CHARACTERIZATION OF SEMIENTIRE GRAPHS WITH CROSSING NUMBER 2

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Abstract. The purpose of this paper is to give characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number 2. In addition, we establish necessary and sufficient conditions in terms of forbidden subgraphs for vertex-semientire graphs and edge-semientire graphs to have crossing number 2.

Keywords: semientire graph, vertex-semientire graph, edge-semientire graph, crossing number, forbidden subgraph, homeomorphic graphs

MSC 2000: 05C50, 05C99

1. Introduction

Graphs considered here are simple graphs (without loops and multiple edges). A graph is said to be embedded in a surface when it is drawn on S so that no two edges intersect. A graph is planar if it can be embedded in the plane. By a plane graph we mean a graph embedded in the plane as opposed to a planar graph.

If there exists an edge $e_1 = uv$ in a plane graph G, we say that the vertices u, v are adjacent to each other and both incident to the edge $e_1 = uv$. The edge $e_1 = uv$ is said to be adjacent to an edge e_2 if and only if $e_2 = uw$ or $e_2 = vw$, where w is a vertex of G distinct from u and v. A region of G is adjacent to the vertices and edges which are on its boundary, and two regions of G are adjacent if their boundaries share a common edge. In this paper, vertices, edges and regions are called the elements of G.

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Kulli and Akka [2] introduced the concepts of a vertex-semientire graph and an edge-semientire graph of a graph. The vertex-semientire graph $e_v(G)$ of a plane graph G is the graph whose vertex set is the union of the vertex set and the region set of G and in which two vertices are adjacent if and only if the corresponding elements (two vertices, two regions or a vertex and a region) of G are adjacent. The edge-semientire graph $e_e(G)$ of a plane graph G is the graph whose vertex set is the union of the edge set and the region set of G and in which two vertices are adjacent if and only if the corresponding elements (two edges, two regions or an edge and a region) of G are adjacent. For other definitions see [1].

In [2], Kulli and Akka established characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs are planar and outerplanar. Further, in [3], Kulli and Muddebihal established characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number one. In addition, they established necessary and sufficient conditions in terms of forbidden subgraphs for vertex-semientire graphs and edge-semientire graphs to have crossing number one.

The main results of this paper are characterizations of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number 2. In addition, we give characterizations in terms of forbidden subgraphs of graphs whose vertex-semientire graphs and edge-semientire graphs have crossing number 2.

The following will be useful for proving our theorems.

Theorem A [2]**.** *Let G be a connected plane graph. Then $e_v(G)$ is planar if and only if G is a tree.*

Theorem B [2]**.** *Let G be a connected plane graph. Then $e_e(G)$ is planar if and only if $\Delta(G) \leqslant 3$ and G is a tree.*

Theorem C [3]**.** *Let G be a connected plane graph. Then $e_v(G)$ has crossing number 1 if and only if G is unicyclic.*

Theorem D [3]**.** *The edge-semientire graph $e_e(G)$ of a connected plane graph G has crossing number 1 if and only if (1) or (2) holds.*

(1) *$\Delta(G) = 3$, G is unicyclic and such that at least one vertex of degree 2 is on the cycle.*
(2) *$\Delta(G) = 4$, G is a tree and has exactly one vertex of degree 4.*

2. Main results

In the next theorem, we present a characterization of graphs whose vertex-semientire graphs have crossing number 2.

Theorem 1. *Let G be a connected plane graph. Then $e_v(G)$ has crossing number 2 if and only if G has exactly two cycles and these cycles are its blocks.*

Proof. Suppose $e_v(G)$ has crossing number 2. Assume that G is a tree. Then by Theorem A, $e_v(G)$ is planar, a contradiction.

Assume that G has at least three cycles. Suppose each cycle is a block of G. Then by Theorem C, each block which is a cycle in G gives at least one crossing in $e_v(G)$. Hence $e_v(G)$ has at least three crossings, a contradiction. Thus G has exactly two cycles.

