RADIO ANTIPODAL COLORINGS OF GRAPHS

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Abstract. A radio antipodal coloring of a connected graph G with diameter d is an assignment of positive integers to the vertices of G, with $x \in V(G)$ assigned $c(x)$, such that

$$d(u,v) + |c(u) - c(v)| \geqslant d$$

for every two distinct vertices u, v of G, where $d(u,v)$ is the distance between u and v in G. The radio antipodal coloring number $\mathrm{ac}(c)$ of a radio antipodal coloring c of G is the maximum color assigned to a vertex of G. The radio antipodal chromatic number $\mathrm{ac}(G)$ of G is $\min\{\mathrm{ac}(c)\}$ over all radio antipodal colorings c of G. Radio antipodal chromatic numbers of paths are discussed and upper and lower bounds are presented. Furthermore, upper and lower bounds for radio antipodal chromatic numbers of graphs are given in terms of their diameter and other invariants.

Keywords: radio antipodal coloring, radio antipodal chromatic number, distance

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1. Introduction

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u–v path in G. A u–v path of length $d(u,v)$ is called a *u–v geodesic.* For a vertex v of a connected graph G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius* $\operatorname{rad} G$ and the maximum eccentricity is its *diameter* $\operatorname{diam} G$. We refer to [3] for graph theory notation and terminology. Let G be a connected graph of diameter d and let k an integer such that $1 \leqslant k \leqslant d$. A *radio*

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k-coloring of G is an assignment of colors (positive integers) to the vertices of G such that

$$d(u,v)+|c(u)-c(v)|\geqslant 1+k$$

for every two distinct vertices u,v of G. The *radio k-coloring number* $\mathrm{rc}_k(c)$ of a radio k-coloring c of G is the maximum color assigned to a vertex of G. The *radio k-chromatic number* $\mathrm{rc}_k(G)$ is $\min\{\mathrm{rc}_k(c)\}$ over all radio k-colorings c of G. A radio k-coloring c of G is a *minimum radio k-coloring* if $\mathrm{rc}_k(c)=\mathrm{rc}_k(G)$. A set S of positive integers is a *radio k-coloring set* if the elements of S are used in a radio k-coloring of some graph G and S is a *minimum radio k-coloring set* if S is a radio k-coloring set of a minimum radio k-coloring of some graph G. The radio 1-chromatic number $\mathrm{rc}_1(G)$ is then the chromatic number $\chi(G)$. A radio d-coloring of G is a *radio labeling* of G, and the radio d-chromatic number $\mathrm{rc}_d(G)$ is the *radio number* $\mathrm{rn}(G)$ that was introduced in [1] and studied further in [2].

A few observations about radio k-colorings will be useful. Let G be a connected graph with $\mathrm{rc}_k(G)=\ell$, where $1\leqslant k\leqslant \operatorname{diam} G$. In any radio k-coloring c of G with $\mathrm{rc}_k(c)=\ell$, certainly some vertex of G is assigned the color ℓ. Also, some vertex of G is colored 1, for otherwise the new k-coloring obtained from c by replacing $c(v)$ by $c(v)-1$ for each vertex v of G is a radio k-coloring of G as well, contradicting the fact that $\mathrm{rc}_k(G)=\ell$. That is, if c is a radio k-coloring of G with $\mathrm{rc}_k(c)=\mathrm{rc}_k(G)$, then there exist vertices u and v of G with $c(u)=1$ and $c(v)=\mathrm{rc}_k(G)$.

Let c be a radio k-coloring of a graph G with $\mathrm{rc}_k(c)=\ell$ and let $v\in V(G)$ with $c(v)=\ell$. For each integer $\ell'>\ell$, define a new coloring c' by $c'(v)=\ell'$ and $c'(u)=c(u)$ for all $u\in V(G)-\{v\}$. Then c' is a radio k-coloring of G with $\mathrm{rn}(c')=\ell'$. This observation yields the following lemma.

Lemma 1.1. *Let G be a connected graph with diameter d and k an integer such that $1\leqslant k\leqslant d$. If c is a radio k-coloring of G with $\mathrm{rc}_k(c)=\ell$, then for each integer $\ell'>\ell$, there exists a radio k-coloring c' of G with $\mathrm{rc}_k(c')=\ell'$.*

Let G be a connected graph of order n and diameter d with $V(G)=\{v_1,v_2,\dots,v_n\}$. For a radio k-coloring c of G, the *complementary coloring* $\overline{c}$ of c is defined by

$$\overline{c}(v_i)=(\mathrm{rc}_k(c)+1)-c(v_i)$$

for all i with $1\leqslant i\leqslant n$. Since $|\overline{c}(v_i)-\overline{c}(v_j)|=|c(v_i)-c(v_j)|$ for all i, j with $1\leqslant i<j\leqslant n$, we have the following result.

