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COMPLETELY SUBDIRECT PRODUCTS OF DIRECTED SETS

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Abstract. This paper deals with directly indecomposable direct factors of a directed set.

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The notion of the completely subdirect product of lattice ordered groups was introduced in [7] and has been used in several articles.

Let L be a directed set and let s^0 be a fixed element of L.

In this paper we define the completely subdirect product decomposition of L with the central element s^0.

In all considerations we are dealing with a fixed s^0; therefore we will often omit the words "with the central element s^0".

If $\{L_i\}_{i\in I}$ is a system of subsets of L, then this system is partially ordered by the set-theoretical inclusion.

We prove the following result (for definitions, cf. Section 1 below):

(A_1) Let L and s^0 be as above. Suppose that there exists a linearly ordered system $\{L_i\}_{i\in I}$ of intervals of L such that
- (i) for each $i \in I$, s^0 belongs to L_i and L_i is a completely subdirect product of directly indecomposable direct factors;
- (ii) $\bigcup_{i\in I} L_i = L$.

Then L is a completely subdirect product of directly indecomposable direct factors.

The results and methods from [5] will be applied.

For related results cf. [2], [3], [4] and [6].

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The relations between completely subdirect product decompositions of directed sets and completely subdirect product decompositions of directed groups will be investigated.

1.

Throughout the paper we suppose that L is a directed set and that s^0 is a fixed element of L.

The notion of the internal direct product decomposition (with the central element s^0) of L will be used in the same sense as in [5]. We suppose that the reader is acquainted with the definitions and the notation from Section 2 in [5].

When considering an internal direct product decomposition of L or of a directed subset of L we always assume that the corresponding central element is s^0.

We recall that $D(L)$ is the set of all internal direct factors of L. Under the partial order defined by the set-theoretical inclusion, $D(L)$ is a Boolean algebra. For $A \in D(L)$ we denote by A' the complement of A in $D(L)$; then $L = A \times A'$.

The Boolean algebra $D(L)$ will be called atomic if for each $A \in D(L)$ with $\operatorname{card} A > 1$ there exists an atom A_0 of $D(L)$ such that $A_0 \leqslant A$.

If $x \in L$ and $A \in D(L)$, then $x(A)$ denotes the component of x in the internal direct factor A.

1.1. Definition. Let L and s^0 be as above. Further, let $\{A_i\}_{i\in I}$ be an indexed system of directed sets such that

1) $A_i \in D(L)$ for each $i \in I$;
2) if $x, y \in L$ and $x(A_i) \leqslant y(A_i)$ for each $i \in I$, then $x \leqslant y$;
3) if i and $i(1)$ are distinct elements of I, then $A_{i(1)} \subseteq A_i'$.

Under these assumptions L is said to be a completely subdirect product of the system $\{A_i\}_{i\in I}$ and we express this fact by writing

$$L = (\mathrm{cs}) \prod_{i\in I} A_i. \tag{1}$$

The relation (1) is called a completely subdirect product decomposition of L.

1.2. Example. Suppose that L^0 is a directed set, $s^0 \in L^0$ and

$$L^0 = \prod_{i\in I} B_i, \tag{2}$$

so that $s^0 \in B_i$ and $\operatorname{card} B_i > 1$ for each $i \in I$. Let α be an infinite cardinal, $\operatorname{card} I > \alpha$. For $x \in L^0$ we put

$$\operatorname{supp}(x) = \{i \in I \colon\, x(B_i) \neq s^0\}.$$

Further, we denote

$$L = \{x \in L^0 \colon\, \operatorname{supp}(x) \leqslant \alpha\}.$$

Then we have

$$L^0 \neq L = (\mathrm{cs}) \prod_{i \in I} B_i. \tag{3}$$

In 1.2, L is a convex subset of L^0. The following example shows that this need not be the case in general for analogous situations.