Suppose two cycles lie in a block. Then G has a subgraph homeomorphic to $K_4 - x$. G has two interior regions r_1 and r_2 and the exterior region R. In $e_v(G)$, the vertices r_1, r_2 and R are mutually adjacent, since the regions r_1, r_2 and R are mutually adjacent in G. Then in each adjacency there exists at least one crossing. Hence $e_v(G)$ has at least 3 crossings, a contradiction. Thus we conclude that G has exactly two cycles as blocks.

Conversely, assume that G has exactly two cycles C_i, $i = 1, 2$, which are both blocks. Also, let each edge which is not on C_i be a block of G. Let r_i, $i = 1, 2$ be two interior regions of C_i and R the exterior region of G. In $e_v(G)$, the vertex r_i is adjacent to each vertex of C_i without crossings, the vertex R is adjacent to each vertex of G without crossings and the vertex R is adjacent to r_i with two crossings.

Thus $e_v(G)$ has crossing number 2. This completes the proof of the theorem.

In the next theorem, we obtain a characterization of graphs whose edge-semientire graphs have crossing number 2. □

Theorem 2. *The edge-semientire graph $e_e(G)$ of a connected plane graph G has crossing number 2 if and only if*

1) $\deg v \leqslant 4$ *for every vertex v of G, and G is a tree and has exactly two vertices of degree 4, or G is not a tree and has exactly one cutvertex of degree 4 and exactly one cycle such that at least one vertex of degree 2 is on the cycle*

or

2) $\deg v \leqslant 3$ *for every vertex v of G and G has exactly two cycles and these cycles are its blocks in which at least one vertex of degree 2 lies on each cycle, or G is unicyclic and such that no vertex of degree 2 is on the cycle.*

P r o o f. Suppose the edge-semientire graph $e_e(G)$ of a connected plane graph G has crossing number 2. Then it is nonplanar. By Theorem B or D, G is a tree with $\Delta(G) \geqslant 4$ or G is not a tree and $\Delta(G) \leqslant 3$.

Suppose G is a tree with deg $\geqslant 4$ for some vertex v of G. We consider the following cases.

Case 1. Suppose $\deg v \geqslant 5$ for some vertex v of the tree G. Then clearly $c(e_e(G)) > 2$, a contradiction. Hence $\Delta(G) \leqslant 4$.

Case 2. Suppose $\deg v = 4$ for some vertex v of G. Assume G has at least 3 vertices of degree 4. Then $L(G)$ has at least 3 subgraphs isomorphic K_4. By the definition of $e_e(G)$, $L(G)$ is a subgraph of $e_e(G)$. The vertex R in $e_e(G)$ which corresponds to the exterior region is adjacent to every vertex of $L(G)$, which gives at least 3 subgraphs isomorphic K_5 in $e_e(G)$. Hence $c(e_e(G)) > 2$, a contradiction. Thus G has at most two vertices of degree 4.

Suppose G is not a tree and assume $\deg v = 4$ for some vertex v of G. We consider 2 cases.

Case 1. Assume G has at least two vertices of degree 4 and at least one cycle C. Then $L(G)$ has at least 2 subgraphs isomorphic to K_4 and at least one subgraph $L(C)$. By the definition of $e_e(G)$, $L(G) \subset e_e(G)$. The vertex r in $e_e(G)$ (which corresponds to an interior region of C) is adjacent to every vertex of $L(C)$. This gives one wheel W. The vertex R in $e_e(G)$ is adjacent to every vertex of two K_4 and W of $L(G)$. This gives at least 3 subgraphs isomorphic to K_5 in $e_e(G)$. Thus $c(e_e(G)) \geqslant 3$, a contradiction.

