REMARKS ON MONGE MATRICES

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Abstract. We present some new results on Monge matrices and their relationship with ultrametric matrices.

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1. Introduction

All vectors and matrices in this paper are real. An $m \times n$ matrix $C = (c_{ik})$ is called (cf. [1], [3]) a *Monge matrix* if it satisfies

$$c_{ij} + c_{kl} \leqslant c_{il} + c_{kj} \quad \text{for all } i, j, k, l,\ i < k,\ j < l. \tag{1}$$

Monge matrices play an important rôle in assignment and transportation problems since the corresponding problems can be solved very efficiently.

In the final part, we shall deal with matrices we will call *matrices with the weak Monge property.* These are square matrices $C = (c_{ij})$ satisfying

$$c_{ij} + c_{kk} \leqslant c_{ik} + c_{kj} \quad \text{for all } i, j, k,\ i < k,\ j < k, \tag{2}$$

or matrices obtained from them by a simultaneous permutation of rows and columns. Such matrices were, in a more general setting, introduced in [1].

Another class of matrices we will investigate are ultrametric matrices.

Strictly ultrametric matrices were defined (cf. [5], [6]) as square nonnegative symmetric matrices $A = (a_{ij})$ satisfying

$$a_{ij} \geqslant \min(a_{ik}, a_{kj}) \quad \text{for all } i, j, k, \tag{3}$$

$$a_{ii} > \max_{j \neq i} a_{ij} \quad \text{for all } i. \tag{4}$$

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The author defined (cf. [2]) *special ultrametric matrices* as matrices satisfying (3) and

$$a_{ii} = \max_{j\neq i} a_{ij} \quad \text{for all } i. \tag{5}$$

The following was shown ([2], Theorem 2.1):

Theorem 1.1. *Let $a_{12}, a_{23}, \ldots, a_{n-1,n}$, $n \geqslant 2$, be nonnegative numbers. Construct a symmetric $n\times n$ matrix $A = (a_{ij})$ by setting*

$$a_{11} = a_{12},\ a_{ii} = \max(a_{i-1,i}, a_{i,i+1}) \text{ for } i = 2,\ldots,n-1,\ a_{nn} = a_{n-1,n}, \tag{6}$$

$$a_{ik} = \min(a_{i,k-1}, a_{i+1,k}) \text{ for all } i,k,\ 1 \leqslant i < k-1 \leqslant n-1, \tag{7}$$

$$a_{ki} = a_{ik} \text{ for all } i,k,i>k. \tag{8}$$

Then A is special ultrametric and every special ultrametric matrix is permutation similar to a matrix obtained in this way.

Remark 1.2. In the sequel, we will call the matrix constructed from the entries $a_{12},\ldots,a_{n-1,n}$ by (6), (7) and (8) an *ordered special ultrametric matrix.*

2. Results

Before investigating Monge matrices, we introduce some notation.

For $k \geqslant 2$, we denote by Z_k the $(k-1)\times k$ matrix

$$Z_k = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & 0 & \ldots & 1 & -1 \end{pmatrix}.$$

Now we can state a lemma the proof of which is straightforward.

Lemma 2.1. *Let $x = (x_1,\ldots,x_n)^T$, $u = (u_1,\ldots,u_{n-1})^T$ be column vectors. Then*

$$Z_n^T u = x$$

if and only if

$$\begin{aligned} x_1 &= u_1, \\ x_1 + x_2 &= u_2, \\ \ldots&\ldots\ldots\ldots \\ x_1 + \ldots + x_{n-1} &= u_{n-1}, \\ x_1 + \ldots + x_n &= 0. \end{aligned}$$

We will also say that a vector $x = (x_1, \dots, x_n)^T$ is *weakly monotone* if

$$\begin{aligned} x_1 &\geqslant 0, \\ x_1 + x_2 &\geqslant 0, \\ \dots\dots&\dots\dots \\ x_1 + \ldots + x_{n-1} &\geqslant 0, \\ x_1 + \ldots + x_n &= 0. \end{aligned}$$

This will be denoted by $x \geqslant_w 0$.