1.3. Example. Let $\mathbb{R}$ be the set of all reals and let L^0 be the set of all real functions defined on $\mathbb{R}$, with the partial order defined coordinatewise. Let $s^0 \in L^0$ be such that $s^0(t) = 0$ for each $t \in \mathbb{R}$. For $i \in \mathbb{R}$ we put

$$B_i = \{f \in L^0 \colon\, f(t) = 0 \quad \text{for each } t \in \mathbb{R} \setminus \{i\}\}.$$

Then (2) is valid. We denote by L_1 the set of all $f \in L^0$ having only a finite number of points of discontinuity. Then the relation

$$L_1 = (\mathrm{cs}) \prod_{i \in I} B_i$$

holds and L_1 is not a convex subset of L^0.

Theorem (A_1) generalizes Theorem (A) of [5] concerning direct product decompositions of a directed set.

In order to have the possibility to compare (A_1) and (A) we consider the following conditions for L:

(α_1) L is an internal direct product of directly indecomposable direct factors.

(α_2) L is a completely subdirect product of directly indecomposable direct factors.

(α_3) The Boolean algebra $D(L)$ is atomic.

(α_4) There exists a linearly ordered system $\{L_i\}_{i \in I}$ of intervals of L such that $\bigcup_{i \in I} L_i = L$ and each L_i satisfies (α_1).

(α_5) There exists a linearly ordered system $\{L_i\}_{i \in I}$ of intervals of L such that $\bigcup_{i \in I} L_i = L$ and each L_i satisfies (α_2).

By using this notation, Theorem (A) of [5] can be expressed as follows:

(A) $(\alpha_4) \Rightarrow (\alpha_3)$.

Similarly, (A_1) can be written in the form

(A_1) $(\alpha_5) \Rightarrow (\alpha_2)$.

It is clear that $(\alpha_1) \Rightarrow (\alpha_2)$ and hence $(\alpha_4) \Rightarrow (\alpha_5)$. Below (cf. Lemma 2.3) we shall show that $(\alpha_2) \Rightarrow (\alpha_3)$. Hence (A) is a particular case of (A_1).

2.

Let us have two completely subdirect product decompositions

$$L = (\mathrm{cs}) \prod_{i \in I} A_i, \tag{1}$$

$$L = (\mathrm{cs}) \prod_{j \in J} B_j. \tag{2}$$

We say that (2) is a refinement of (1) if for each $i \in I$ there exists a subset $J(i)$ of J such that

$$A_i = (\mathrm{cs}) \prod_{j \in J(i)} B_j.$$

Now let (1) and (2) be arbitrary (i.e., we do not suppose that (2) is a refinement of (1)). For each $i \in I$ and $j \in J$ we put

$$A_i \cap B_j = C_{ij}.$$

Then (by applying the results from [5]) we obtain

$$\begin{aligned} &L = A_1 \times A_i', \quad L = B_j \times B_j', \\ &L = (A_i \cap B_j) \times (A_i \cap B_j') \times (A_i' \cap B_j) \times (A_i' \cap B_j'). \end{aligned}$$

Thus $C_{ij} \in D(L)$. Hence the condition 1) from 1.1. holds for C_{ij}. Moreover,

$$C_{ij}' = (A_i \cap B_j') \times (A_i' \cap B_j) \times (A_i' \cap B_j'). \tag{3}$$

Let $x, y \in L$ and suppose that $x(C_{ij}) \leqslant y(C_{ij})$ is valid for each $(i,j) \in I \times J$. We have (cf. [5])

$$x(C_{ij}) = (x(A_i))(B_j).$$

To simplify the notation we shall write $x(A_i)(B_j)$ rather than $(x(A_i))(B_j)$.

Thus, if i is fixed, then

$$x(A_i)(B_j) \leqslant y(A_i)(B_j)$$

is valid for each $j \in J$. Therefore in view of 1.1

$$x(A_i) \leqslant y(A_i).$$

By using 1.1 again we get $x \leqslant y$. We have verified the validity of the condition 2) from 1.1 for the system $\{C_{ij}\}_{(i,j)\in I\times J}$.