Case 2. Assume G has at least one vertex of degree 4, at least two cycles C_i, $i = 1, 2$ as blocks and let r_i be the interior regions of C_i. Then $L(G)$ has at least one subgraph isomorphic to K_4 and at least two subgraphs $L(C_i)$. In $e_e(G)$, r_i is adjacent to every vertex of $L(C_i)$, which gives a wheel W_i. Since $L(G) \subset e_e(G)$, the vertex R in $e_e(G)$ which corresponds to the exterior region is adjacent to every vertex of $L(G)$ and r_i. This gives at least 3 subgraphs isomorphic to K_5 in $e_e(G)$. Hence $c(e_e(G)) > 2$, a contradiction.

From cases 1 and 2 we conclude that G has exactly one vertex of degree 4 and exactly one cycle.

Suppose G has exactly one vertex v of degree 4 and a cycle C. Assume that every vertex of C has degree at least three. Let e_i, $i = 1, 2, 3$ and 4 be edges adjacent to v. Then $L(G)$ has exactly one subgraph isomorphic to K_4 and exactly one cycle $L(C)$. Let r be the interior region of C and R the exterior region of G. In $e_e(G)$, the vertex r is adjacent to every vertex of $L(C)$ without crossing, which gives $e_e(G) - R$. We get two wheels $L(C) + r$ and $K_3 + e_i$ $(= K_4)$, $i = 1, 2, 3$ or 4 in $e_e(G) - R$. In $e_e(G) - \{rR, Re_i\}$, the vertex R is adjacent to every vertex of $e_e(G) - \{r, e_i\}$ without crossings. In $e_e(G)$ it is easy to see that the edges Re_i and rR cross respectively at least one edge and at least 2 edges of $e_e(G) - \{rR, re_i\}$. Thus $e_e(G)$ has at least 3 crossings, a contradiction. This proves (1).

Assume G is not a tree and $\deg v \leqslant 3$ for every vertex v of G. We consider three cases.

Case 1. Assume G has at least 3 cycles. Suppose each cycle has at least one vertex of degree two and each cycle is a block of G. Let R and r_i, $i = 1, 2, 3$ be vertices in $e_e(G)$ which correspond to the exterior and interior regions of G. Then $e_e(G) - R$ has at least 3 blocks each of which is a wheel. In $e_e(G)$, R is adjacent to each wheel. We get at least 1 crossing in each case. It is clear that $e_e(G)$ has at least 3 crossings, a contradiction.

Case 2. Suppose G has at least two cycles in a block. Then G has a subgraph homeomorphic to $K_4 - x$. Obviously G has 2 interior regions, say r_1 and r_2, and the exterior region R. Clearly $e_e(G) - R$ has a block in which the edge joining the vertices r_1 and r_2 has two crossings. Also in $e_e(G)$, the vertex R is adjacent to r_1 and r_2, which makes two more crossings. Thus $c(e_e(G)) \geqslant 4$, a contradiction.

From the above cases, we conclude that G has at most two cycles C_i as blocks.

Assume G has no vertex of degree 2 on each cycle C_i. The interior regions r_1 and r_2 are adjacent respectively to every vertex of C_1 and C_2 without crossings and this gives $e_e(G) - R$ where R is the exterior region. The vertex R is adjacent to each vertex of $e_e(G) - \{r_1, r_2\}$ without crossings. In $e_e(G)$, r_1R and r_2R are edges. Clearly each r_iR crosses at least 2 edges in $e_e(G) - \{r_1R, r_2R\}$. Thus $c(e_e(G)) \geqslant 4$, a contradiction.

Suppose G is unicyclic and all vertices of the cycle C are of degree less than 3. Assume that at least one vertex of the cycle C of G has degree 2. Then by condition (1) of Theorem D, $e_e(G)$ has exactly one crossing, a contradiction. This proves (2).

Conversely, suppose G is a graph satisfying conditions (1) or (2). Then by Theorem B or D, $e_e(G)$ has crossing number at least 2. We now show that its crossing number is at most 2. Assume first that G satisfies condition (1). We consider 3 cases.