We have now:

Theorem 2.2. *Let $C = (c_{ij})$ be an $m \times n$ matrix, $m \geqslant 2, n \geqslant 2$. Then the following are equivalent:*

(i) *C is a Monge matrix.*

(ii) *$Z_m C Z_n^T \leqslant 0$.*

(iii) *Whenever x, y, respectively, are column vectors with m and n coordinates satisfying $x \geqslant_w 0$, $y \geqslant_w 0$, then*

$$x^T C y \leqslant 0.$$

P r o o f. The equivalence of (i) and (ii) is immediate since the entries b_{kl}, $k = 1, \dots, m$, $l = 1, \dots, n$ of the matrix $B = Z_m C Z_n^T$ satisfy

$$b_{kl} = c_{kl} + c_{k+1,l+1} - c_{k,l+1} - c_{k+1,l},$$

and the system of inequalities in $B \leqslant 0$ is equivalent to the system in (1) as is easily shown by induction with respect to $k + l - i - j$.

Now suppose (ii). Let $x \geqslant_w 0$, $y \geqslant_w 0$ be column vectors with respectively m and n coordinates. By Lemma 2.1, there exist nonnegative vectors u and v for which

$$Z_m^T u = x, \ \ Z_n^T v = y.$$

We have then

$$x^T C y = u^T Z_m C Z_n^T v$$

so that (iii) is fulfilled.

If (iii) holds, we choose vectors u, v as vectors with one coordinate 1 and all the others zero. The corresponding vectors x, y are weakly monotone. By (iii), $x^T C y \leqslant 0$, which implies that $Z_m C Z_n^T$ is nonpositive. $\square$

Lemma 2.3. *If $x = (x_1, \ldots, x_n)^T$ is a column vector satisfying $\sum_{i=1}^{n} x_i = 0$, then there exists a permutation matrix P such that $Px \geqslant_w 0$.*

Proof. There exists a permutation matrix P such that $Px = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})^T$, $x_{i_1} \geqslant x_{i_2} \geqslant \ldots \geqslant x_{i_n}$. It is easily checked that then $Px \geqslant_w 0$. □

Corollary 2.4. *Let C be an $m \times n$ Monge matrix. Whenever column vectors x, y with m, n coordinates, respectively, have sum of their coordinates equal to zero, then there exist permutation matrices P_1 and P_2 such that $x^T P_2 C P_1 y \leqslant 0$.*

In the following theorems, a matrix will be called *equilibrated* if all its row sums and all its column sums are equal to zero. The following is easy.

Lemma 2.5. *Let $C = (c_{ij})$ be an $m \times n$ Monge matrix. Let e_m and e_n mean the column vectors of all ones of dimensions m and n, respectively. Then there exist column vectors u of dimension m and v of dimension n such that the matrix*

$$\widehat{C} = C + u e_n^T + e_m v^T \tag{9}$$

is an equilibrated Monge matrix.

Proof. Denote $y := Ce_n$, $z := C^T e_m$. The number $K = e_m^T C e_n$ fulfils then

$$y^T e_m = K, \quad z^T e_n = K.$$

If we choose

$$u = \frac{K}{2mn} e_m - \frac{1}{n} y, \quad v = \frac{K}{2mn} e_n - \frac{1}{m} z,$$

we obtain from (9)

$$\widehat{C} e_n = y + \frac{K}{2m} e_m - y + e_m \Big[\frac{K}{2m} - \frac{K}{m}\Big],$$

which is zero; similarly $\widehat{C}^T e_m = 0$. Since the transformation (9) preserves Monge matrices, the proof is complete. □

Remark 2.6. One can easily show that the vectors u and v are unique up to the transformation $u \to u + \alpha e_m$, $v \to v - \alpha e_n$ for some α.

In the next theorem, we say that a column vector $u = (u_1, \ldots, u_k)^T$ is *monotone* if either $u_1 \geqslant u_2 \ldots \geqslant u_k$, or $u_1 \leqslant u_2 \ldots \leqslant u_k$. (If these inequalities are strict, we say that u is *strictly monotone*.)

Theorem 2.7. *Let C be an $n \times n$ equilibrated Monge matrix. Then C has a nonpositive eigenvalue of maximum modulus, and the corresponding eigenvectors of both C and C^T can be chosen monotone.*

P r o o f. Suppose first that all inequalities (1) for C are strict. Let ε be a positive number, later to be specified, and let J be the matrix of all ones. If $\gamma_0 = 0, \gamma_1, \dots, \gamma_{n-1}$ are all eigenvalues of C, possibly also complex, then the matrix $B = J - \varepsilon C$ is easily seen to have eigenvalues $n, -\varepsilon\gamma_1, \dots, -\varepsilon\gamma_{n-1}$, the first eigenvalue corresponding to the eigenvector $(1, \dots, 1)^T$. Since C is equilibrated, both the matrices B and C have the same system of eigenvectors.

