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# ON THE LINEAR CAPACITY OF ALGEBRAIC CONES

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*Abstract.* We define the linear capacity of an algebraic cone, give basic properties of the notion and new formulations of certain known results of the Matrix Theory. We derive in an explicit way the formula for the linear capacity of an irreducible component of the zero cone of a quadratic form over an algebraically closed field. We also give a formula for the linear capacity of the cone over the conjugacy class of a "generic" non-nilpotent matrix.

*Keywords*: irreducible algebraic cone, linear subspace, conjugacy class of a matrix, quadratic form

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## Introduction and preliminaries

Papers [1], [2], [5], [7], [8], [9] are partially devoted to computing the maximal dimension of a linear space consisting of matrices and satisfying some special conditions (for instance, nilpotency of its elements, restrictions on their ranks, restrictions on the number of their eigenvalues). The papers lead us to introducing the notion of linear capacity of subsets of a finite dimensional linear space. In the present note, we propose a general definition of the linear capacity of an irreducible algebraic cone and give a listing of its basic properties. We also propose new formulations (perhaps simplest) of certain results stated in the papers. We explicitly compute the linear capacity of the zero cone of a quadratic form over an algebraically closed field. Moreover, we give a formula for the linear capacity of the cone over the conjugacy class of a "generic" non-nilpotent matrix. The formula completes, in some sense, Gerstenhaber's important General Theorem on Linear Spaces of Nilpotent Matrices.

Throughout the note we work over an algebraically closed field $\mathbb{F}$ of characteristic zero. We set $\mathbb{F}^* = \mathbb{F} \setminus \{0\}$. We denote by $\mathbb{X}$ a non-zero finite dimensional linear space over $\mathbb{F}$.

A non-empty set $C \subseteq \mathbb{X}$ is a *cone* if $\mathbb{F}C \subseteq C$. (In particular, $0 \in C$.) A function $\psi\colon \mathbb{X} \longrightarrow \mathbb{F}$ is regular if $\psi \circ f \in \mathbb{F}[T_1, \ldots, T_d]$ for each (or, equivalently, for a fixed) linear isomorphism $f\colon \mathbb{F}^d \longrightarrow \mathbb{X}$, where $d = \dim \mathbb{X}$. Having defined the regular functions, we define the algebraic subsets of $\mathbb{X}$ and the Zariski topology on $\mathbb{X}$ in the usual way. For $E \subseteq \mathbb{X}$ we denote by $\overline{E}$ the closure of $E$ in the Zariski topology on $\mathbb{X}$ and by $\mathbb{F}[E]$ the coordinate ring of $E$. We refer to [10] for all the information needed about notions, terminology, and facts of Algebraic Geometry.

For a quadratic form $\varphi\colon \mathbb{X} \longrightarrow \mathbb{F}$ we define $\widetilde{\varphi}\colon \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{F}$ to be the polar form of $\varphi$ (i.e. $\widetilde{\varphi}$ is the unique symmetric bilinear form such that $\varphi(x) = \widetilde{\varphi}(x, x)$ for all $x \in \mathbb{X}$). We set

$$\operatorname{Ker} \varphi = \{x \in \mathbb{X};\ \widetilde{\varphi}(x, y) = 0 \text{ for all } y \in \mathbb{X}\}.$$

The form $\varphi$ is *non-degenerate* if $\operatorname{Ker} \varphi = \{0\}$. We define the *complement* $E^{\varphi}$ of a set $E \subseteq \mathbb{X}$ with respect to the form $\varphi$ by the formula

$$E^{\varphi} = \{x \in \mathbb{X};\ \widetilde{\varphi}(x, y) = 0 \text{ for all } y \in E\}.$$

(In particular, $\operatorname{Ker} \varphi = \mathbb{X}^{\varphi}$. The complement $E^{\varphi}$ is a linear subspace of $\mathbb{X}$.) We refer to [6] for further information about quadratic forms.

The set of all non-negative integers will be denoted by $\mathbb{N}$. We define $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. The cardinality of a finite set $S$ will be denoted by $\#S$.

We shall need some terminology, notations, and facts of Matrix Theory. We define $\mathcal{M}_{m \times n}$ to be the set of all $m \times n$-matrices whose entries are elements of the field $\mathbb{F}$. We shall write $\mathcal{M}_n$ instead of $\mathcal{M}_{n \times n}$. We denote by $\mathcal{GL}_n$ the full linear group contained in $\mathcal{M}_n$, by $I_n$ the unit matrix of $\mathcal{M}_n$, and by $O_{m \times n}$ the zero matrix of $\mathcal{M}_{m \times n}$. (Of course, $O_n = O_{n \times n}$.)

