WHAT'S THE PRICE OF A NONMEASURABLE SET?

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Abstract. In this note, we prove that the countable compactness of $\{0,1\}^{\mathbb{R}}$ together with the Countable Axiom of Choice yields the existence of a nonmeasurable subset of $\mathbb{R}$. This is done by providing a family of nonmeasurable subsets of $\mathbb{R}$ whose intersection with every non-negligible Lebesgue measurable set is still not Lebesgue measurable. We develop this note in three sections: the first presents the main result, the second recalls known results concerning non-Lebesgue measurability and its relations with the Axiom of Choice, the third is devoted to the proofs.

Keywords: Lebesgue measure, nonmeasurable set, axiom of choice

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1. Introduction

Throughout this paper we assume that the axioms of the standard Zermelo-Frenkel (ZF) set theory hold. The only choice assumption that we make is the Countable Axiom of Choice $(\mathrm{AC})_{\aleph_0}$, since "without the Countable Axiom of Choice it is impossible to define satisfactorily Lebesgue measure, or even Borel sets" ([8], p. 144).* We will call "Greek fret functions" the functions $c_k(\xi)$ defined as follows. For every $\xi \in [0,1[$ and $k \in \mathbb{N}$ let $c_k(\xi)$ be the k-th digit in the binary expansion of ξ, i.e., the sequence $(c_k(\xi))_{k\in\mathbb{N}}$ is the only sequence in $\{0,1\}$ which is not eventually 1 and such that

$$\xi = \sum_{k=1}^{\infty} c_k(\xi) 2^{-k}.$$

* Indeed, without assuming $(\mathrm{AC})_{\aleph_0}$, the set of real numbers could be a countable union of countable sets (see [8], Th. 10.6): in that case the basic assumption of the countable additivity of Lebesgue measure would trivialize the theory. We refer the reader to the appendix of [4] which explores how to avoid $(\mathrm{AC})_{\aleph_0}$ in many applications using "coded" Borel sets.

We shall consider the functions c_k as elements of $\{0,1\}^{\mathbb{R}}$ by extending them by 1-periodicity, i.e., $c_k(\xi) = c_k(\xi - [\xi])$ where $[\xi]$ denotes the integer part of ξ. Another way to introduce the functions c_k, suitable for graphical representation, is as follows. Let

$$c(x) := \begin{cases} 0 & \text{if } x \in \bigcup\limits_{k\in\mathbb{Z}} [2k, 2k+1[, \\ 1 & \text{if } x \in \bigcup\limits_{k\in\mathbb{Z}} [2k+1, 2k+2[. \end{cases}$$

Then $c_k(x) := c(2^k x)$. Let us set $G := \{c_k \colon\ k \in \mathbb{N}\}$.

We denote by D the set of dyadic numbers (the set of numbers whose binary digits are eventually 0) and by $D(G)$ the set of cluster points of G with respect to the product topology in $\{0,1\}^{\mathbb{R}}$. Moreover, in what follows, μ_* (μ^*) denotes the usual outer (inner) measure on $\mathbb{R}$.

Note that each element of $D(G)$ is the characteristic function χ_E of some subset E of $\mathbb{R}$. In order that $D(G)$ be actually non-empty, the assumption that $\{0,1\}^{\mathbb{R}}$ is countably compact (see [9]) is sufficient. Then, without any further hypothesis, we can state

Main Result. *Consider the family $\mathcal{E} := \{E \subset \mathbb{R} \colon\ \chi_E \in D(G)\}$. Then the following properties hold true for any element $E \in \mathcal{E}$:*

(a) *E is invariant under translations by any dyadic number;*

(b) *the subsets E, $-E$ and D give a partition of $\mathbb{R}$;*

(c) *E is a saturated nonmeasurable set (i.e., by definitions $\mu_*(E) = 0 = \mu_*(E^c)$).*

R e m a r k 1.1. Property (c) of Main Result admits the following equivalent formulations [7]:

(i) the intersection of E with every measurable set of positive measure is nonmeasurable;

(ii) $\mu^*(E \cap A) = \mu(A) = \mu^*(E^c \cap A)$ for every measurable subset A of $\mathbb{R}$;

(iii) $\mu_*(E \cap A) = 0$ and $\mu^*(E \cap A) = \mu(A)$ for every measurable subset A of $\mathbb{R}$.

As an immediate consequence of Main Result, we get

Theorem 1.2. *Let us assume that (the axioms of the standard Zermelo-Fraenkel set theory together with) the Countable Axiom of Choice* $(\mathrm{AC})_{\aleph_0}$ *hold true.*

If $\{0,1\}^{\mathbb{R}}$ is countably compact, then there exists a saturated nonmeasurable set.