Let (i, j) and $(i(1), j(1))$ be distinct elements of the set $I \times J$. Without loss of generality we can suppose that $i \neq i(1)$. Thus according to 1.1 we get

$$A_{i(1)} \subseteq A'_i.$$

Now we distinguish two cases. If $j = j(1)$, then $C_{i(1)j(1)} \subseteq A'_i \cap B_j$ and hence in view of (3), $C_{i(1)j(1)} \subseteq C'_{ij}$. If $j \neq j(1)$, then $B_{j(1)} \subseteq B'_j$, whence

$$C_{i(1)j(1)} \subseteq A'_i \cap B'_j$$

and by using (3) again we infer that also in this case we have $C_{i(1)j(1)} \subseteq C'_{ij}$. Thus the condition 3) from 1.1 holds. Therefore

$$L = (\mathrm{cs}) \prod_{(i,j)\in I\times J} C_{ij}. \tag{4}$$

Let i be a fixed element of I. We intend to prove that the relation

$$A_i = (\mathrm{cs}) \prod_{j\in J} C_{ij} \tag{5}$$

is valid. Again, we assume the conditions 1), 2) and 3) from 1.1. In view of 2.7, [5], for each $j \in J$ we have

$$A_i = (A_i \cap B_j) \times (A_i \cap B'_j).$$

Hence C_{ij} belongs to $D(A_i)$ and its complement in $D(A_i)$ is $A_i \cap B'_j$.

Let $x, y \in A_i$ and suppose that $x(C_{ij}) \leqslant y(C_{ij})$ for each $j \in J$. This means that

$$x(A_i)(B_j) \leqslant y(A_i)(B_j)$$

for each $j \in J$. But in view of the relations $x \in A_i$ and $y \in A_i$ we get $x(A_i) = x$, $y(A_i) = y$, whence $x(B_j) \leqslant y(B_j)$ for each $j \in J$. Therefore $x \leqslant y$.

Let j and $j(1)$ be distinct elements of J. Then $B_{j(1)} \subseteq B'_j$, hence

$$C_{ij(1)} = A_i \cap B_{j(1)} \subseteq A_i \cap B'_j.$$

Thus $C_{ij(1)}$ is a subset of the complement of C_{ij} in $D(A_i)$. We have verified that (5) holds. Analogously we can verify

$$B_j = (\mathrm{cs}) \prod_{i \in I} C_{ij}. \tag{6}$$

By summarizing we obtain

2.1. Proposition. *Let* (1) *and* (2) *be completely subdirect product decompositions of* L. *Then* (5) *is also a completely subdirect product decomposition of* L. *Moreover,* (5) *is a refinement of both* (1) *and* (2).

2.2. Lemma. *Let* (1) *be valid and let* C *be an interval of* L, $s^0 \in C$. *Put* $C_i = A_i \cap C$ *for each* $i \in I$. *Then*

$$C = (\mathrm{cs}) \prod_{i \in I} C_i. \tag{7}$$

Proof. Let $i \in I$. Then $L = A_i \times A'_i$. Thus in view of [5], Lemma 2.8 we have

$$C = (A_i \cap C) \times (A'_i \cap C).$$

Hence $C_i \in D(C)$. Moreover, the complement C'_i of C_i in $D(C)$ is $A'_i \cap C$.

If $x \in C$, then by applying 2.8 of [5] again we get that the relation

$$x(C_i) = x(A_i) \tag{8}$$

is valid.

Let $x, y \in C$ and suppose that $x(C_i) \leqslant y(C_i)$ holds for each $i \in I$. Hence $x(A_i) \leqslant y(A_i)$ for each $i \in I$ and thus $x \leqslant y$.

