{d}x = \mathrm{d}w^1_1 - w^1_2\,\mathrm{d}x = \mathrm{d}w^2_0 - w^2_1\,\mathrm{d}x = \mathrm{d}w^3_0 - g\,\mathrm{d}x = 0$$

on the domain $\mathcal{D}(E,G)$. The additional equations

$$\mathrm{d}w^1_2 - G\,\mathrm{d}x = \mathrm{d}w^2_1 - E\,\mathrm{d}x = \mathrm{d}w^2_2 - F\,\mathrm{d}x = 0$$

define the $\mathcal{C}$-curves.

Concerning the Weierstrass-Hilbert method, we may restrict ourselves to a few brief notes, see also I 8. Since Adj $\mathrm{d}\breve{\alpha}$ is a six-dimensional module there exist four-dimensional Lagrangian subspaces $\mathbf{l}\colon \mathbf{L} \subset \mathcal{D}(E,G)$. They satisfy the $\mathcal{HJ}$ condition $\mathbf{l}^*\,\mathrm{d}\breve{\alpha} = 0$. If x, w^1_0, w^2_0, w^3_0 may be taken for coordinates on $\mathbf{L}$, this condition locally reads $\mathbf{l}^*\breve{\alpha} = \mathrm{d}W$ where $W = W(x, w^1_0, w^2_0, w^3_0)$ is an unknown function. In more detail,

$$(29) \qquad \mathbf{l}^*\Big(f - cg - \sum(f^i_j - cg^i_1)w^i_1\Big) = W_x, \quad \mathbf{l}^*(f^i_1 - cg^i_1) = W^i_0, \mathbf{l}^*c = W^3_0$$

($i = 1, 2$) and alternatively, in terms of the new local coordinates

$$(30) \qquad x, w^1_0, w^2_0, w^3_0, p = f^1_1 - cg^1_1, q = f^2_1 - cg^2_1, r = c$$

and of the Hamilton function defined by

$$H(x,w_0^1,w_0^2,w_0^3,p,q,r)=-f+cg+\sum(f_1^i-cg_1^i)w_1^i,$$

we obtain the $\mathcal{HJ}$ equation $W_x+H(\dots,W_0^1,W_0^2,W_0^3)=0$ and the embedding equations $p=W_0^1$, $q=W_0^2$, $r=W_0^3$ for the subspace $\mathbf{L}\subset\mathcal{D}(E,G)$. The decomposition (10) for the "fixed endpoints" leading to the Weierstrass function

$$\begin{aligned}\mathcal{E}=f(\dots,w_1^1,w_1^2)-f(\dots,r_1^1,r_1^2)-\sum f_1^i(\dots,r_1^1,r_1^2)(w_1^i-r_1^i)\\ -c(\dots,r_1^1,r_1^2,r_2^1)\Big(g(\dots,w_1^1,w_1^2)-g(\dots,r_1^1,r_1^2)\\ -\sum g_1^i(\dots,r_1^1,r_1^2)(w_1^i-r_1^i)\Big)\end{aligned}\tag{31}$$

does not require any comment.

Observation. All "generic cases" of our extremality problems admit a uniform approach and the sufficient extremality conditions resemble the classical results for the case of trivial constraints.

8. A strange nondegenerate subcase. Let us suppose that e^1 is of the second order at most. In virtue of (24_1) this is expressed by $c_1^1=c_2^2=0$ or, more explicitly, by

$$f_{11}^{i2}\equiv cg_{11}^{i2}(i=1,2),\ B^2=cA^2\tag{32}$$

where the expressions

$$A^i\equiv g_0^i+g_1^ig_0^3-g_{1x}^i-\sum w_1^jg_{10}^{ij},\ B^i\equiv f_0^i+g_1^if_0^3-f_{1x}^i-\sum w_1^jf_{10}^{ij}$$

involve all lower order terms of the coefficients a^i and b^i, respectively. Then $(24_2,25)$ simplify to

$$e_{\ 2}^{11}=-c_1^1,e_{\ 2}^{12}=-c_1^2,e_{\ 2}^{21}=-f_{11}^{11}+cg_{11}^{11},e_{\ 2}^{22}=0.\tag{33}$$