2. Nonmeasurable sets versus the axiom of choice

The aim of this section is to examine the most important results concerning the connection between the Axiom of Choice in its various forms and the existence of nonmeasurable sets. In the sequel we will denote

$$\text{(NM)} := \text{“there exists a nonmeasurable subset of the real line } \mathbb{R}\text{”},$$

by (M) its negation, i.e.,

$$\text{(M)} := \text{“every subset of the real line is Lebesgue-measurable”}.$$

Actually, for each $n \in \mathbb{N}$, (NM) is equivalent to the fact that “there exists a nonmeasurable subset of $\mathbb{R}^n$” [4].

The existence of a nonmeasurable subset of $\mathbb{R}$ was first proved by G. Vitali [25] by assuming the Axiom of Choice for a continuous family of sets $(\mathrm{AC})_{\mathrm{c}}$. The same assumption was further used in the paradoxical decomposition of the ball in $\mathbb{R}^3$ due to S. Banach and A. Tarski [1] (see also [8], [26]).

Another classical result is due to W. Sierpiński [19] who proved that Zermelo's Theorem for $\mathbb{R}$ (i.e., the real line can be well-ordered) implies the existence of a nonmeasurable subset of $\mathbb{R}^2$ whose intersection with every line consists at most of two points. One can prove in a constructive way that (NM) is equivalent to “there exists a saturated nonmeasurable subset of the real line $\mathbb{R}$” (see, e.g., [27]). Therefore, (NM) is also equivalent to the proposition “there exist two measurable functions f and g whose composition $f \circ g$ is nonmeasurable” (see, e.g., [5]).

The usual derivations of saturated nonmeasurable sets are obtained by assuming either $(\mathrm{AC})_{\mathrm{c}}$, see P. H. Halmos [6]; or Zermelo's Theorem for $\mathbb{R}$, cf. A. Simoson [21]; or the existence of a Hamel basis on $\mathbb{Q}$ for $\mathbb{R}$, see M. Kuczma [11] (cf. [3], [18]).

Since we can prove that properties (a) and (b) of Main Result implies property (c), one of the “cheaper” way to the existence of a saturated nonmeasurable set is to spend (both the Countable Axiom of Choice and) the axiom C_2 (i.e., for every family $\mathcal{F}$ of pairs there exists a choice function).

Proposition 2.1. *If we assume* $(\mathrm{AC})_{\aleph_0} + \mathrm{C}_2$*, then the cardinality of the class of saturated nonmeasurable sets is exactly equal to* $2^{\mathfrak{c}}$.

Proof. If we consider the quotient set $X := \mathbb{R}/D$ and the family $\mathcal{F} := \{([x],[-x])\colon x \in \mathbb{R}\}$, a choice function v for $\mathcal{F}$ leads to the existence of a set $\widetilde{E}$ composed by the classes of equivalence chosen from each pair of $\mathcal{F}$ by v. Hence, it is easy to see that the set E, union of the classes of $\widetilde{E}$, satisfies both (a) and (b) of

Main Result, i.e., E is a saturated nonmeasurable set. Moreover, if Y is an arbitrary subset of $\widetilde{E}$ then the set E_Y, obtained by the union of the classes $[x] \in Y$ and the classes $[-x]$ where $[x] \in \widetilde{E} \setminus Y$, satisfies in the same way (a) and (b): it follows that the cardinality of the class of saturated nonmeasurable sets is exactly 2^c. $\square$

Actually, we can prove that if we assume both the Continuum Hypothesis (CH) and the Axiom of Choice, then also the cardinality of $\mathcal{E}$ equals 2^c. The rather technical proof, not in the spirit of the paper, will not be presented.

The existence of a set E enjoying property (c) was explicitly obtained by W. Sierpiński [20]. He proved his result by exploiting a result of S. Ulam [24], who proved that Zermelo's Theorem for $\mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) implies

$$\text{(U)} := \text{“there exists a finitely additive function on } \mathcal{P}(\mathbb{N}) \\ \text{which is not countably additive”}.$$

Actually, it can be proved that (U) is equivalent to the assumption that the set $D(G)$ is not empty.

In a sense, the present paper gives a "geometrical interpretation" of the Ulam-Sierpiński result together with a topological interpretation of the hypothesis.

A noteworthy result is due to I. Halperin [7]. He established that "the existence of a discontinuous solution of $f(x+y) = f(x) + f(y)$ which assumes only a countable number of distinct values implies the existence of a partition of $\mathbb{R}$ into a countable number of disjoint subsets which are saturated nonmeasurable and congruent under translation". The existence of such a function follows from the existence of a Hamel basis on $\mathbb{Q}$ for $\mathbb{R}$.