Let i and $i(1)$ be distinct elements of I. Then

$$C_{i(1)} = A_{i(1)} \cap C \subseteq A'_i \cap C = C'_i.$$

Therefore according to 1.1 the relation (7) holds. $\square$

Let (α_2) and (α_3) be as in Section 2.

2.3. Lemma. $(\alpha_2) \Rightarrow (\alpha_3)$.

Proof. Suppose that (1) holds and that all A_i are directly indecomposable. The case $L = \{s^0\}$ being trivial we can suppose that $\operatorname{card} L > 1$. Hence we can assume that $\operatorname{card} A_i > 1$ for each $i \in I$. According to 4.11 in [5], all A_i are atoms of $D(L)$. Let $B \in D(L)$ be such that B fails to be the least element of $D(L)$, i.e., $B \neq \{s^0\}$. In view of 2.1 we have

$$B = (\mathrm{cs}) \prod_{i \in I} (A_i \cap B).$$

Hence there exists $i(1) \in I$ such that $A_{i(1)} \cap B \neq \{s^0\}$. At the same time,

$$A_{i(1)} = (A_{i(1)} \cap B) \times (A_{i(1)} \cap B').$$

Since $A_{i(1)}$ is directly indecomposable we infer that $A_{i(1)} \cap B = A_{i(1)}$. Hence $A_{i(1)} \subseteq B$. Thus (α_3) is valid. □

3.

In this section we suppose that the condition (α_5) is satisfied. It suffices to consider the case when $L \neq \{s^0\}$ and $L_i \neq \{s^0\}$ for each $i \in I$. We can also assume that I is linearly ordered and whenever $i(1), i(2) \in I$, $i(1) \leqslant i(2)$, then $L_{i(1)} \subseteq L_{i(2)}$.

Let $i(1) \in I$. There exists a completely subdirect product decomposition

$$L_{i(1)} = (\mathrm{cs}) \prod_{j \in J(i(1))} A_{i(1)j} \tag{1}$$

such that all $A_{i(1)j}$ are directly indecomposable and $A_{i(1)j} \neq \{s^0\}$. Then 2.1 implies that the completely subdirect product decomposition (1) is uniquely determined.

Now we apply the same method as in [5], Section 4 with the distinction that

(a) instead of the relation (1) from [5] we use the relation (1) above;

(b) instead of 2.8 from [5] we use 2.2 above (including the relation (8) from Section 2);

(c) the internal direct product decompositions (e.g. in (10′) and in analogous subsequent places of [5]) are replaced by completely subdirect product decompositions.

In this way we obtain a system $\{C_k\}_{k \in K}$ of directly indecomposable elements of $D(L)$ (cf. [5], 4.8 and 4.10).

Hence the condition 1) from 1.1 is valid for the system $\{C_k\}_{k\in K}$.

Let $x, y \in L$ and $x(C_k) \leqslant y(C_k)$ for each $k \in K$. There exists $i(1) \in I$ such that both x and y belong to $L_{i(1)}$. From the definition of C_k and from 4.4 in [5] we get that

$$x(A_{i(1)j}) \leqslant y(A_{i(1)j})$$

is valid for each $j \in J(i(1))$. Thus $x \leqslant y$. Hence the condition 2) from 1.1 holds for the system $\{C_k\}_{k\in K}$.

Finally, let k and $k(1)$ be distinct elements of K and let $x \in C_{k(1)}$. There exists $i(1) \in I$ with $x \in L_{i(1)}$. In view of the construction of the system $\{C_k\}_{k\in K}$ in [5] we infer that there are j and $j(1)$ in $J(i(1))$ such that

$$A_{i(1)j} = L_{i(1)} \cap C_k, \quad A_{i(1)j(1)} = L_{i(1)} \cap C_{k(1)}.$$

We have $x \in A_{i(1)j(1)}$, thus $x \in A'_{i(1)j}$, where $A'_{i(1)j}$ is the complement of $A_{i(1)}$ in $D(L_{i(1)})$. Then under the notation as in [5], x belongs to C_k^*. According to 4.8 in [5], C_k^* is the complement of C_k in $D(L)$. Hence the condition 3) from 1.1 holds for the system $\{C_k\}_{k\in K}$.