Given a matrix $A \in \mathcal{M}_n$, we set $\mathcal{O}(A) = \{U^{-1}AU;\ U \in \mathcal{GL}_n\}$ and

$$\sigma(A) = \{\lambda \in \mathbb{F};\ \lambda \text{ is an eigenvalue of } A\}.$$

Furthermore, we define $\mathrm{s}_n^j(A)$, with $j \in \mathbb{N}^*$ such that $j \leqslant n$, to be the sum of all principal minors of $A$ of size $j$. (In other words, $T^n + \sum\limits_{j=1}^{n} (-1)^j \mathrm{s}_n^j(A) T^{n-j}$ is the characteristic polynomial of the matrix $A$.) For an arbitrary $j \in \mathbb{N}$ we shall write $r_A(j)$ instead of $\operatorname{rank}(A^j)$. (In particular, $r_A(0) = n$.)

The following fact is quite important. A function $\varrho\colon \mathbb{N} \longrightarrow \mathbb{N}$ such that $\varrho(0) = n$ is weakly decreasing and satisfies the convexity condition

$$\varrho(j) + \varrho(j+2) \geqslant 2\varrho(j+1) \text{ for all } j \in \mathbb{N}$$

if and only if there is a matrix $B \in \mathcal{M}_n$ such that $r_B(j) = \varrho(j)$ for all $j \in \mathbb{N}$ (cf. [11, Theorem 2 and Theorem 3]).

We define the *Jordan partition* of a matrix $A \in \mathcal{M}_n$ to be the sequence $(a_k)_{k=1}^{\infty}$ such that $a_k$ is equal to the size of the $k$th nilpotent block contained in the Jordan canonical form of $A$, if $k$ is not greater than the number of the nilpotent blocks, and to 0, if $k$ is greater than the number of the nilpotent blocks. (We assume that the nilpotent blocks of the canonical form of a matrix are ordered with respect to their sizes, with the largest one in the upper left corner.) Given the Jordan partition $(a_k)_{k=1}^{\infty}$ of a matrix $A \in \mathcal{M}_n$, we define its *conjugate* to be the sequence $(p_j)_{j=1}^{\infty}$ such that

$$p_j = \#\{k \in \mathbb{N}^* \,;\ a_k \geqslant j\}.$$

It is worth noticing that

$$(\bullet) \qquad p_j = r_A(j-1) - r_A(j) \text{ for all } j \in \mathbb{N}^*.$$

For Jordan partitions $\mathbf{a} = (a_k)_{k=1}^{\infty}$ and $\mathbf{b} = (b_k)_{k=1}^{\infty}$ of n i l p o t e n t matrices $A, B \in \mathcal{M}_n$ we set

$$\mathbf{b} \preceq \mathbf{a} \text{ if and only if, } \sum_{k=1}^{l} b_k \leqslant \sum_{k=1}^{l} a_k \text{ for all } l \in \mathbb{N}^*.$$

We shall need the following version of the well-known Gerstenhaber theorem on the closure of the conjugacy class of a nilpotent matrix (cf. [4] and [1, Section 3]).

**Theorem 0.1** (Gerstenhaber)**.** *Let $A, B \in \mathcal{M}_n$ be nilpotent matrices. Then the following conditions are equivalent:*
(1) $B \in \overline{\mathcal{O}(A)}$,
(2) $\mathbf{b} \preceq \mathbf{a}$,
(3) $r_B(j) \leqslant r_A(j)$ *for all* $j \in \mathbb{N}$,
*where* $\mathbf{a}$ *and* $\mathbf{b}$ *are the Jordan partitions of $A$ and of $B$, respectively.*

## 1. Definition, basic properties, examples

We begin with the fundamental definition (which is a slight modification of the definition proposed in [12, Section 3]).

Let $E \subseteq \mathbb{X}$ be a set such that the closure $\overline{E}$ is an irreducible cone. We define the *linear capacity* of the set, $\Lambda(E)$, by the formula

$$\Lambda(E) = \max\{\dim L \,;\ L \text{ is a linear subspace of } \mathbb{X},\ L \subseteq \overline{E}\}.$$

The simplest properties of the linear capacity are the following.