D. Pincus [15] (cf. [16]) asked whether the Hahn-Banach Theorem (HB) implies (NM); the question was answered affirmatively by M. Foreman and F. Wehrung [4]; later on, J. Pawlikowski, exploiting ideas from [4], proved that (HB) actually implies the Banach-Tarski paradox!

Moreover, J. Mycielski and S. Swierczowski [13] proved that the property "every infinite positional game with perfect information and a denumerable set of positions is determinated", an equivalent form of the Axiom of Determinateness (AD) (see [8]), implies (M).

Finally, R. Solovay [23] exhibited a model of the usual Zermelo-Frenkel set theory (ZF) in which the principle of Dependent Choices (DC) holds and, nevertheless, every subset of $\mathbb{R}$ is Lebesgue-measurable. In his proof, he used an additional axiom "there exists a weakly inaccessible cardinal" (WIC), (see [8]). S. Shelah [22] showed that one cannot get rid of this hypothesis (cf. [17] or [26] p. 209). In Solovay's model the Axiom of Determinateness (AD) fails.

P r o o f o f Main Result. First of all we make the following remarks (whose simple proofs are left to the reader):

(i) $E \in \mathcal{E}$ if and only if for each finite $J \subset \mathbb{R}$ there exists $k \in \mathbb{N}$ such that $\chi_E = c_k$ on J. Since $\{0,1\}^{\mathbb{R}}$ is a Hausdorff topological space, the set of such integers k is necessarily infinite.

(ii) Let $E \in \mathcal{E}$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$, assume that $c_k(x) = c_k(y)$ for large $k \in \mathbb{N}$; then either both x and y belong to E, or both x and y do not belong to E.

(iii) Let $E \in \mathcal{E}$, $x \in \mathbb{R}$ and $y \in \mathbb{R}$, assume that $c_k(x) \neq c_k(y)$ for large $k \in \mathbb{N}$; then either $x \in E$ and $y \notin E$, or $x \notin E$ and $y \in E$.

Fix $E \in \mathcal{E}$; we first prove (a). Let $x \in E$ and $d \in D$; since $c_k(d) = 0$ for large $k \in \mathbb{N}$, one has $c_k(x) = c_k(x+d)$ for large $k \in \mathbb{N}$; by (ii) it follows that $x + d \in E$.

Proof of (b). Let $d \in D$: by (i) we have $\chi_E(d) = c_k(d)$ for infinitely many $k \in \mathbb{N}$, hence $\chi_E(d) = 0$. This proves that $E \cap D = \emptyset$. Moreover, $(-E)\cap D = (-E)\cap(-D) = -(E\cap D) = \emptyset$. Let $x \in \mathbb{R}\setminus D$: then $[x]+1-x = 1-\sum\limits_{k=1}^{\infty} c_k(x)2^{-k} = \sum\limits_{k=1}^{\infty} (1-c_k(x))2^{-k}$. Since $1-c_k(x)$ does not eventually equal 1, we have $c_k(x) = c_k([x]+1-x) = 1-c_k(x)$ for each $k \in \mathbb{N}$. In view of (iii), we conclude that (if $x \notin D$ then) $x \in E$ if and only if $-x \notin E$. This proves that $E \cap (-E) = \emptyset$ and $E \cup (-E) = \mathbb{R} \setminus D$. By summing up, $E \cup (-E) \cup D$ gives a partition of $\mathbb{R}$.

As to (c), it is a consequence of the following lemma:

Lemma 3.1. *Let $E \subset \mathbb{R}$ be such that $\mu^*(E) > 0$ and let D be a dense subset of $\mathbb{R}$ such that $E + D = E$. Then $\mu^*(E \cap A) = \mu^*(A)$ for each measurable set A.*

Indeed, both E and $\mathbb{R}\setminus E$ satisfy the hypothesis of the lemma; so we get $\mu^*(E\cap A) = \mu(A) = \mu^*(\mathbb{R} \setminus E \cap A)$ for each measurable set A, which is equivalent to (c).

R e m a r k 3.2. Proof of Lemma 3.1 can be achieved by means of standard measure theory arguments (see Appendix 1) despite the fact that it is commonly gained via Lebesgue's point theorem (see [10]).

4. Appendix 1

P r o o f o f L e m m a 3.1. Let us fix $0 < \alpha < 1$.

S t e p 1. For every $\sigma > 0$ there exist a, b (in D) with $0 < b - a < \sigma$ and $\mu^*(E \cap [a, b[) \geqslant \alpha(b-a)$.