Therefore we obtain

$$L = (\mathrm{cs}) \prod_{k\in K} C_k,$$

completing the proof of (A_1).

4.

For a directed group G we denote the group operation by $+$, though we do not assume that G is abelian; 0 is the neutral element of G.

4.1. Definition. Suppose that

$$\varphi\colon\ G \to \prod_{i\in I} G_i$$

is an isomorphism of a directed group G into the direct product of directed groups G_i $(i \in I)$. Assume that for each $i(1) \in I$ and each $x^{i(1)} \in G_{i(1)}$ there exists $g \in G$ such that

$$\varphi(g)_{i(1)} = x^{i(1)}, \quad \varphi(g)_i = 0 \quad \text{for each } i \in I \setminus \{i(1)\}.$$

Then the morphism φ is said to be a completely subdirect product decomposition of G.

Let G and φ be as in 4.1 and let $i(1) \in I$. We put

$$G^0_{i(1)} = \{g \in G\colon\ \varphi(g) = 0 \quad \text{for each } i \in I \setminus \{i(1)\}\}.$$

Hence $G^0_{i(1)}$ is a directed subgroup of G. For each $g \in G^0_{i(1)}$ we set

$$\varphi_{i(1)}(g) = \varphi(g)_{i(1)}.$$

Then $\varphi_{i(1)}$ is an isomorphism of $G^0_{i(1)}$ onto $G_{i(1)}$. Thus without loss of generality the directed groups $G^0_{i(1)}$ and $G_{i(1)}$ can be identified (such that the element $g \in G^0_{i(1)}$ is identified with $\varphi(g)_{i(1)}$). Under this supposition we write

$$G = (\mathrm{cs}) \prod_{i \in I} G_i. \tag{1}$$

Let (1) be valid and $i(1) \in I$. For $g \in G$ we denote

$$\varphi(g)_{i(1)} = g(i(1)).$$

Then (1) yields that $g(i(1)) \in G$. Further, we denote

$$G^*_{i(1)} = \{g \in G\colon\ g(i(1)) = 0\}.$$

Then $G^*_{i(1)}$ is a convex subgroup of G. We set

$$g^*_{i(1)} = g - g(i(1)).$$

Hence $g^*_{i(1)} \in G^*_{i(1)}$.

Let $x \in G_{i(1)}$ and $y \in G^*_{i(1)}$. Put $g = x + y$. Then we clearly have

$$g(i(1)) = x.$$

From this construction we immediately obtain

4.2. Lemma. *Under the above notation,* $G = G_{i(1)} \times G^*_{i(1)}$.

For each directed group G we denote by $L(G)$ the underlying lattice. Let (1) be valid; put $L(G) = L$, $L(G_i) = L_i$, $L(G^*_i) = L^*_i$ for each $i \in I$, $s^0 = 0$. Then 4.2 yields

4.3. Lemma. $L = L_{i(1)} \times L^*_{i(1)}$ *for each* $i(1) \in I$.

Consider the system $\{L_i\}_{i\in I}$ and the conditions 1), 2), 3) from 1.1. In view of 4.3, this system satisfies the condition 1). From (1) we infer that 2) and 3) are also valid. Thus we have

4.4. Lemma. *Let* (1) *be valid. Then*

$$L(G) = (\mathrm{cs})\prod_{i\in I} L(G_i). \tag{2}$$

4.5. Proposition. *Suppose that G is a directed group, $L = L(G)$, $s^0 = 0$ and that a completely subdirect product decomposition*

$$L = (\mathrm{cs})\prod_{i\in I} L_i \tag{2'}$$

is given. Then for each $i \in I$, L_i is a subgroup of G and (1) *is valid, where $G_i = L_i$.*