**Proposition 1.1.** *Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be non-zero finite dimensional linear spaces over $\mathbb{F}$, let $E_1 \subseteq \mathbb{X}_1$ and $E_2, G \subseteq \mathbb{X}_2$ be such that the closures $\overline{E_1}$, $\overline{E_2}$, and $\overline{G}$ are irreducible cones, and let $f\colon \mathbb{X}_1 \longrightarrow \mathbb{X}_2$ be a linear map such that $f(E_1) \subseteq G$. Then*
(i) $\Lambda(E_1) = \Lambda(\overline{E_1})$,
(ii) $\Lambda(E_1) \leqslant \dim \overline{E_1}$,
(iii) $\Lambda(E_1) = \dim \overline{E_1}$ *if and only if* $\overline{E_1}$ *is a linear subspace of* $\mathbb{X}_1$,
(iv) *if* $E_2 \subseteq G$, *then* $\Lambda(E_2) \leqslant \Lambda(G)$,
(v) *if both* $E_1$ *and* $E_2$ *are algebraic cones, then* $\Lambda(E_1 \times E_2) = \Lambda(E_1) + \Lambda(E_2)$,
(vi) *if* $f|_{\overline{E_1}}\colon \overline{E_1} \longrightarrow \overline{G}$ *is an injection, then* $\Lambda(E_1) \leqslant \Lambda(G)$,
(vii) *if* $f|_{\overline{E_1}}$ *is a bijection and* $(f|_{\overline{E_1}})^{-1}\colon \overline{G} \longrightarrow \overline{E_1}$ *is the restriction of a linear map* $\mathbb{X}_2 \longrightarrow \mathbb{X}_1$, *then* $\Lambda(E_1) = \Lambda(G)$.

We omit the obvious proof.

Let us recall that a matrix $A \in \mathcal{M}_n$ is nilpotent if and only if the closure $\overline{\mathcal{O}(A)}$ is a cone (cf. [11, Proposition 1]). This cone is evidently irreducible. Now, we may formulate Gerstenhaber's General Theorem on Linear Spaces of Nilpotent Matrices (cf. [5], [1]) in the language of linear capacity.

**Theorem 1.2** (Gerstenhaber)**.** *Let $A \in \mathcal{M}_n$ be a nilpotent matrix. Then*

$$\Lambda(\mathcal{O}(A)) = \frac{1}{2}\left(n^2 - \sum_{j=1}^{\infty} p_j^2\right),$$

*where $(p_j)_{j=1}^{\infty}$ is the conjugate of the Jordan partition of $A$.*

Let us point out that in order to derive the above formulation we have to apply Gerstenhaber's Theorem 0.1 and remarks from [1, Section 3].

We shall see also how the notion of linear capacity works in the statement and proof of [8, Theorem 1].

From now on, the letter $n$ stands for an integer not smaller than 2.

Example 1.3. We consider the set $\mathcal{V} = \{A \in \mathcal{M}_n ;\ \#\sigma(A) = 1\}$ and the linear map $\Psi\colon \mathbb{F} \times \mathcal{M}_n \ni (\lambda, A) \mapsto A + \lambda I_n \in \mathcal{M}_n$. The set $\mathcal{V}$ is a cone. We define $\mathcal{N}_n = \{A \in \mathcal{M}_n ;\ A \text{ is nilpotent}\}$. By Gerstenhaber's Theorem 0.1, $\mathcal{N}_n = \overline{\mathcal{O}(B_n)}$, where $B_n \in \mathcal{M}_n$ is the nilpotent Jordan block of size $n$. It is evident that $\Psi(\mathbb{F} \times \mathcal{N}_n) \subseteq \mathcal{V}$. Moreover, the restriction

$$\Psi|_{\mathbb{F}\times\mathcal{N}_n}\colon\ \mathbb{F} \times \mathcal{N}_n \longrightarrow \mathcal{V}$$

is a bijective map (with the inverse $\mathcal{V} \ni A \mapsto (\frac{1}{n}\mathrm{tr}(A),\ A - \frac{1}{n}\mathrm{tr}(A)I_n) \in \mathbb{F} \times \mathcal{N}_n$). Consequently, $\mathcal{V}$ is an irreducible algebraic set. So, we may try to compute its linear capacity. By properties (vii) and (v) of Proposition 1.1, and by Gerstenhaber's Theorem 1.2, we have

$$\Lambda(\mathcal{V}) = \Lambda(\mathbb{F} \times \mathcal{N}_n) = \Lambda(\mathbb{F}) + \Lambda(\mathcal{N}_n) = 1 + \Lambda(\mathcal{O}(B_n)) = 1 + \tfrac{1}{2}n(n-1).$$