We will suppose $e_{\ 2}^{12}\neq0$ and $e_{\ 2}^{21}\neq0$, hence

$$c_1^2\neq0,\ f_{11}^{11}\neq cg_{11}^{11}\tag{34}$$

in this section. Then the $\mathcal{EL}$ system $e^1=e^2=0$ can be adapted as

$$w_2^1=E,\ w_2^2=F,\tag{35}$$

where E, F are certain functions of $x, w_0^1, w_0^3, w_1^1, w_1^2$. Our reasonings will be carried out in the relevant definition domain $\mathcal{D}(E,F) \subset \mathbb{R}^6$. (The requirements (32, 33) can be fulfilled; e.g., (32_1) is satisfied if $f_1^2 = G(g_1^2)$ and then (32, 33) simplify accordingly.)

The $\mathcal{A}$-curves have the second order continuous derivatives and satisfy the system

$$\mathrm{d}w_0^1 - w_1^1\,\mathrm{d}x = \mathrm{d}w_0^2 - w_1^2\,\mathrm{d}x = \mathrm{d}w_0^3 - g\,\mathrm{d}x = 0, \tag{36}$$

by definition. Then, moreover, $\mathrm{d}w_1^1 - E\,\mathrm{d}x = \mathrm{d}w_1^2 - F\,\mathrm{d}x = 0$ holds for the $\mathcal{C}$-curves.

The $\mathcal{PC}$ form (18) may be regarded as a form on the space $\mathcal{D}(E,F)$. Owing to (32), Adj $\mathrm{d}\breve{\alpha}$ is a four-dimensional module and there are four-dimensional Lagrangian subspaces $\mathbf{l}$: $\mathbf{L} \subset \mathcal{D}(E,F)$. If a given $\mathcal{C}$-curve $P(t)$ can be embedded into such a subspace equipped with coordinates x, w_0^1, w_0^2, w_0^3 ($P(t)$ is embedded into a Mayer field of extremals) then the decomposition analogous to (10) can be employed to obtain the already known Weierstrass function (31) which resolves the extremality problem. (The only change in our case is that the coefficient c in (31) does not depend on the variable r_2^1.)

Concerning the determination of the Lagrangian subspace $\mathbf{L} \subset \mathcal{D}(E,F)$, the $\mathcal{HJ}$ conditions (29) do not formally change but, instead of functions (30), already the restricted family

$$x, w_0^1, w_0^2, w_0^3, p = f_1^1 - cg_1^1, q = f_1^2 - cg_1^2 \tag{37}$$

is good enough for the alternative local coordinates on $\mathcal{D}(E,F)$. (Inequalities (31) ensure a nonvanishing Jacobian.) It follows that the functions

$$H = -f + cg + \sum (f_1^i - cg_1^i) w_1^i,\ K = c \tag{38}$$

can be expressed in terms of new coordinates and we obtain the *involutive system*

$$W_x + H(\ldots, W_0^1, W_0^2) = 0,\ W_0^3 = K(\ldots, W_0^1, W_0^2) \tag{39}$$

determining the Lagrangian subspace.

O b s e r v a t i o n. The $\mathcal{HJ}$ conditions may be occasionally expressed by an involutive system (instead of the common single equation) but this strange fact need not affect the Weierstrass-Hilbert method.

9. A simple degenerate subcase. Let us again suppose e^1 to be of the second order at most, hence conditions (32) are true. However, unlike Section 8, let $e_2^{12} = -c_1^2 = 0$ be identically vanishing, see (33_2).