Case 1. Suppose G is a tree and has exactly two vertices of degree 4. Then clearly $e_e(G)$ has exactly two subgraphs, each isomorphic to K_5, and hence $e_e(G)$ can be drawn with exactly two crossings.

Case 2. Suppose G is not a tree and has exactly one vertex of degree 4 and exactly one cycle C such that at least one vertex of degree 2 is on the cycle. Then it is easy to see that $e_e(G)$ has exactly two crossings.

Now assume (2). Then G has exactly two cycles C_i as blocks in which at least one vertex of degree 2 lies on each cycle. Let r_i, $i = 1, 2$ be the interior regions of two circles C_i of G. The vertex r_i is adjacent to every vertex of $L(C_i)$ without crossings, which gives $e_e(G) - R$ where R is the exterior region of G. Obviously $e_e(G) - R$

has at least two blocks each of which is a wheel with at least one boundary edge. In $e_e(G) - \{r_1R, r_2R\}$ the vertex R is adjacent to every vertex of $e_e(G) - \{r_1, r_2\}$ without crossings. By the definition of $e_e(G)$, r_1R and r_2R are edges. Hence either of r_1R and r_2R crosses exactly one edge of $e_e(G) - \{r_1R, r_2R\}$ and gives $e_e(G)$. Hence $e_e(G)$ has exactly two crossings.

Suppose G is unicyclic in which no vertex of degree 2 is on the cycle C. Let the vertices r and R correspond to the interior and exterior regions of G, respectively. The vertex r is adjacent to every vertex of $L(C)$ and gives one wheel together with a triangle on each side (in $e_e(G) - R$) without crossings. In $e_e(G) - rR$, the vertex R is adjacent to every vertex of $e_e(G) - r$ without crossings. Thus the edge rR crosses exactly two boundary edges of $e_e(G) - rR$ and gives $e_e(G)$. Hence $c(e_e(G)) = 2$. This completes the proof of the theorem. □

3. Forbidden subgraphs

With help of Theorems 1 and 2 we now characterize graphs whose semientire graphs have crossing number 2, in terms of forbidden subgraphs.

Theorem 3. *Suppose a connected plane graph G has at least two cycles as blocks. The vertex-semientire graph $e_v(G)$ has crossing number 2 if and only if it has no subgraph homeomorphic to G_i, $i = 12, 13, 14, 16, \ldots, 19$ or 20 (Fig. 1).*

Proof. Assume a connected plane graph G has at least two cycles. Suppose $c(e_v(G)) = 2$. Then by Theorem 1, G has at most two cycles as blocks. It follows that G has no subgraph homeomorphic to $G_{12}, G_{13}, G_{14}, G_{16}, G_{17}, G_{18}, G_{19}$ or G_{20}.

Conversely, suppose G has at least two cycles as blocks and has no subgraph homeomorphic to $G_{12}, G_{13}, G_{14}, G_{16}, G_{17}, G_{18}, G_{19}$ or G_{20}.

Suppose G has at least 3 cycles each of them being a block of G. Then G has a subgraph homeomorphic to $G_{12}, G_{13}, G_{16}, G_{17}, G_{18}, G_{19}$ or G_{20}, a contradiction.

Suppose G has a block which contains at least two cycles. Then G has a subgraph homeomorphic to G_{14}, a contradiction.

In each case we have arrived at a contradiction. Thus Theorem 1 implies that $c(e_v(G)) = 2$. This completes proof. □

Theorem 4. *The edge-semientire graph $e_e(G)$ of a connected plane graph G (with at least 5 vertices and 5 edges and $\Delta(G) \leqslant 4$) has crossing number 2 if and only if G has no subgraph homeomorphic to G_i, $1 = 1, 2, \ldots, 14$ or 15 (Fig. 1).*

