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# ON FREDHOLM ALTERNATIVE FOR CERTAIN QUASILINEAR BOUNDARY VALUE PROBLEMS

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*Abstract.* We study the Dirichlet boundary value problem for the $p$-Laplacian of the form

$$-\Delta_p u - \lambda_1 |u|^{p-2}u = f \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $N \geqslant 1$, $p > 1$, $f \in C(\overline{\Omega})$ and $\lambda_1 > 0$ is the first eigenvalue of $\Delta_p$. We study the geometry of the energy functional

$$E_p(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega fu$$

and show the difference between the case $1 < p < 2$ and the case $p > 2$. We also give the characterization of the right hand sides $f$ for which the above Dirichlet problem is solvable and has multiple solutions.

*Keywords*: $p$-Laplacian, variational methods, PS condition, Fredholm alternative, upper and lower solutions

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## 1. Statement of the results

Our aim is to study the solvability of the Dirichlet boundary value problem

$$\begin{cases} -\Delta_p u - \lambda_1 |u|^{p-2}u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $p > 1$ is a real number, $\Omega$ is a bounded domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$ [1], $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian and $f \in C(\overline{\Omega})$. By $\lambda_1$ we

[1] We assume that if $N \geqslant 2$ then $\partial\Omega$ is a compact connected manifold of class $C^2$.

denote the first eigenvalue of the related homogeneous eigenvalue problem

$$\begin{cases} -\Delta_p u - \lambda |u|^{p-2} u = 0 \text{ in } \Omega, \\ \qquad\qquad\qquad u = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.2}$$

In this paper, the function $u$ is said to be a (*weak*) *solution* of (1.1) if $u \in W_0^{1,p}(\Omega)$ and the integral identity

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v - \lambda_1 \int_\Omega |u|^{p-2} uv = \int_\Omega fv \tag{1.3}$$

holds for all $v \in W_0^{1,p}(\Omega)$.

As for the properties of $\lambda_1$ (see e.g. [1], [15]), let us mention that $\lambda_1$ is positive, simple and isolated and the corresponding eigenfunction $\varphi_1$ (associated with $\lambda_1$) satisfies $\varphi_1 > 0$ in $\Omega, \partial\varphi_1/\partial n < 0$ on $\partial\Omega$, where $n$ denotes the exterior unit normal to $\partial\Omega$. One also has $\varphi_1 \in C^{1,\nu}(\overline{\Omega})$ with some $\nu \in (0,1)$ (see e.g. [8, Lemma 2.1, p. 115]). Moreover, $\lambda_1$ can be characterized as the best (the greatest) constant $C > 0$ in the Poincaré inequality

$$\int_\Omega |\nabla u|^p \geqslant C \int_\Omega |u|^p \tag{1.4}$$

for all $u \in W_0^{1,p}(\Omega)$, where the identity

$$\int_\Omega |\nabla u|^p - \lambda_1 \int_\Omega |u|^p = 0$$

holds exactly for the multiples of the first eigenfunction $\varphi_1$.

In our further considerations we will use the standard spaces $W_0^{1,p}(\Omega)$, $L^p(\Omega)$, $C(\overline{\Omega})$ and $C^1(\overline{\Omega})$ (or $C_0^1(\overline{\Omega})$, respectively), with the corresponding norms $\|u\| = (\int_\Omega |\nabla u|^p)^{\frac{1}{p}}, \|u\|_{L^p} = (\int_\Omega |u|^p)^{1/p}, \|u\|_C = \max\limits_{x\in\Omega} |u(x)|, \|u\|_{C^1} = \|u\|_C + \max\limits_{x\in\Omega} |\nabla u(x)|$, respectively (here $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}$ or $\mathbb{R}^N$). The subscript 0 indicates that the traces (or values) of functions equal zero on $\partial\Omega$. Moreover, for element $h$ we use the ($L^2$-orthogonal) decomposition

$$h(x) = \tilde{h}(x) + \overline{h}\varphi_1(x),$$

and also the $L^2$-nonorthogonal decomposition

$$h(x) = \tilde{h}(x) + \hat{h},$$

where $\bar{h}, \hat{h} \in \mathbb{R}$ and

$$\int_\Omega \tilde{h}(x)\varphi_1(x)\,\mathrm{d}x = 0.$$

The particular subspace formed by $\tilde{h}(x)$ will be denoted by $\widetilde{C}(\overline{\Omega})$.

By $B_C(\tilde{f}, \varrho)$ we denote the open ball in the space $C(\overline{\Omega})$ with the center $\tilde{f}$ and radius $\varrho$.

We introduce the energy functional associated with (1.1):

$$E_f(u) \colon = \frac{1}{p}\int_\Omega |\nabla u|^p - \frac{\lambda_1}{p}\int_\Omega |u|^p - \int_\Omega fu, \ u \in W_0^{1,p}(\Omega).$$

This functional is continuously Fréchet differentiable on $W_0^{1,p}(\Omega)$ and its *critical points* correspond one-to-one to *solutions* of (1.1).

Our main results concern the geometry of $E_f$ and the structure of the set of its critical points on the one hand and the solvability properties of (1.1) on the other. They are formulated in theorems below.