Omitting the case when $g^{i2}_{11} = f^{i2}_{11} \equiv 0$ identically, (32_1) ensures the dependence $f^2_1 = F(x, w^1_0, w^2_0, w^3_0, g^2_1)$ which implies $c = \partial F/\partial g^2_1$, hence $c^2_1 = g^{22}_{11}\partial^2 F/(\partial g^2_1)^2 = 0$. This implies the linearity $F = Kg^2_1 + L$ and therefore

$$f = K(\ldots)g + L(\ldots)w^2_1 + M(\ldots, w^1_1) \quad (\ldots = x, w^1_0, w^2_0, w^3_0).$$

It follows easily that $K = c$ and

$$\breve{\alpha} = M\,\mathrm{d}x + M^1_1\omega^1_0 + L\,\mathrm{d}w^2_0 + K\,\mathrm{d}w^3_0,$$
$$e^1 = (L^3_0 - K^2_0)w^2_1 - K^1_0 w^1_1 - K_x + M^3_0,\ e^2 = -M^{11}_{11}w^1_2 + \ldots.$$

The remaining condition (32_2) reads $e^1 g^1_2 - (L^3_0 - K^2_0)g = L_x + w^1_1 L^1_0 - M^2_0$ and may be regarded as a differential equation for the function g.

At this place, we take a *technical measure*: let us suppose $K = 0$, $M = M(w^1_1)$ in order to clarify the following construction, moreover let $L^3_0 \neq 0$, $M'' \neq 0$ $(' = \mathrm{d}/\mathrm{d}w^1_1)$ for certainty. Recalling the simplified data, we may state the formulae

$$f = L(\ldots)w^2_1 + M(w^1_1), \breve{\alpha} = M\,\mathrm{d}x + M'\omega^1_0 + L\,\mathrm{d}w^2_0,$$

$$e^1 = L^3_0 w^2_1, e^2 = -M'' w^1_2 + (L^1_0 + g^1_1 L^2_0)w^2_1, \tag{40}$$

$$g = -G + F(\ldots, w^1_1)w^2_1 \quad \Big(G = \frac{L_x + w^1_1 L^1_0}{L^3_0}\Big). \tag{41}$$

This choice of g with an arbitrary function F ensures the validity of condition (32_2).

We will suppose that the functions L, M, M' have second order derivatives in an open domain $\mathcal{D}(L) \times \mathcal{D}(M) \times \mathbb{R} \subset \mathbb{R}^6$ of variables $x, w^1_0, w^2_0, w^3_0, w^1_1, w^2_1$. The $\mathcal{A}$-curves satisfy the system

$$\mathrm{d}w^1_0 - w^1_1\,\mathrm{d}x = \mathrm{d}w^2_0 - w^2_1\,\mathrm{d}x = \mathrm{d}w^3_0 + G\,\mathrm{d}x = 0$$

by definition (we suppose the existence of the relevant derivatives). The $\mathcal{C}$-curves moreover satisfy the $\mathcal{EL}$ system $e^1 = e^2 = 0$. In virtue of (40, 41) we have the "almost explicit" equations

$$w^1_0 = Ax + B, w^2_0 = C, \frac{\mathrm{d}w^3_0}{\mathrm{d}x} = G(x, Ax + B, C, w^3_0, A) \tag{42}$$

where A, B, C are constants, to determine them.

Passing to the extremality properties, one can observe that the Lagrangian subspace are of little use. Briefly: $\breve{\alpha}$ is expressed by $x, w^1_0, w^2_0, w^3_0, w^1_1$ and the $\mathcal{HJ}$ condition $\mathbf{l}^*\,\mathrm{d}\breve{\alpha} = 0$ yields two interrelations, hence x, w^1_0, w^2_0, w^3_0 cannot be taken for coordinates on $\mathbf{L}$. We need *one more dimension.* The subspace $\mathbf{k}\colon\ \mathbf{K} \subset \mathcal{D}(E) \times \mathcal{D}(M) \times \mathbb{R}$

defined by $w^1_1 - A = 0$ and depending on the choice of the constant A will be employed for a substitute. Clearly

$$\text{(43)} \qquad \mathrm{d}\breve{\alpha} = M'' \,\mathrm{d}w^1_1 \wedge \omega^1_0 + \,\mathrm{d}L \wedge \omega^2_0, \mathbf{k}^* \,\mathrm{d}\breve{\alpha} = \,\mathrm{d}L \wedge \omega^2_0,$$