To conclude the section, let us recall the definiton of the determinantal variety. For $m \in \mathbb{N}^*$ and an integer $k$ satisfying the inequalities $0 \leqslant k \leqslant \min\{m,n\}$ we define the *(generic) determinantal variety* $\mathcal{H}^k_{m\times n}$ by the formula

$$\mathcal{H}^k_{m\times n} = \{A \in \mathcal{M}_{m\times n};\ \operatorname{rank}(A) \leqslant k\}.$$

It is not hard to verify that $\mathcal{H}^k_{m\times n}$ is an irreducible algebraic cone. The theorem on linear subspaces of a determinantal variety (cf. [2], [9]) may be formulated in the following way.

**Theorem 1.4** (Flanders). *Let the integers $m$, $n$, and $k$ be as above. Then $\Lambda(\mathcal{H}^k_{m\times n}) = k\max\{m,n\}$.*

## 2. Linear capacity and quadratic forms

We shall compute the linear capacity of an irreducible component of the zero cone of a quadratic form. The formula can be deduced from, for instance, the information given in [6] but it seems to be worth stating and proving in an explicit way.

First, we establish certain bounds for that capacity.

**Theorem 2.1.** *Let $\varphi\colon \mathbb{X} \longrightarrow \mathbb{F}$ be an arbitrary quadratic form, let $V$ be an irreducible component of the zero cone $\varphi^{-1}(0)$, and let $L$ be a linear subspace of $\mathbb{X}$ such that $L \subseteq V$ and $\dim L = \Lambda(V)$. Then*

(i) $\operatorname{Ker}\varphi \subseteq L$,

(ii) $c \leqslant \Lambda(V) \leqslant \frac{1}{2}(d+c)$,

*where $d = \dim\mathbb{X}$ and $c = \dim\operatorname{Ker}\varphi$ (in other words, $c$ is equal to the corank of the form $\varphi$).*

Proof. If $\varphi$ is identically equal to 0, then inclusion (i) is obvious. If $\varphi$ is a non-zero reducible form, i.e. if $\varphi(x) = \varphi_1(x)\varphi_2(x)$ for all $x \in \mathbb{X}$ and for certain non-zero linear forms $\varphi_1, \varphi_2\colon \mathbb{X} \longrightarrow \mathbb{F}$, then $\operatorname{Ker}\varphi = \varphi_1^{-1}(0) \cap \varphi_2^{-1}(0)$ which yields inclusion (i). So, we assume that $\varphi$ is an irreducible form. Then $V = \varphi^{-1}(0) \neq \mathbb{X}$. It is evident that $\varphi(x+y) = \varphi(x) + 2\widetilde{\varphi}(x,y) + \varphi(y) = 0$ for all $x \in L$ and for all $y \in \operatorname{Ker}\varphi$. Consequently, $L + \operatorname{Ker}\varphi \subset V$ which yields $L + \operatorname{Ker}\varphi = L$ (by the definition of $L$). The proof of inclusion (i) is complete.

The lower bound of (ii) immediately follows from (i).

If $\varphi$ is identically equal to 0, then the upper bound holds in a trivial way. Now, let us assume that $\varphi$ is a non-degenerate form. Then $c = 0$. By the polarization formula, we have $\widetilde{\varphi}(w, z) = \frac{1}{2}(\varphi(w + z) - \varphi(w) - \varphi(z)) = 0$ for all $w, z \in L$, which implies $L \subseteq L^{\varphi}$. In virtue of the non-degenerateness, we get $d = \dim L + \dim L^{\varphi} \geqslant 2 \dim L$. This yields the desired inequality $\Lambda(V) \leqslant \frac{1}{2}d$. Finally, let us consider the case of an arbitrary non-zero $\varphi$. We set $\mathbb{Y} = \mathbb{X}/\operatorname{Ker}\varphi$ and $\Phi\colon\ \mathbb{Y} \ni [x] \mapsto \varphi(x) \in \mathbb{F}$. Then $\Phi$ is a well defined non-degenerate quadratic form on $\mathbb{Y}$. We define $\Pi$ to be the natural projection $\mathbb{X} \longrightarrow \mathbb{Y}$. By (i) and by the above "non-degenerate case", we obtain

$$\Lambda(V) - c = \dim L - \dim \operatorname{Ker}\varphi = \dim \Pi(L) \leqslant \tfrac{1}{2} \dim \mathbb{Y} = \tfrac{1}{2}(d - c).$$