**Theorem 1.1** ([5])**.** *Let $1 < p < 2$ and $0 \neq \tilde{f} \in \widetilde{C}(\overline{\Omega})$. Then there exists $\varrho = \varrho(\tilde{f}) > 0$ such that for any $f \in B_C(\tilde{f}, \varrho)$ the functional $E_f$ is unbounded from below and has at least one critical point (which is the saddle point). Moreover, for $f \in B_C(\tilde{f}, \varrho) \setminus \widetilde{C}(\overline{\Omega})$ the functional $E_f$ has at least two distinct critical points.*

**Theorem 1.2** ([5])**.** *Let $p > 2$ and $0 \neq \tilde{f} \in \widetilde{C}(\overline{\Omega})$. Then the functional $E_{\tilde{f}}$ is bounded from below and has at least one critical point (which is the global minimizer). Moreover, there exists $\varrho = \varrho(\tilde{f}) > 0$ such that for $f \in B_C(\tilde{f}, \varrho) \setminus \widetilde{C}(\overline{\Omega})$ the functional $E_f$ has at least two distinct critical points.*

**Theorem 1.3** ([5])**.** *Let $p > 1, p \neq 2, \tilde{f} \in \widetilde{C}(\overline{\Omega})$. Then the problem (1.1) has at least one solution if $f = \tilde{f}$. For $0 \neq \tilde{f} \in \widetilde{C}(\overline{\Omega})$ there exists $\varrho = \varrho(\tilde{f}) > 0$ such that (1.1) has at least one solution for any $f \in B_C(\tilde{f}, \varrho)$. Moreover, there exist real numbers $F_- < 0 < F_+$ (see Fig. 1) such that the problem (1.1) with $f = \tilde{f} + \hat{f}$ has*

(i) *no solution for $\hat{f} \notin [F_-, F_+]$;*

(ii) *at least two distinct solutions for $\hat{f} \in (F_-, 0) \cup (0, F_+)$;*

(iii) *at least one solution for $\hat{f} \in \{F_-, 0, F_+\}$.*

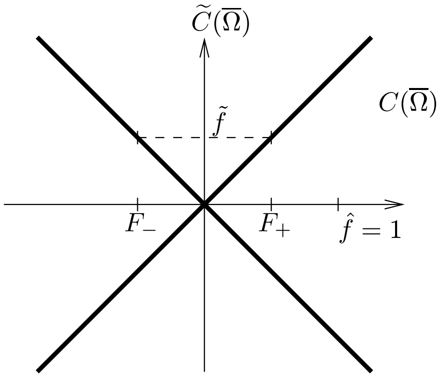


Fig. 1. "Slice" of $C(\overline{\Omega})$ containing all constants and one fixed $\tilde{f} \in \widetilde{C}(\overline{\Omega})$.

## 2. Remarks

Remark 2.1. Note that a standard bootstrap regularity argument implies that any solution from Theorems 1.1–1.3 belongs to $L^\infty(\Omega)$ (cf. Drábek, Kufner, Nicolosi [9]). It follows then from the regularity results of Tolksdorf [19] (see also Di Benedetto [4] and Liebermann [14]) that it belongs to $C^{1,\nu}(\overline{\Omega})$ with some $\nu \in (0,1)$. In particular, our solution is an element of $C_0^1(\overline{\Omega})$.

Remark 2.2. In particular, it follows from our results that the set of $f \in C(\overline{\Omega})$ for which (1.1) with $p \neq 2$ has at least one solution has a nonempty interior in $C(\overline{\Omega})$.

Remark 2.3. Note that Theorem 1.3 provides a necessary and sufficient condition for solvability of the problem (1.1). This condition is in fact of Landesman-Lazer type (see [13], cf. also [10]). Indeed, given $\tilde{f} \in \widetilde{C}(\overline{\Omega})$, $\tilde{f} \neq 0$, the problem (1.1) with the right hand side $f(x) = \tilde{f}(x) + \hat{f}$ has a solution if and only if

$$F_-(\tilde{f}) \leqslant \frac{1}{\|\varphi_1\|_{L^1}} \int_\Omega f(x)\varphi_1(x)\,\mathrm{d}x \leqslant F_+(\tilde{f}).$$

However, it should be pointed out that this condition differs from the original condition of Landesman and Lazer due to the fact that $F_-$ and $F_+$ depend on the component $\tilde{f}$ of the right hand side $f$ and not on the perturbation term (which is actually not present in our problem (1.1)). By homogeneity we have that for any $t > 0$,

$$F_\pm(t\tilde{f}) = tF_\pm(\tilde{f}).$$

Our proofs can be found in paper [5] and rely on the combination of the variational approach and the method of lower and upper solutions. We also use essentially the results obtained by Drábek and Holubová [7], Takáč [17] and Fleckinger-Pellé and Takáč [12]. In fact, Theorem 1.1 was proved already in [7], however, here a different approach is used. During the preparation of this manuscript the author received a preprint of Takáč [18], where a result similar to our Theorem 1.3 is proved. However, the approach used in [18] is very different from ours.

Our objective in this paper is to *avoid* complicated *technical assumptions*. For this reason we restrict to rather special domains $\Omega$ and right hand sides $f$. On the other hand, we believe that in our approach the main ideas appear more clearly and that a possible generalization of $\Omega$ or $f$ will bring new insight neither into the geometry of $E_f$ nor to the solvability of (1.1).

It should be mentioned that our approach covers also the case $N = 1$, and completes thus the previous results in this direction proved by Del Pino, Drábek and Manásevich [3], Drábek, Girg and Manásevich [6], Manásevich and Takáč [16], Binding, Drábek and Huang [2], Drábek and Takáč [11]. In fact, the first relevant result which led to better understanding of the problem appeared in [3].

Note also that our Theorems 1.1, 1.2 and 1.3 express not only the difference between the linear case $p = 2$ and the nonlinear case $p \neq 2$ but also the striking difference between the case $1 < p < 2$ and the case $p > 2$. The main goal of this paper is actually to emphasize this fact.

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