so we have a "nearly Lagrangian" subspace. (One can observe that $w^1_1 - A$ is a first integral of the $\mathcal{EL}$ system. Any other first integral e.g., the functions $w^2_0 - C$ or L=const. can be principle employed as well.) After this preparation, let

$$Q(t) = (x(t), w^1_0(t), w^2_0(t), w^3_0(t), w^1_1(t), w^2_1(t)) \in \mathcal{D}(L) \times \mathcal{D}(M) \times \mathbb{R}, \;\; 0 \leqslant t \leqslant 1,$$

be an $\mathcal{A}$-curve and

$$R(t) = (x(t), w^1_0(t), w^2_0(t), w^3_0(t), C, w^2_1(t)) \in \mathbf{K}, \;\; 0 \leqslant t \leqslant 1,$$

its "projection" into $\mathbf{K}$. If a $\mathcal{C}$-curve $P(t) \in \mathbf{K}$, $0 \leqslant t \leqslant 1$, has the same endpoints as the curve $Q(t)$, hence as the curve $P(t)$, the decompositions (10) can be applied with the result

$$\text{(44)} \qquad \int_Q \alpha - \int_P \alpha = \int_0^1 \mathcal{E} x'(t) \,\mathrm{d}t + \iint \,\mathrm{d}L \wedge \,\mathrm{d}w^2_0$$

which follows from (40, 43_2), where

$$\mathcal{E} = M(w^1_1) - M(A) - M'(A)(w^1_1 - A)$$

is a "partial Weierstrass function". To determine the sign of the difference (39), only the double integral causes some difficulties. It is however well-adapted for the "rotation principle".

We wish to determine the sign of the expression

$$\text{(45)} \qquad \iint \,\mathrm{d}L \wedge \,\mathrm{d}w^2_0 = \oint L \,\mathrm{d}w^2_0 = \int_R L \,\mathrm{d}w^2_0 = \int_Q L \,\mathrm{d}w^2_0.$$

The $\mathcal{C}$-curve $Q(t)$ satisfies (36) and consequently also the equation

$$\begin{aligned} 0 &= L^3_0(\,\mathrm{d}w^3_0 + G \,\mathrm{d}x) = L^3_0 \,\mathrm{d}w^3_0 + (L_x + L^1_0 - L^3_0 F w^2_1) \,\mathrm{d}x \\ &= L^3_0 \,\mathrm{d}w^3_0 + L_x \,\mathrm{d}x + L^1_0 \,\mathrm{d}w^1_0 - L^3_0 F \,\mathrm{d}w^2_0 = \,\mathrm{d}L - (L^2_0 + L^3_0) \,\mathrm{d}w^2_0. \end{aligned}$$

So, we may introduce the three-dimensional space of variables $x, u = w^2_0 v = L$, the Pfaffian equation $U \,\mathrm{d}u = V \,\mathrm{d}v$ where $u = L^2_0 + L^3_0 F$, $V = 1$, and $(U/V)_x =$

$(L_0^2 + L_0^3 F)_x$ is nonvanishing. Then the Theorem III 2 ensures the constant sign of the value (45).

Altogether taken, the extremum is ensured if the signs of both summands on the right hand side of (44) are in the needful accordance, e.g., both are nonnegative for the case of the minimum.

Observation. If the common Weierstrass-Hilbert method fails, we may speak of a degenerate variational problem. (A more precise definition in terms of the $\mathcal{PC}$ form $\breve{\alpha}$ and the $\mathcal{EL}$ space $\mathbf{E}$ is not appropriate at this place.) We have seen that certain (*coisotropic*) subspaces $\mathbf{k}\colon\ \mathbf{K} \subset \mathbf{E}$ can replace with success the useless Lagrangian subspaces $\mathbf{l}\colon\ \mathbf{L} \subset \mathbf{E}$ to achieve sufficient extremality conditions. See also II 2, III 10–15 where $\mathbf{k} = \mathrm{id.}$, $\mathbf{K} = \mathbf{E}$ was employed for analogous aims.