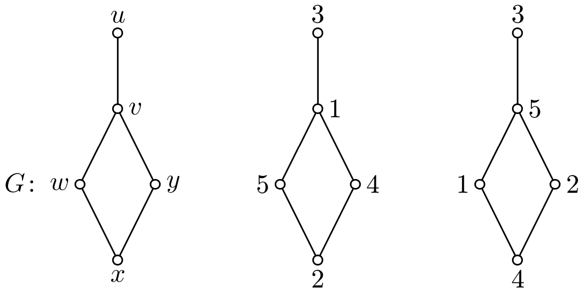
Proposition 1.2. *Let G be a connected graph with diameter d and k an integer such that $1\leqslant k\leqslant d$. If c is a radio k-coloring of G, then $\overline{c}$ is also a radio k-coloring of G and $\mathrm{rc}_k(c)=\mathrm{rc}_k(\overline{c})$.*

In this paper, we study the radio $(d-1)$-colorings of a connected graph G with diameter d, that is, those radio colorings c of G for which

$$d(u,v) + |c(u) - c(v)| \geqslant \operatorname{diam} G$$

for every two distinct vertices u, v of G. Thus in a radio $(d-1)$-coloring of G, it is possible for two vertices u and v to be colored the same, but only if they are antipodal, that is, $d(u,v) = \operatorname{diam} G$. For this reason, we also refer to a radio $(d-1)$-coloring as a *radio antipodal coloring* or, more simply, as an *antipodal coloring*. The *antipodal coloring number* $\operatorname{ac}(c)$ of an antipodal coloring c of G is the maximum color assigned to a vertex of G. The *antipodal chromatic number* $\operatorname{ac}(G)$ of G is $\min\{\operatorname{ac}(c)\}$ over all antipodal colorings c of G. An antipodal coloring c of G is a *minimum antipodal coloring* if $\operatorname{ac}(c) = \operatorname{ac}(G)$.

To illustrate these concepts, consider the graph G of Figure 1(a). Since $\operatorname{diam} G = 3$, it follows that in any antipodal coloring of G, the colors of every two adjacent vertices must differ by at least 2, the colors of every two vertices whose distance is 2 must differ by at least 1, and the colors of two antipodal vertices can be the same. Thus the coloring of G given in Figure 1(b) is an antipodal coloring. In fact, it can be verified that the antipodal coloring of G given in Figure 1(b) is a minimum antipodal coloring of G and so $\operatorname{ac}(G) = 5$. The complementary antipodal coloring of c is illustrated in Figure 1(c).



(a) (b) (c)

Figure 1. Minimum antipodal colorings of a graph

2. Antipodal colorings of paths

We now consider antipodal colorings of paths. We denote the path of order n by P_n. Thus $\operatorname{diam} P_n = n - 1$. Antipodal colorings of P_n for $1 \leqslant n \leqslant 6$ are shown in Figure 2. In fact, all these colorings are minimum antipodal colorings of the respective paths. Thus, $\operatorname{ac}(P_1) = 1$, $\operatorname{ac}(P_2) = 1$, $\operatorname{ac}(P_3) = 2$, $\operatorname{ac}(P_4) = 4$, $\operatorname{ac}(P_5) = 7$, and $\operatorname{ac}(P_6) = 11$. We verify $\operatorname{ac}(P_5) = 7$ only.

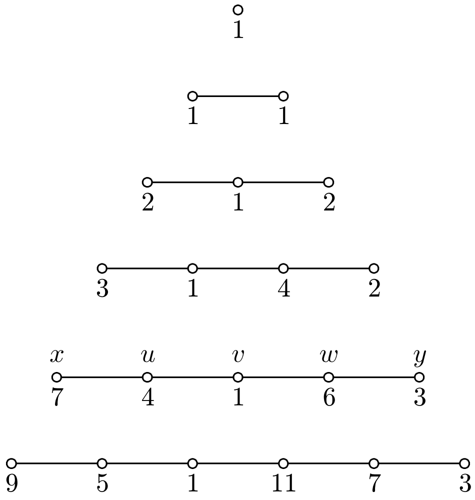


Figure 2. Antipodal colorings of P_n for $1 \leqslant n \leqslant 6$

Example 2.1. $\operatorname{ac}(P_5) = 7$.

Proof. Since the coloring of P_5 shown in Figure 2 has antipodal coloring number 7, $\operatorname{ac}(P_5) \leqslant 7$. Assume, to the contrary, that $\operatorname{ac}(P_5) < 7$. By Lemma 1.1, there is an antipodal coloring c of P_5 with $\operatorname{ac}(c) = 6$. Let $P_5\colon\ x, u, v, w, y$ (shown in Figure 2). We consider the colors of u, v, w. Since the distance between any two of u, v, and w is at most 2 and $\operatorname{diam} P_5 = 4$, it follows that $c(u), c(v)$, and $c(w)$ are distinct. Let $m = \min\{c(u), c(v), c(w)\}$ and $M = \max\{c(u), c(v), c(w)\}$. There are two cases.