We shall now use the theory of compound matrices [4]. The second compound $B^{(2)}$ of B whose rows and columns are indexed by pairs (i,j), $i < j$, and (k,l), $k < l$, has entries

$$B^{(2)}_{(i,j)(k,l)} = \det \begin{pmatrix} 1 - \varepsilon c_{ik} & 1 - \varepsilon c_{il} \\ 1 - \varepsilon c_{jk} & 1 - \varepsilon c_{jl} \end{pmatrix} = -\varepsilon(c_{ik} + c_{jl} - c_{jk} - c_{il}) + Q_{(ij)(kl)}\varepsilon^2$$

for some matrix Q.

Thus $B^{(2)}$ is a positive matrix for ε sufficiently small. In addition, by the well known spectral properties of compound matrices, the eigenvalues of $B^{(2)}$ are $-n\varepsilon\gamma_1, \dots, -n\varepsilon\gamma_{n-1}$ and $\binom{n-1}{2}$ eigenvalues of the form $\varepsilon^2\gamma_i\gamma_j$, $1 \leqslant i < j \leqslant n-1$.

By the Perron-Frobenius theorem, one of the numbers $-n\varepsilon\gamma_k$, say, $-n\varepsilon\gamma_1$, is the Perron eigenvalue of $B^{(2)}$ (provided ε satisfies $\varepsilon|\gamma_j| < n$ for all j), and the corresponding eigenvector can be chosen positive. This eigenvector has, independently of ε, the form

$$(u, e)^{(2)}$$

which means, u being the eigenvector of B corresponding to $-\varepsilon\gamma_1$, the second compound matrix of the $n \times 2$ matrix (u, e). Therefore, the coordinates of u are either decreasing or increasing, thus strictly monotone. Since B and C have the same system of eigenvectors, u is a monotone eigenvector of C corresponding to the nonpositive eigenvalue γ_1 which, in addition, has the maximum modulus.

Now let the inequalities (1) be not all strict. Denote by $V(\xi)$ the matrix (ξ^{i+j}), $i, j = 1, \dots, n$, where ξ is a positive number less than one. Let η be a small positive number. Then the matrix $C - \eta(V(\xi) - dxe^T - dex^T + d^2ee^T)$, where $x = (1, \xi, \dots, \xi^{n-1})^T$ and $d = \frac{1}{n}\sum\limits_{k=0}^{n-1}\xi^k$, is an equilibrated Monge matrix for which all inequalities in (1) are already strict and which converges to C if $\eta \searrow 0$. By continuity of the (simple!) Perron eigenvalue and the corresponding eigenvector, it follows from the previous part that monotonicity again holds, even if it, in the limit, need not be strict. $\square$

Now we shall find the relationship between special ultrametric matrices and matrices with the weak Monge property.

Theorem 2.8. *The negative of every special ultrametric matrix has the weak Monge property.*

P r o o f. Let first A be an $n \times n$ ordered special ultrametric matrix generated by its entries $a_{12}, \ldots, a_{n-1,n}$. Let indices i, j, k satisfy $i > j$, $i > k$. By (3),

$$a_{jk} \geqslant \min(a_{ji}, a_{ik}).$$

If $a_{jk} \geqslant a_{ji}$, then

$$a_{ii} \geqslant a_{ji} + a_{ik} - a_{jk} \tag{10}$$

since $a_{ii} \geqslant a_{ik}$; if $a_{jk} < a_{ji}$, (10) is also true since then $a_{jk} \geqslant a_{ik}$ as well as $a_{ii} \geqslant a_{ji}$. This means that the matrix $-A$ satisfies (2).

Since every ultrametric matrix is permutation similar to an ordered special ultrametric matrix by Theorem 1.1, the result follows. □

Corollary 2.9. *The negative of every strictly ultrametric matrix has the weak Monge property.*

P r o o f. It follows immediately from the fact [2] that every strictly ultrametric matrix can be obtained from a special ultrametric matrix by increasing some diagonal entries. □

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