10. A curious nondegenerate subcase. With the data of Section 6, we will suppose

$$f_{11}^{ij} \equiv C_{11}^{ij}, f_{11}^{11} f_{11}^{22} = (f_{11}^{12})^2, g_{11}^{11} g_{11}^{22} = (g_{11}^{12})^2 \tag{46}$$

for a certain coefficient $C \neq c$, which means that the genericity condition (28) is not satisfied. Let moreover $f_{11}^{11} \neq 0$ (hence $g_{11}^{11} \neq 0$, $C \neq 0$). Then

$$f_{11}^{i2} = b f_{11}^{i1}, g_{11}^{i2} = b g_{11}^{i1}, c_2^2 = b c_2^1 \tag{47}$$

for a certain factor b, which follows from $(46_{2,3})$ and (25_2). Moreover

$$e{}^{12}_{\ 2} = b e{}^{11}_{\ 3}, (Xe^2)_3^2 e{}^{22}_{\ 2} = b e{}^{21}_{\ 2} = (Xe^2)_3^1 \tag{48}$$

in virtue of $(47, 24_1, 25_1)$. One can observe that $e{}^{11}_{\ 3} = -c_2^1 \neq 0$ (otherwise $c_2^1 = c_2^2 = 0$ by (47_3), which implies (32_1) and the contradiction $C = c$) and *we shall suppose that* $e{}^{21}_{\ 2} \neq 0$. It follows that the third order summands e^1 and Xe^2 are proportional, therefore the function

$$e = e^1 - KXe^2 \quad \left(K = \frac{e{}^{11}_{\ 3}}{e{}^{21}_{\ 2}} = -\frac{c_2^1}{c_2^1 a^1 - (C-c) g_{11}^{11}} \right) \tag{49}$$

(use (21_1) as concerns the coefficient K) is of a lower order. We will see that e *is of order two* and $e_2^2 - b e_2^1 \neq 0$, see Section 12 for the proof. It follows that the $\mathcal{EL}$ system $e^1 = e^2 = 0$ is equivalent to the second order system $e = e^2 = 0$ which can be moreover adapted as

$$w_2^1 = E(\ldots), w_2^2 = F(\ldots) \quad (\ldots = x, w_0^1, w_0^2, w_0^3, w_1^1, w_1^2). \tag{50}$$

Formally the system (35) appears here again and some concepts may be introduced analogously as in Section 8: the domain $\mathcal{D}(E,F)\subset\mathbb{R}^6$, the $\mathcal{A}$-curves satisfying (36) and the $\mathcal{C}$-curves satisfying moreover $\mathrm{d}w_1^1 - E\,\mathrm{d}x = \mathrm{d}w_1^2 - F\,\mathrm{d}x = 0$.

A certain distinction however occurs: $\breve{\alpha}$ and $\mathrm{d}\breve{\alpha}$ cannot be regarded as forms on $\mathcal{D}(E,F)$ at the present time due to the coefficient c which depends on the variables w_2^1, w_2^2. We will nevertheless see that the Weierstrass-Hilbert method does not fail if the arguments of Section 8 are slightly modified.

If (45) is inserted for the variables w_2^1, w_2^2 occurring in c, then $\breve{\alpha}$ becomes a differential form on $\mathcal{D}(E,F)$. (In terms of the previous parts, we use the restriction $\mathbf{e}^*\breve{\alpha}$ on the $\mathcal{EL}$ subspace $\mathbf{E}\subset\mathbf{L}$.) Then $\mathrm{d}\breve{\alpha}$ regarded as a form on $\mathcal{D}(E,F)$ *has four-dimensional Lagrangian subspaces* $\mathbf{l}\colon \mathbf{L}\subset\mathcal{D}(E,F)$; we again refer to the next sections. Assume that x, w_0^1, w_0^2, w_0^3 can be taken for local coordinates on $\mathbf{L}$, and a $\mathcal{C}$-curve $P(t)\in\mathbf{L}$ $(0\leqslant t\leqslant 1)$ is embedded into $\mathbf{I}$ (in classical terms: in the relevant Mayer field). Then the same reasoning as in Section 8 (i.e., the decomposition (10) with the "projection" $R(t)\in\mathbf{L}$ of a $\mathcal{C}$-curve $Q(t)$ with the same endpoint) leads to the common Weierstrass function