Case 1. $m < c(v) < M$. We may assume that $1 \leqslant c(u) < c(v) < c(w) \leqslant 6$. Since $c(v) - c(u) \geqslant 3$ and $c(w) - c(v) \geqslant 3$, it follows that $c(w) \geqslant 7$, which is a contradiction.

Case 2. $m < c(u) < M$. Assume first that $1 \leqslant c(v) < c(u) < c(w) \leqslant 6$. Since $c(u) - c(v) \geqslant 3$ and $c(w) - c(u) \geqslant 3$, it follows that $4 \leqslant c(u) \leqslant 5$. If $c(u) = 4$, then $c(v) = 1$ and consequently there is no color for x. If $c(u) = 5$, then $c(w) = 6$. Since $d(u, w) = 2$, it follows that $d(u, w) + |c(w) - c(u)| = 3$, which is a contradiction. Next assume that $1 \leqslant c(w) < c(u) < c(v) \leqslant 6$. Then $c(u) \geqslant 3$. On the other hand, since $d(u,v)=1$ and $c(v)\leqslant 6$, it follows that $c(u)\leqslant 3$. Thus $c(u)=3$. Necessarily, $c(w)=1$ and $c(v)=6$. Thus there is no color for x. $\square$

In order to provide additional information about the antipodal chromatic numbers of paths, we present a sufficient condition for the antipodal chromatic number of a connected subgraph of a connected graph G of diameter d to be bounded above by $\mathrm{ac}(G)$.

Theorem 2.2. *Let G be a connected graph and H a connected subgraph of G. If $d_G(u,v)-d_H(u,v)\leqslant \operatorname{diam} G-\operatorname{diam} H$ for all $u,v\in V(H)$, then $\mathrm{ac}(H)\leqslant \mathrm{ac}(G)$.*

Proof. Let c be an antipodal coloring of G and $u,v\in V(H)$. Since $d_G(u,v)-d_H(u,v)\leqslant \operatorname{diam} G-\operatorname{diam} H$, it follows that

$$|c(u)-c(v)|\geqslant \operatorname{diam} G-d_G(u,v)\geqslant \operatorname{diam} H-d_H(u,v),$$

so $d_H(u,v)+|c(u)-c(v)|\geqslant \operatorname{diam} H$. Hence the restriction of c to H is an antipodal coloring of H and so $\mathrm{ac}(H)\leqslant \mathrm{ac}(G)$. $\square$

With an additional restriction on the subgraph H in Theorem 2.2, the conclusion can be strengthened. A subgraph H of a connected G is *distance-preserving* if $d_H(u,v)=d_G(u,v)$ for all $u,v\in V(H)$.

Theorem 2.3. *Let H be a distance-preserving connected subgraph of a connected graph G. If $\operatorname{diam} H<\operatorname{diam} G$, then $\mathrm{ac}(H)<\mathrm{ac}(G)$.*

Proof. Let c be an antipodal coloring of G. Let $W\subseteq V(G)$ such that $c(w)=1$ for all $w\in W$. If $W\cap V(H)=\emptyset$, then the result is immediate. So we may assume that $W\cap V(H)\neq\emptyset$. Define a function $c'\colon V(H)\to\mathbb{N}$ by

$$c'(v)=\begin{cases} c(v)-1 & \text{if } v\notin W,\\ 1 & \text{if } v\in W.\end{cases}$$

We now show that c' is an antipodal coloring of H. Let $u,v\in V(H)$. If neither u nor v is in W, then $|c'(u)-c'(v)|=|c(u)-c(v)|$. Since H is distance-preserving and $\operatorname{diam} G-\operatorname{diam} H>0$, it follows that $d_G(u,v)-d_H(u,v)=0<\operatorname{diam} G-\operatorname{diam} H$. Thus $|c'(u)-c'(v)|=|c(u)-c(v)|\geqslant \operatorname{diam} G-d_G(u,v)\geqslant \operatorname{diam} H-d_H(u,v)$. If $u,v\in W$, then $d_G(u,v)=\operatorname{diam} G$. Since H is distance-preserving, $d_G(u,v)=d_H(u,v)$, implying that $\operatorname{diam} G=\operatorname{diam} H$, which contradicts the fact that $\operatorname{diam} G>\operatorname{diam} H$. So we assume that exactly one of u,v is in W, say $u\in W$ and $v\notin W$. Since $\operatorname{diam} G\geqslant \operatorname{diam} H+1$ and H is distance-preserving,

$$\begin{aligned}|c'(v)-c'(w)|&=|c(v)-1|-1\geqslant \operatorname{diam} G-d_G(v,w)-1\\&\geqslant (\operatorname{diam} H+1)-d_H(v,w)-1=\operatorname{diam} H+d_H(v,w).\end{aligned}$$