Proof. Let $i(1) \in I$. From (2′) we infer that

$$L = L_{i(1)} \times L'_{i(1)}. \tag{3}$$

Thus according to Theorem 3, [1], both $L_{i(1)}$ and $L'_{i(1)}$ are subgroups of the group G and for the directed group G the internal direct product decomposition

$$G = L_{i(1)} \times L'_{i(1)} \tag{4}$$

is valid. Moreover, from the proof of the above mentioned theorem of [1] it follows that for each $x \in G$ the component of x in $L_{i(1)}$ with respect to (3) is the same as the component of x in $L_{i(1)}$ with respect to (4).

Consider the mapping φ of G into $\prod\limits_{i\in I} L_i$ defined by $\varphi(g) = (\dots, g(L_i), \dots)_{i\in I}$. Then φ is a homomorphism with respect to the group operation. In view of (2′), φ is a monomorphism. Therefore φ is an isomorphism with respect to the group operation. This yields that (1) is valid, where $G_i = L_i$ for each $i \in I$. $\square$

5.

We conclude by considering the conditions (α_1)–(α_5) from Section 1; the following implications between them were given above:

$$(\alpha_4) \Rightarrow (\alpha_3), \quad (\alpha_5) \Rightarrow (\alpha_2), \quad (\alpha_1) \Rightarrow (\alpha_2),$$
$$(\alpha_4) \Rightarrow (\alpha_5), \quad (\alpha_2) \Rightarrow (\alpha_3).$$

The natural question arises which of these implications can be reversed.

It is clear that $(\alpha_2) \Rightarrow (\alpha_5)$. Hence in view of (A_1), the conditions (α_2) and (α_5) are equivalent. The following two examples show that this is the only such case.

5.1. Example. Let L be the system of all finite subsets of an infinite set M, $s^0 = \emptyset$. The system L is partially ordered by the set-theoretical inclusion. Then L is a directed set. It was remarked already in [5], Example 5.1 (under another terminology) that L satisfies (α_3) and that it does not satisfy (α_1). Thus, since $(\alpha_2)\Rightarrow(\alpha_3)$, we obtain that the relation $(\alpha_2)\Rightarrow(\alpha_1)$ fails to be valid. It is easy to verify that the condition (α_4) does not hold for L; hence (α_4) is not implied by (α_3). Further, the condition (α_5) is valid for L and thus (α_5) does not imply (α_4).

5.2. Example. Let L_1 be as in 1.3 and let L be the set of all $f \in L_1$ such that, whenever $t_0 \in R$ and f is not continuous at the point t_0, then $t_0 < 0$. The condition (α_3) is valid for L, but (α_2) does not hold for L. Hence (α_2) is not implied by (α_3).

References

[1] *J. Jakubík*: Direct decompositions of partially ordered groups. Czechoslovak Math. J. *10* (1960), 231–243. (In Russian.)
[2] *J. Jakubík*: Weak product decompositions of discrete lattices. Czechoslovak Math. J. *21* (1971), 399–412.
[3] *J. Jakubík*: Weak product decompositions of partially ordered sets. Colloquium Math. *25* (1972), 177–190.
[4] *J. Jakubík*: Directly indecomposable direct factors of a lattice. Math. Bohem. *121* (1996), 281–292.
[5] *J. Jakubík*: Atomicity of the Boolean algebra of direct factors of a directed set. Math. Bohem. *132* (1998), 145–162.
[6] *L. Libkin*: Direct decompositions of atomistic algebraic lattices. Algebra Univ. *33* (1995), 127–135.
[7] *F. Šik*: Über subdirekte Summen geordneter Gruppen. Czechoslovak Math. J. *10* (1960), 400–424.

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