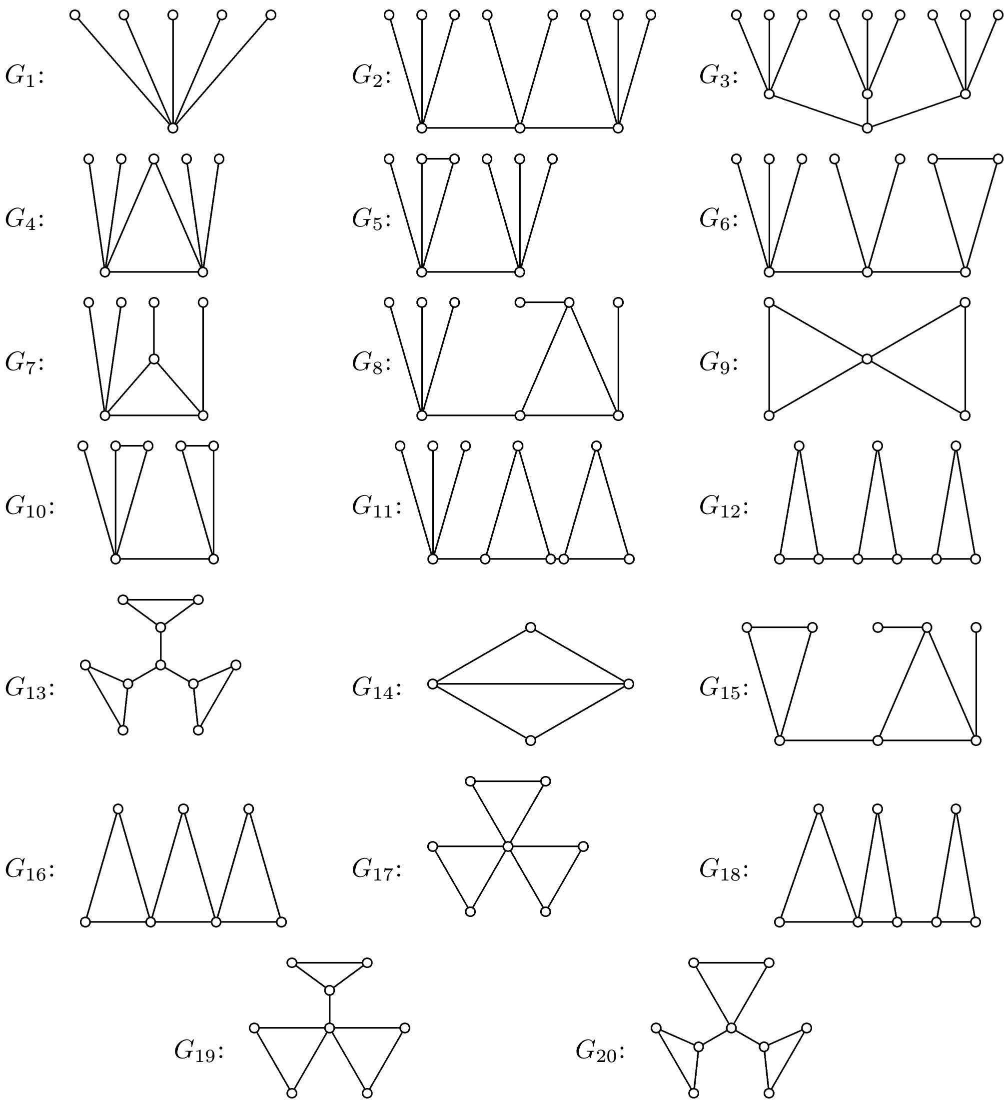

Fig. 1

P r o o f. Assume G is a connected plane graph whose edge-semientire graph $e_e(G)$ has crossing number 2. We prove that all graphs homeomorphic to G_i, $i = 1, 2, \ldots, 14$ or 15 have $c(e_e(G_i)) > 2$. By Theorem 2, we have (1) $\deg v \leqslant 4$ for every vertex v of G and G is a tree and has exactly two vertices of degree 4 or G is not a tree and has exactly one vertex of degree 4 and exactly one cycle such that at least one vertex of degree 2 is on the cycle. Or (2) $\deg v \leqslant 3$ for every vertex v of G and G has exactly two cycles as blocks in which at least one vertex of degree 2 is on each cycle or G is unicyclic and such that no vertex of degree 2 is on the cycle. From (1) or (2) it follows that G has no subgraph homeomorphic to any one of the graphs G_i, $i = 1, 2, \dots, 15$.