Thus, c' is an antipodal coloring of H with $\mathrm{ac}(c') < \mathrm{ac}(c)$. Therefore, $\mathrm{ac}(H) < \mathrm{ac}(G)$. $\square$

Let $n \geqslant 4$. Since P_n is a distance-preserving subgraph of P_{n+1} and $\operatorname{diam} P_{n+1} > \operatorname{diam} P_n$, the following corollary is an immediate consequence of Theorem 2.3.

Corollary 2.4. *Let $n \geqslant 4$. Then $\mathrm{ac}(P_n) < \mathrm{ac}(P_{n+1})$.*

We now establish an upper bound for the antipodal chromatic number of paths.

Theorem 2.5. *For every integer $n \geqslant 1$,*

$$\mathrm{ac}(P_n) \leqslant \binom{n-1}{2} + 1.$$

Proof. Since $\mathrm{ac}(P_1) = \mathrm{ac}(P_2) = 1$, $\mathrm{ac}(P_3) = 2$, $\mathrm{ac}(P_4) = 4$, $\mathrm{ac}(P_5) = 7$, and $\mathrm{ac}(P_6) = 11$, it actually follows that $\mathrm{ac}(P_n) = \binom{n-1}{2} + 1$ for $1 \leqslant n \leqslant 6$. So we may assume that $n \geqslant 7$. Let $P_n \colon v_1, v_2, \ldots, v_n$. We consider two cases, according to whether n is even or n is odd.

Case 1. $n = 2k$ is even for some integer $k \geqslant 4$. Define a coloring c of P_{2k} by

$$c(v_i) = 1 + (k-i)(n-2)$$

for all i with $1 \leqslant i \leqslant k$ and

$$c(v_{k+j}) = \binom{n-1}{2} + 1 - (j-1)(n-2)$$

for all j with $1 \leqslant j \leqslant k$. Then $c(v_k) = 1$, $c(v_{k+1}) = \binom{n-1}{2} + 1$,

$$c(v_1) > c(v_2) > \ldots > c(v_k), \text{ and } c(v_{k+1}) > c(v_{k+2}) > \ldots > c(v_{2k}).$$

Since $c(v_1) = 1 + (k-1)(n-2) = 1 + (n-2)^2/2 < \binom{n-1}{2} + 1 = c(v_{k+1})$, it follows that $\mathrm{ac}(c) = c(v_{k+1}) = \binom{n-1}{2} + 1$. Next we show that c is an antipodal coloring of P_{2k}. Let $u, v \in V(P_{2k})$. If $u, v \in \{v_1, v_2, \ldots, v_k\}$ or $u, v \in \{v_{k+1}, v_{k+2}, \ldots, v_{2k}\}$, then $|c(u) - c(v)| \geqslant n-2$ by the definition of c. So we assume that $u \in \{v_1, v_2, \ldots, v_k\}$ and $v \in \{v_{k+1}, v_{k+2}, \ldots, v_{2k}\}$. Let $u = v_i$ for some i with $1 \leqslant i \leqslant k$. The only possibilities for a vertex v for which $|c(u) - c(v)| < n-3$ are $v = v_{i+k}$ or $v = v_{i+k+1}$ (where the latter situation occurs only if $1 \leqslant i < k$), in which case $|c(u) - c(v)| = k-1$. However, in these cases, $|c(u) - c(v)| + d(u, v) \geqslant (k-1) + k = n-1 = \operatorname{diam} P_n$.

Case 2. $n = 2k + 1$ is odd for some integer $k \geqslant 3$. Define a coloring c of P_{2k+1} by