$$\mathcal{E} = F(\ldots,w_1^1,w_1^2) - F(\ldots,r_1^1,r_1^2) - \sum F_1^i(\ldots,r_1^1,r_1^2)(w_1^i - r_1^i) \tag{51}$$

where $\ldots = x, w_0^1, w_0^2, w_0^3$ and $F = f + \mathbf{l}^* c\cdot g$. (That means, the variables

$$w_1^1 = r_1^1, w_1^2 = r_1^2, w_2^1 = E(\ldots,r_1^1,r_1^2), w_2^2 = F(\ldots,r_1^1,r_1^2)$$

are "freezed" in the coefficient c.) The function $\mathcal{E}$ resolves the extremality, e.g., the minimum is guaranteed if $\mathcal{E}\geqslant 0$. This is a curious result since $\mathcal{E}$ is not a "strongly definite" function: $F = F(\ldots,w_1^1,w_1^2)$ represents the graph of a developable surface (depending on parameters $\ldots = x, w_0^1, w_0^2, w_0^3$) and $\mathcal{E}\equiv 0$ is vanishing along the generating lines.

Observation. Even some "semi-definite" variational problems (with a not definite second differential) may be regarded as nondegenerate from our point of view.

11. Continuation. Let us briefly mention the Lagrangian subspaces $\mathbf{l}\colon\mathbf{L}\subset\mathcal{D}(E,F)$ satisfying $\mathbf{l}^*\breve{\alpha} = \mathrm{d}W$ where $W = W(x,w_0^1,w_0^2,w_0^3)$ is an unknown function, i.e., we again have the conditions (29). In our case, however, (30) cannot be taken for local coordinates on the six-dimensional space $\mathcal{D}(E,F)$. By a lucky accident, the functions $f_1^1 - cg_1^1$, $f_1^2 - cg_1^2$ are functionally dependent on every level set $\mathrm{d}x = \mathrm{d}w_0^1 = \mathrm{d}w_0^2 = \mathrm{d}w_0^3 = \mathrm{d}c = 0$ as follows from the identity $\det\,(f_{11}^{ij} - cg_{11}^{ij}) = 0$. It follows that locally

$$f_1^2 - cg_1^2 = F(\ldots,f_1^1 - cg_1^1, c) \quad (\ldots = x, w_0^1, w_0^2, w_0^3)$$

for an appropriate function F. Analogously $H = H(\ldots, f_1^1 - cg_1^1, c)$ for the function (38_1). Altogether taken, equations (29) imply the involutive system

$$W_x + H(\ldots, W_0^1, W_0^3), W_0^2 = F(\ldots, W_0^1, W_0^3)$$

for the function W. This is a substitute for the classical $\mathcal{HJ}$ equation.

O b s e r v a t i o n. As in Section 3.

12. Complements. In order to cope with the references of Section 10, let us denote $\omega_s \equiv \omega_s^1 + b\omega_s^2$ and let us state the congruence

$$\mathrm{d}\breve{\alpha} \cong (e^1\pi + e^2\omega_0^1) \wedge \mathrm{d}x + \left(\sum c_1^i\omega_1^i + c_2^1\omega_2\right) \wedge \pi + (C-c)\omega_1 \wedge \omega_0 \tag{52}$$

which is a mere transcription of (19) if the identities $(46_1, 47_{2,3})$ are applied. Recalling that $C \neq c$ and $c_2^1 \neq 0$, we have the module

$$\begin{aligned}\mathrm{Adj}\ \mathrm{d}\breve{\alpha} &= \left\{\pi, \omega_0, e^2\omega_0^1, \omega_1, e^1\,\mathrm{d}x - \sum c_1^i\omega_1^i - c_2^i\omega_2, e^2\,\mathrm{d}x\right\}\\ &= \left\{\mathrm{d}x,\ \mathrm{d}w_0^1,\ \mathrm{d}w_0^2,\ \mathrm{d}w_0^3,\ \mathrm{d}w_1^1 + b\,\mathrm{d}w_1^2, \sum c_1^i\,\mathrm{d}w_1^i + c_2^1(\mathrm{d}w_2^1 + b\,\mathrm{d}w_2^2)\right\}\end{aligned}$$