Conversely, assume that G is a connected plane graph and does not contain a subgraph homeomorphic to any one of the graphs G_i, $i = 1, \dots, 15$. We shall show that G satisfies (1) or (2) and hence by Theorem 2, $e_e(G)$ has crossing number 2. Suppose $\deg v \geqslant 5$ for some vertex v of G. Then G contains a subgraph homeomorphic to G_1, a contradiction. Hence $\deg v \leqslant 4$ for every vertex v of G. We consider the following two cases.

Case 1. Suppose G is a tree. Assume there exist at least three vertices of degree 4. Then G has a subgraph homeomorphic to G_2 or G_3, a contradiction. Hence G has exactly two vertices of degree 4.

Case 2. Suppose G is not a tree. Then we consider two subcases.

Subcase 2.1. Suppose G is unicyclic C. Assume G has exactly two vertices v_1 and v_2 of degree 4. Then we consider 3 possibilities.

a) If $v_1, v_2 \in C$, then G has a subgraph homeomorphic to G_4.

b) If v_1 or $v_2 \in C$, then G has a subgraph homeomorphic to G_5.

c) If $v_1, v_2 \notin C$, then G has a subgraph homeomorphic to G_6.

In each case we have a contradiction. Thus G has exactly one vertex of degree 4 and exactly one cycle.

Suppose G has exactly one vertex v of degree 4 and exactly one cycle C such that no vertex of degree 2 is on the cycle. Then we consider two possibilities.

a) If $v \in C$, then G has a subgraph homeomorphic to G_7, a contradiction.

b) If $v \notin C$, then G has a subgraph homeomorphic to G_8, a contradiction.

Thus G has exactly one vertex of degree 4 and exactly one cycle such that at least one vertex of degree 2 is on the cycle, or G is unicyclic with every vertex of degree 3 on the cycle.

Subcase 2.2. Assume G is not a unicyclic graph. Suppose G has exactly one vertex v of degree 4 and at least two cycles C_1 and C_2, each of which has at least one vertex of degree 2. We consider the following three possibilities.

a) If $v \in C_1$ and C_2, then G has a subgraph homeomorphic to G_9.

b) If $v \in C_1$ or C_2, then G has a subgraph homeomorphic to G_{10}.

c) If $v \notin C_1$ and C_2, then G has a subgraph homeomorphic to G_{11}.

In each case we have a contradiction. Thus G has at least 2 cycles each of which has at least one vertex of degree 2. Assume $\deg v \leqslant 3$ for every vertex v of G. Then we consider 3 cases.

Case 1. Suppose G has at least 3 cycles as blocks such that each block has at least one vertex of degree two. Then G has a subgraph homeomorphic to G_{12} or G_{13}, a contradiction.

Case 2. Suppose G has a block which contains at least two cycles. Then G has a subgraph homeomorphic to G_{14}, a contradiction.

Thus G has at most two cycles as blocks.

Case 3. Suppose G has exactly two cycles as blocks such that one block has no vertex of degree 2. Then G has a subgraph homeomorphic to G_{15}, a contradiction. Thus G has exactly two cycles such that each cycle has at least one vertex of degree 2, or G has exactly one cycle such that each vertex on the cycle is of degree 3.

We have exhausted all possibilities. In each case we found that G contains a subgraph homeomorphic to some of the forbidden subgraphs G_i, $i = 1, \ldots, 15$. Hence by Theorem 2, $e_e(G)$ has crossing number 2. This completes the proof of the theorem. $\square$

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