$$c(v_i) = \binom{n-1}{2} + 1 - (i-1)(n-2)$$

for all i with $1 \leqslant i \leqslant k+1$ and

$$c(v_{k+j+1}) = \binom{n-1}{2} + 2 - k - (j-1)(n-2)$$

for all j with $1 \leqslant j \leqslant k$. Then $\mathrm{ac}(c) = c(v_1) = \binom{n-1}{2} + 1$. Next we show that c is an antipodal coloring of P_{2k+1}. Let $u, v \in V(P_{2k+1})$. If $u, v \in \{v_1, v_2, \ldots, v_{k+1}\}$ or $u, v \in \{v_{k+2}, v_{k+3}, \ldots, v_{2k+1}\}$, then $|c(u) - c(v)| \geqslant n - 2$. So we assume that $u \in \{v_1, v_2, \ldots, v_{k+1}\}$ and $v \in \{v_{k+2}, v_{k+3}, \ldots, v_{2k+1}\}$. Let $u = v_i$ for some i with $1 \leqslant i \leqslant k+1$. The only possibilities for a vertex v for which $|c(u) - c(v)| < n - 3$ are $v = v_{i+k}$ or $v = v_{i+k+1}$ (where the former case can occur only if $2 \leqslant i \leqslant k+1$ and the latter case occurs only if $1 \leqslant i \leqslant k$). If $v = v_{i+k}$, then $|c(u) - c(v)| = d(u, v) = k$; while if $v = v_{i+k+1}$, then $|c(u) - c(v)| = k - 1$ and $d(u, v) = k + 1$. Thus, in each case, $d(u, v) + |c(u) - c(v)| = 2k = n - 1$. Therefore, c is an antipodal coloring of P_n for n odd and $\mathrm{ac}(P_n) \leqslant \binom{n-1}{2} + 1$. $\square$

Although we have shown only that $\mathrm{ac}(P_n) \leqslant \binom{n-1}{2} + 1$ for all $n \geqslant 1$, with equality holding for $1 \leqslant n \leqslant 6$, there is reason to believe that we may have equality for all n. We state this as a conjecture.

Conjecture 2.6. *For every positive integer n,*

$$\mathrm{ac}(P_n) = \binom{n-1}{2} + 1.$$

Regardless of whether this conjecture is true or not, it is useful to know the values of $\mathrm{ac}(P_n)$ for every positive integer n.

3. Bounds for radio antipodal chromatic numbers

Let G be a connected graph of order n and diameter d and let $V(G) = \{v_1, v_2, \ldots, v_n\}$, where $d(v_1, v_n) = d$. Then the coloring c of G defined by $c(v_1) = c(v_n) = 1$ and $c(v_i) = 1 + (i-1)(d-1)$ $(2 \leqslant i \leqslant n-1)$ is an antipodal coloring of G with $\mathrm{ac}(c) = 1 + (n-2)(d-1)$. This observation yields the following upper bound for $\mathrm{ac}(G)$ in terms of the order and diameter of G.

Observation 3.1. *If G is a connected graph of order n and diameter d, then*

$$\operatorname{ac}(G) \leqslant (n-2)(d-1)+1.$$

Since the antipodal chromatic number of a connected graph of diameter 2 is the chromatic number of G, we will restrict our attention to connected graphs of diameter at least 3.

If G is a graph containing an edge e such that $\operatorname{diam} G = \operatorname{diam}(G-e)$, then every antipodal coloring of G is an antipodal coloring of $G-e$. This observation yields the following.

Proposition 3.2. *If G is a graph containing an edge e such that* $\operatorname{diam} G = \operatorname{diam}(G-e)$*, then* $\operatorname{ac}(G-e) \leqslant \operatorname{ac}(G)$.

Theorem 3.3. *If G is a connected graph of order n and diameter d, then*

$$\operatorname{ac}(P_{d+1}) \leqslant \operatorname{ac}(G) \leqslant \operatorname{ac}(P_{d+1}) + (d-1)(n-d-1).$$

P r o o f. Let $P_{d+1}\colon v_1, v_2, \ldots, v_{d+1}$ be a geodesic in G. We first verify the lower bound. Let c be an antipodal coloring of G. Then an antipodal coloring c' of P_{d+1} can be obtained from c by defining $c'(v_i) = c(v_i)$ for all $1 \leqslant i \leqslant d+1$. Since $\operatorname{ac}(c) \geqslant \operatorname{ac}(c') \geqslant \operatorname{ac}(P_{d+1})$, it follows that $\operatorname{ac}(G) \geqslant \operatorname{ac}(P_{d+1})$.

For the upper bound, let c'' be an antipodal coloring of P_{d+1} with $\operatorname{rn}(c'') = \operatorname{ac}(P_{d+1})$. Let $V(G) - V(P_{d+1}) = \{u_1, u_2, \ldots, u_{n-d-1}\}$. Define a coloring c of G by $c(v_i) = c''(v_i)$ for $1 \leqslant i \leqslant d+1$ and $c(u_j) = \operatorname{ac}(P_{d+1}) + (d-1)j$ for $1 \leqslant j \leqslant n-d-1$. Since c is an antipodal coloring of G, it follows that

$$\operatorname{ac}(G) \leqslant \operatorname{ac}(c) = c(u_{n-d-1}) = \operatorname{ac}(P_{d+1}) + (d-1)(n-d-1)$$

as desired. □

The following is a consequence of Theorems 2.5 and 3.3.