in the region where $e^2 \neq 0$. This is a completely integrable module whence

$$\mathrm{d}(\mathrm{d}w_2^1 + b\,\mathrm{d}w_2^2) \cong \mathrm{d}b \wedge \mathrm{d}w_2^2 \cong 0\ (\bmod\,\mathrm{Adj}\ \mathrm{d}\breve{\alpha})$$

and it follows that

$$\mathrm{d}b \cong b_1^1\,\mathrm{d}w_1^1 + b_1^2\,\mathrm{d}w_1^2 \in \mathrm{Adj}\ \mathrm{d}\breve{\alpha}, b_1^2 = bb_1^1. \tag{53}$$

Of course, the identity (53_2) holds true also at the exceptional points where $e^2 = 0$.

Let us restrict the form $\mathrm{d}\breve{\alpha}$ to the subspace defined by $e^2 = 0$ which moreover implies the identity

$$0 = \mathrm{d}e^2 \cong Xe^2\,\mathrm{d}x + \sum e^{2i}{}_1\omega_1^i + e^{21}{}_2\omega_2 \quad (\bmod\,\omega_0^1, \omega_0^2, \omega_0^3), \tag{54}$$

see (48_2) as concerns the last summand. After this restriction, the congruence (42) again simplifies to

$$\mathrm{d}\breve{\alpha} \cong \pi \wedge \left(e\,\mathrm{d}x - \sum K^i\omega_1^i\right) + (C-c)g_{11}^{11}\omega_1 \wedge \omega_0 \quad (K^i \equiv c_1^i - Ke^{2i}_1). \tag{55}$$

One can observe with pleasure that the function (49) is engaged. Then the identity $\mathrm{d}^2\breve{\alpha} = 0$ provides the useful formula

$$e_2^2 - be_2^1 = K^2 - bK^1$$

by comparison of the coefficients of the summands $\pi \wedge \omega_2^2 \wedge \mathrm{d}x$. We should like to prove that this is a nonvanishing expression. However, assuming $K^2 = bK^1$, (55) can be rewritten as

$$\mathrm{d}\breve{\alpha} \cong \pi \wedge (e\,\mathrm{d}x + K^1\omega_1) + (C - c)g_{11}^{11}\omega_1 \wedge \omega_0$$

and (assuming $b_1^1 \neq 0$, which is the general case) the summand

$$(C - c)g_{11}^{11}\,\mathrm{d}\omega_1 \wedge \omega_0 \cong (C - c)g_{11}^{11}\,\mathrm{d}b \wedge \omega_1^2 \wedge \omega_0 \cong (C - c)g_{11}^{11}b_1^1\omega_1 \wedge \omega_1^2 \wedge \omega_0$$

occurring in the identity $\mathrm{d}\breve{\alpha} = 0$ leads to the contradiction $C = c$. (We have tacitly employed several simple formulae, e.g.,

$$\mathrm{d}\omega_s = \mathrm{d}x \wedge \omega_{s+1} + \mathrm{d}b \wedge \omega_s^2,\ \mathrm{d}b \cong b_1^1\omega_1^1 + b_1^2\omega_1^2 = b_1^1\omega_1,$$

(22, 23) and (54) to eliminate the form ω_2.)

Since K^1, K^2 cannot both vanish, one can easily see that

$$\mathrm{Adj}\ \mathrm{d}\breve{\alpha} = \left\{\omega_0, \omega_1, K^1\pi, K^2\pi, e\,\mathrm{d}x - \sum K^i\omega_1^i\right\}$$

is a four-dimensional module even after performing the total restriction by the conditions $e = e^2 = 0$ (hence after the substitution (50)). Therefore two requirements are needful to obtain the Lagrange subspace in the six-dimensional space $\mathcal{D}(E, F)$ in good accordance with Section 10.