Let us argue by contradiction and fix $\sigma > 0$ such that $\mu^*(E \cap [a, b[) < \alpha(b-a)$ whenever $a, b \in D$ and $0 < b-a < \sigma$. Let $\widetilde{E} := E \cap [0,1]$. Note that, since D is dense in $\mathbb{R}$, we can take $\mu^*(\widetilde{E})$ as the infimum of the set of numbers $\sum\limits_{i=1}^{\infty} |I_i|$, where $(I_i)_{i\in\mathbb{N}}$ is a sequence of non-degenerate intervals of the form $]a,b]$ with ends belonging to D and lengths $|I_i| < \sigma$, and such that $\widetilde{E} \subset \bigcup\limits_{i\in\mathbb{N}} I_i$. Let us fix one of these sequences $(I_i)_{i\in\mathbb{N}}$. Then $\mu^*(\widetilde{E}) \leqslant \sum\limits_{i=1}^{\infty} \mu^*(E\cap I_i) < \alpha \sum\limits_{i=1}^{\infty} |I_i|$. It follows that $\mu^*(\widetilde{E}) \leqslant \alpha\mu^*(\widetilde{E})$. On the other hand, from the hypothesis we have $\mu^*(E) > 0$, thus reaching a contradiction.

Step 2. $\mu^*(E\cap]x,y[) = |y-x|$ for each real numbers x, y.

Let $x,y\in\mathbb{R}$ with $x<y$. Fix $\varepsilon \in]0, y-x[$ and σ such that $0<\sigma<y-x-\varepsilon$, let a and b as in Step 1. Let $d_1\in D$ be such that $a+d_1 \in [x, x+\varepsilon/2[$, then $b+d_1<y$. If $y-(b+d_1) \geqslant \sigma+\varepsilon/4$, then let $d_2\in D$ be such that $a+d_2 \in [b+d_1, b+d_1+\varepsilon/4[$. It follows that $0<(a+d_2)-(b+d_1)<\varepsilon/4$ and $b+d_2<y$.

If we repeat these choices, we can find a finite set $\{d_1,\dots,d_n\} \subset D$ such that

(i) $[a,b]+d_i \subset [x,y]$ $(i=1,\dots,n)$;

(ii) $0<(a+d_{i+1})-(b+d_i)<\varepsilon/2^{i+1}$ $(i=1,\dots,n-1)$ and $0<y-(b+d_n)<\sigma+\varepsilon/2^n$;

(iii) $\mu\big(\bigcup\limits_{i=1}^{n} ([a,b]+d_i)\big) > y-x-(\varepsilon+\sigma)$.

Actually, since the intervals $[a,b]+d_i$ are disjoint, from (iii) and $E+D=E$ we get

$$\mu^*\left(E\cap \bigcup_{i=1}^{n} ([a,b]+d_i)\right) \geqslant \alpha\mu\left(\bigcup_{i=1}^{n} ([a,b]+d_i)\right) > \alpha(y-x-(\varepsilon+\sigma)).$$

This leads, due to (i) and the arbitrariness of ε, σ, α, to $\mu^*(E\cap[x,y]) \geqslant |y-x|$.

The converse inequality is trivial.

Step 3. If A is an arbitrary measurable subset of $\mathbb{R}$ we have $\mu^*(E\cap A)=\mu(A)$.

If A is open then the proof is trivial since, by using Carathéodory, we have $\mu^*\big(E\cap \bigcup\limits_{i=1}^{n} I_i\big) = \sum\limits_{i=1}^{n} \mu^*(E\cap I_i)$ and then the equality holds true whenever A is a finite union of open intervals, hence also when A is an open subset of $\mathbb{R}$. Let now $A=K$ be a compact subset of $\mathbb{R}$ and G a bounded open set containing K. Then by subaddictivity and the result for open sets, we have

$$\mu^*(E\cap K) \geqslant \mu^*(E\cap G) - \mu^*(E\cap(G\setminus K)) = \mu^*(G)-\mu^*(G\setminus K) = \mu^*(K).$$

Finally, let A be a measurable subset of $\mathbb{R}$ with positive measure. Let $0<\varepsilon<\mu^*(A)$. Then there exists a compact set $F\subset A$ such that $\mu^*(F)>\varepsilon$. Then $\varepsilon<\mu(F)=\mu^*(F\cap E) \leqslant \mu^*(A\cap E)$ so that $\mu^*(E\cap A)\geqslant \mu^*(A)$ by the arbitrariness of ε.

Indeed, without assuming $(\mathrm{AC})_{\aleph_0}$, the set of real numbers could be a countable union of countable sets (see [8], Th. 10.6): in that case the basic assumption of the countable additivity of Lebesgue measure would trivialize the theory. We refer the reader to the appendix of [4] which explores how to avoid $(\mathrm{AC})_{\aleph_0}$ in many applications using "coded" Borel sets.

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