Corollary 3.4. *If G is a connected graph of order n and diameter d, then*

$$\operatorname{ac}(G) \leqslant \frac{(d-1)(2n-d-2)}{2} + 1.$$

Next we present a lower bound for $\operatorname{ac}(G)$ for a connected graph G in terms of its diameter only. For a set S of vertices of a connected graph G, we define the *diameter of a set S* by

$$\operatorname{diam}(S) = \max\{d(u,v)\colon\ u, v \in S\}.$$

Thus $\operatorname{diam}(V(G)) = \operatorname{diam} G$.

Lemma 3.5. *For every connected graph G,*

$$\operatorname{ac}(G) \geqslant 1 + \max_{S \subseteq V(G)} \{(|S| - 1)(\operatorname{diam} G - \operatorname{diam}(S))\}.$$

P r o o f. Let G be a connected graph with $\operatorname{diam} G = d$, let c be an antipodal coloring of G, and let $S = \{v_1, v_2, \ldots, v_k\} \subseteq V(G)$. It then suffices to show that

$$\operatorname{ac}(c) \geqslant 1 + (|S| - 1)(d - \operatorname{diam}(S)). \tag{1}$$

We may assume that $c(v_1) \leqslant c(v_2) \leqslant \ldots \leqslant c(v_k)$. Since $\operatorname{ac}(c) \geqslant 1$, it follows that (1) holds for $k = 1$; so we may assume that $k \geqslant 2$. Since $c(v_{i+1}) - c(v_i) \geqslant d - d(v_{i+1}, v_i) \geqslant d - \operatorname{diam}(S)$ for all i with $1 \leqslant i \leqslant k - 1$, it follows that

$$c(v_k) \geqslant c(v_1) + (k - 1)(d - \operatorname{diam}(S)) \geqslant 1 + (k - 1)(d - \operatorname{diam}(S)).$$

Thus $\operatorname{ac}(c) \geqslant c(v_k) \geqslant 1 + (k - 1)(d - \operatorname{diam}(S))$, as desired. □

Proposition 3.6. *If G is a connected graph of diameter $d \geqslant 2$, then*

$$\operatorname{ac}(G) \geqslant 1 + \frac{d^2}{4}.$$

P r o o f. Let u and v be two antipodal vertices of G, that is, $d(u, v) = d$, and let $P\colon\ u = v_1, v_2, \ldots, v_{d+1} = v$ be a u–v geodesic. For each integer k with $1 \leqslant k \leqslant d+1$, let P_k be the subpath $v_1, v_2, \ldots, v_k$ of P. Since P is a u–v geodesic, $d(v_i, v_j) = |i - j|$ for all $v_i, v_j \in V(P_k)$ and so $d(V(P_k)) = k - 1$. Define $f(k) = 1 + (k - 1)(d - (k - 1))$, where $1 \leqslant k \leqslant d + 1$. It then follows from Lemma 3.5 that

$$\operatorname{ac}(G) \geqslant \max\{f(k)\colon\ 1 \leqslant k \leqslant d + 1\} = f\Big(\frac{d}{2} + 1\Big)$$

and we obtain the desired result. □

The lower bound for the antipodal chromatic number of a graph G given in Proposition 3.6 in terms of the diameter of G does not appear to be especially good, suggesting that knowing only the diameter of G is not sufficient to provide potentially useful information about $\operatorname{ac}(G)$. For this reason, we now present lower bounds for $\operatorname{ac}(G)$ in terms of the diameter and (1) the maximum degree of G and (2) the clique number of G, beginning with the first of these.

Theorem 3.7. *If G is a connected graph of diameter $d \geqslant 3$ and maximum degree Δ, then*

$$\operatorname{ac}(G) \geqslant 2 + \Delta(d - 2).$$

P r o o f. Let c be an antipodal coloring of G. Let v be a vertex of degree Δ in G and let $u_1, u_2, \ldots, u_\Delta$ be the neighbors of v, where $c(u_1) \leqslant c(u_2) \leqslant \ldots \leqslant c(u_\Delta)$. Since $d(v, u_i) = 1$ for all i, $1 \leqslant i \leqslant \Delta$, it follows that $|c(v) - c(u_i)| \geqslant d-1$. Moreover, since $d(u_i, u_j) \leqslant 2$ for each pair i, j with $1 \leqslant i \neq j \leqslant \Delta$, we have $|c(u_i) - c(u_j)| \geqslant d - 2$. We now consider three cases, according to how $c(v)$ compares with the numbers $c(u_1), c(u_2), \ldots, c(u_\Delta)$.

Case 1. $c(v) < c(u_1)$. Then

$$\mathrm{ac}(c) \geqslant c(u_\Delta) \geqslant c(v) + (d-1) + (\Delta - 1)(d-2) \geqslant 2 + \Delta(d-2). \tag{2}$$

Case 2. $c(v) > c(u_\Delta)$. Here too we obtain $\mathrm{rn}(c) \geqslant 2 + \Delta(d-2)$ by interchanging $c(u_\Delta)$ and $c(v)$ in (2).