Observation. Some formal properties of the problem cannot be always easily obtained by direct calculations and our generalized $\mathcal{PC}$ forms may provide an indispensable tool in this respect.

13. Implementation. We will be interested in the existence of functions f, g satisfying (46). The condition (46_1) are equivalent to the congruences

$$\mathrm{d}f_1^i \cong C\,\mathrm{d}g_1^i \quad (\mathrm{mod}\ \mathrm{d}x,\ \mathrm{d}w_0^1,\ \mathrm{d}w_0^2,\ \mathrm{d}w_0^3).$$

It follows that $\mathrm{d}C \wedge \mathrm{d}g_1^i \cong 0$, consequently either $\mathrm{d}C \cong 0$ (mod $\mathrm{d}w_1^1$, $\mathrm{d}w_1^2$) or $\mathrm{d}g_2^1 \wedge \mathrm{d}g_2^2 = 0$. The first case is easy: $f = Aw_1^1 + Bw_1^2 + Cg + D$ with coefficients independent of w_1^1, w_2^2 and it does not need any comments. The second case automatically implies (46_2) hence (46_3) and it will be mentioned in more detail.

It is well-known (cf. II 5) that a function g satisfying (46_2) appears if we take a system of equations

$$(56) \qquad P + Qw_1^1 + Rw_1^2 = g, \;\; P' + Q'w_1^1 + R'w_1^2 = 0 \quad ('= \partial/\partial z),$$

where P, Q, R are in principle arbitrary functions of the variables x, w_0^1, w_0^2, w_0^3 and a new parameter z. Assuming $P'' + Q''w_1^1 + R''w_1^2 \neq 0$, then $z = z(x, w_0^1, w_0^2, w_0^3, w_1^1, w_1^2)$ calculated from (56_2) and substituted into (56_1) provides the sought function g. Clearly

$$(57) \qquad g_1^1 = Q, g_1^2 = R, g_{11}^{11} = Q'z_1^1, g_{11}^{12} = Q'z_1^2 = R'z_1^1, g_{11}^{22} = R'z_1^2.$$

Assuming $\mathrm{d}g_1^1 \neq 0$, $\mathrm{d}g_1^2 \neq 0$ for better clarity, (46_1) implies the existence of local identities of the kind

$$(58) \qquad f_1^1 = G(\ldots, g_1^1), f_1^2 = F(\ldots, g_1^2) \qquad (\ldots = x, w_0^1, w_0^2, w_0^3)$$

with the compatibility $\partial G/\partial g_1^1 = \partial F/\partial g_1^2$. Also the identities

$$(59) \qquad \overline{P} + Gw_1^1 + Fw_1^2 = f, \;\; \overline{P}' + \frac{\partial G}{\partial g_1^1}w_1^1 + \frac{\partial F}{\partial g_1^2}w_1^2 = 0$$

(where $\overline{P} = \overline{P}(\ldots, z)$, $G = G(\ldots, Q)$, $F = F(\ldots, R)$) analogous to (56) immediately follow. If $\overline{P}$ is chosen such that

$$(60) \qquad \overline{P} = P'\frac{\partial G}{\partial g_1^1} = P'\frac{\partial F}{\partial g_1^2},$$

then $(56_2, 59_2)$ determine the same function z. Conversely, if $\overline{P}, F, G$ are chosen to satisfy the compatibility and (60), the conditions (46) are satisfied with $C = \partial G/\partial g_1^1 = \partial F/\partial g_1^2$.

O b s e r v a t i o n. We have two developable surfaces $u = f(\ldots, w_1^1, w_1^2)$, $u = F(\ldots, w_1^1, w_1^2)$ in the three-dimensional space of the variables u, w_1^1, w_1^2 (where $\ldots = x, w_0^1, w_0^2, w_0^3$ are parameters). Equations $(56_2, 59_2)$ determine the vertical projections of the generating lines (depending on the parameter z) and are identical by virtue of (60).

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Author's address: *Jan Chrastina*, Masaryk University, Janáčkovo nám. 2a, 662 95 Brno, Czech Republic.