Case 3. $c(u_1) < c(v) < c(u_\Delta)$. Then there is a positive integer $t < \Delta$ such that $c(u_t) < c(v) < c(u_{t+1})$. Observe that

$$\begin{aligned} c(u_\Delta) - c(u_1) &= \sum_{i=1}^{t-1} [c(u_{i+1}) - c(u_i)] + [c(v) - c(u_t)] + [c(u_t) - c(v)] \\ &\quad + \sum_{i=t+1}^{\Delta-1} [c(u_{i+1}) - c(u_i)] \\ &\geqslant (t-1)(d-2) + 2(d-1) + (\Delta - t - 1)(d-2). \end{aligned}$$

Thus

$$\begin{aligned} \mathrm{ac}(c) \geqslant c(u_\Delta) &\geqslant 1 + (t-1)(d-2) + 2(d-1) + (\Delta - t - 1)(d-2) \\ &= 2d - 1 + (\Delta - 2)(d-2) = 3 + \Delta(d-2). \end{aligned}$$

Therefore, $\mathrm{ac}(G) \geqslant 2 + \Delta(d-2)$. □

The bound in Theorem 3.7 is sharp for graphs with diameter 3. By Theorem 3.7, if G is a connected graph of diameter 3 and maximum degree Δ, then $\mathrm{ac}(G) \geqslant 2+\Delta$. Let G be obtained from the path $P_3\colon\ u, v, w$ by adding $a \geqslant 1$ new vertices $w_1, w_2, \ldots, w_a$ and joining each w_i, $1 \leqslant i \leqslant a$, to the vertex w. Then the maximum degree Δ of G is $a+1$ and the diameter of G is 3. Define a coloring $c\colon V(G) \to \mathbb{N}$ by $c(u) = 2$, $c(v) = 4$, $c(w) = 1$, $c(w_1) = 3$, and $c(w_i) = 5 + (i-2)$ for all i with $2 \leqslant i \leqslant a$. The graph G and the coloring c are shown in Figure 3. It can be verified that c is an antipodal coloring of G. If $a = 1$, then G is the path P_4 and $\mathrm{ac}(c) = c(v) = 4$. If $a \geqslant 2$, then $\mathrm{ac}(c) = c(w_a) = 2+\Delta$. Thus in all cases, $\mathrm{ac}(c) = 2+\Delta$; so $\mathrm{ac}(G) = 2+\Delta$ by Theorem 3.7.

If G is a connected graph of diameter 4 and maximum degree 2, then $G = P_5, C_8, C_9$ and $\mathrm{ac}(G) \geqslant 6$ by Theorem 3.7. We have seen that $\mathrm{ac}(P_5) = 7$ and it

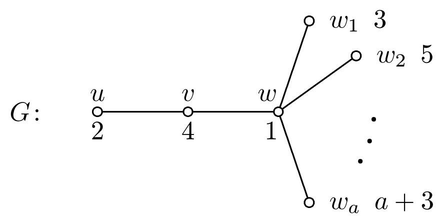


Figure 3. A graph G with diameter 3 and $\mathrm{ac}(G) = 2 + \Delta$

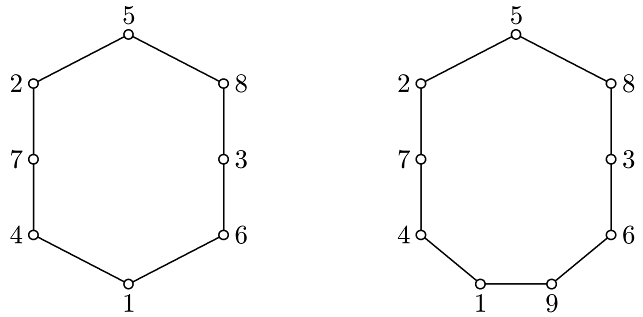


Figure 4. Minimum antipodal colorings of C_8 and C_9

can be shown that $\mathrm{ac}(C_8) = 8$ and $\mathrm{ac}(C_9) = 9$. Minimum antipodal colorings of C_8 and C_9 are shown in Figure 4.

Hence the bound in Theorem 3.7 is not sharp for graphs with diameter 4 and maximum degree 2. However, the bound in Theorem 3.7 is sharp for graphs with diameter 4 and maximum degree at least 3. By Theorem 3.7, if G is a connected graph of diameter 4 and maximum degree Δ, then $\mathrm{ac}(G) \geqslant 2+2\Delta$. Let G be obtained from the path $P_4\colon\ u, v, w, x, y$ by adding $a \geqslant 1$ new vertices $w_1, w_2, \ldots, w_a$ and joining each w_i, $1 \leqslant i \leqslant a$, to the vertex w. Then the maximum degree Δ of G is $a + 2 \geqslant 3$ and the diameter of G is 4. Define a coloring $c\colon V(G) \to \mathbb{N}$ by $c(u) = 7$, $c(v) = 4$, $c(w) = 1$, $c(x) = 6$, $c(y) = 3$, and $c(w_i) = 8 + 2(i - 1)$ for all i with $1 \leqslant i \leqslant a$. The graph G and the coloring c are shown in Figure 5. The coloring c is an antipodal coloring of G with $\mathrm{ac}(c) = \mathrm{ac}(w_a) = 8+(a-1)2 = 8+2(\Delta-3) = 2+2\Delta$. Therefore, $\mathrm{ac}(G) = 2 + 2\Delta$ by Theorem 3.7.

The lower bound for $\mathrm{ac}(G)$ does not appear to be sharp for connected graphs of diameter 5 or more and so an improved bound for such graphs is anticipated. The *clique number* $\omega(G)$ is the maximum order of a complete subgraph of G.

Theorem 3.8. *Let G be a connected graph of diameter $d \geqslant 3$ and clique number ω. Then*

$$\mathrm{ac}(G) \geqslant 1 + (d - 1)(\omega - 1).$$

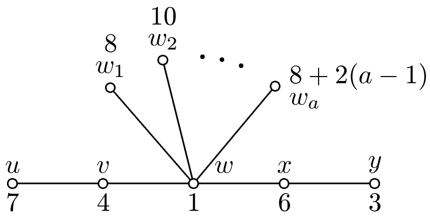


Figure 5. A graph G with diameter 4 and $\mathrm{ac}(G) = 2 + 2\Delta$

P r o o f. Let $U = \{u_1, u_2, \ldots, u_\omega\}$ be the set of vertices in some complete subgraph of G and let c be an antipodal coloring of G. We may assume that $c(u_1) \leqslant c(u_2) \leqslant \ldots \leqslant c(u_\omega)$. Since $d(u_i, u_j) = 1$ for each pair i, j of integers with $1 \leqslant i \neq j \leqslant \omega$, it follows that $|c(u_i) - c(u_j)| \geqslant d-1$. Hence $\mathrm{ac}(G) \geqslant \mathrm{ac}(c) \geqslant 1 + (d-1)(\omega-1)$. $\square$

We now consider a class of graphs of diameter 3. The *corona* $\mathrm{cor}(G)$ of a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$ is that graph of order $2n$ obtained from G by adding n new vertices $u_1, u_2, \ldots, u_n$ and the n new edges u_iv_i $(1 \leqslant i \leqslant n)$. We now determine $\mathrm{ac}(\mathrm{cor}(K_n))$ for all $n \geqslant 3$.

For $n \geqslant 4$, define a coloring c of $\mathrm{cor}(K_n)$ by letting $c(v_i) = 1 + 2(i-1)$ for all i with $1 \leqslant i \leqslant n$ and letting $c(u_1) = c(u_2) = 2n-2$ and $c(u_i) = 2 + 2(i-3)$ for all $3 \leqslant i \leqslant n$. Since c is an antipodal coloring of $\mathrm{cor}(K_n)$ with $\mathrm{ac}(c) = c(v_n) = 2n-1$, it follows that $\mathrm{ac}(\mathrm{cor}(K_n)) \leqslant 2n-1$. On the other hand, $\mathrm{ac}(\mathrm{cor}(K_n)) \geqslant 2n-1$ by Theorem 3.8. Thus $\mathrm{ac}(\mathrm{cor}(K_n)) = 2n-1$ for $n \geqslant 4$. Figure 6 shows an antipodal coloring c of $\mathrm{cor}(K_3)$ with $\mathrm{ac}(c) = 6$ and so $\mathrm{ac}(\mathrm{cor}(K_3)) \leqslant 6$.

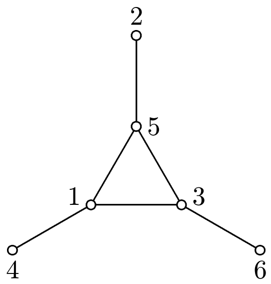


Figure 6. An antipodal coloring of $\mathrm{cor}(K_3)$

We show, in fact, that $\mathrm{ac}(\mathrm{cor}(K_3)) = 6$. If this were not the case, then there exists an antipodal coloring c' of $\mathrm{cor}(K_3)$ with $\mathrm{ac}(c') = 5$. Necessarily, then, the vertices of K_3 are colored 1, 3, and 5 by the coloring c'. However, then the end-vertex adjacent to the vertex colored 3 cannot be colored by any of the colors from the set $\{1, 2, \ldots, 5\}$, which is a contradiction.

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