(Let us note that the linear subspace $\Pi(L)$ is contained in the zero cone $\Phi^{-1}(0)$.) Consequently, $\Lambda(V) \leqslant \frac{1}{2}(d + c)$. The proof of the theorem is complete. □

We are in a position to establish the main formula of the section. We denote by $[\tau]$ the largest integer not greater than a real number $\tau$.

**Theorem 2.2.** *Let $\varphi$, $V$, $d$, and $c$ be as in Theorem* 2.1. *Then*

(i) $\Lambda(V) = d - 1$ *whenever $\varphi$ is reducible and not identically equal to* 0,

(ii) $\Lambda(V) = [\frac{1}{2}(d + c)]$ *whenever $\varphi$ is an irreducible form.*

Proof. Assertion (i) is obvious.

We shall prove (ii). By the irreducibility of $\varphi$ and by the algebraic closedness of $\mathbb{F}$, we have $d \geqslant 3$. Let us additionally assume that $\varphi$ is a non-degenerate form. Then we may also assume without loss of generality that $\mathbb{X} = \mathbb{F}^d$ and that there are elements $\lambda_1, \ldots, \lambda_d \in \mathbb{F}^*$ such that $\varphi(x) = \sum\limits_{j=1}^{d} \lambda_j x_j^2$ for all $x = (x_1, \ldots, x_d) \in \mathbb{F}^d$. Let $\mu_j \in \mathbb{F}$, with $j = 1, \ldots, d$, be an element such that $\mu_j^2 = \lambda_j$, let $\iota \in \mathbb{F}$ be such that $\iota^2 + 1 = 0$, and let $h = [\frac{1}{2}d]$. If $d$ is even, then the linear subspace $\widehat{L}$ of $\mathbb{F}^d$ defined by $\widehat{L} = \{(\mu_{h+1}y_1, \ldots, \mu_d y_h, \iota\mu_1 y_1, \ldots, \iota\mu_h y_h);\ y_1, \ldots, y_h \in \mathbb{F}\}$ is contained in $V = \varphi^{-1}(0)$. Since $\dim \widehat{L} = h$ and, due to Theorem 2.1, $\Lambda(V) \leqslant h$, we get the desired formula. If $d$ is odd, we obtain $\Lambda(V) = h$ taking into consideration the subspace

$$\{(\mu_{h+1}y_1, \ldots, \mu_{d-1}y_h, \iota\mu_1 y_1, \ldots, \iota\mu_h y_h, 0);\ y_1, \ldots, y_h \in \mathbb{F}\} \subset \mathbb{F}^d.$$

Now we turn to the general case of (ii). Let $\Phi\colon\ \mathbb{Y} = \mathbb{X}/\operatorname{Ker}\varphi \longrightarrow \mathbb{F}$ be the non-degenerate quadratic form defined as in the proof of Theorem 2.1 and let $\Pi\colon\ \mathbb{X} \longrightarrow \mathbb{Y}$ be the natural projection. Since the form $\varphi$ is irreducible, so is $\Phi$. Let $L_0$ be a linear subspace of $\mathbb{Y}$ such that $L_0 \subset \Phi^{-1}(0)$ and $\dim L_0 = \Lambda(\Phi^{-1}(0))$. By the above "non-degenerate case", we have $\dim L_0 = [\frac{1}{2}(d-c)]$. The inverse image $\Pi^{-1}(L_0)$ is a linear subspace of $\mathbb{X}$. Moreover, $\Pi^{-1}(L_0) \subset V$ and $\dim \Pi^{-1}(L_0) = [\frac{1}{2}(d-c)]+c = [\frac{1}{2}(d+c)]$. Combining these with inequality (ii) of Theorem 2.1, we get $\Lambda(V) = [\frac{1}{2}(d+c)]$. $\square$

## 3. Linear capacity of the cone over the conjugacy class of a "generic" matrix

In the sequel we shall need the following

**Lemma 3.1.** *Let $A \in \mathcal{M}_n$ be such that $q = r_A(n) \geqslant 1$ and let $B \in \overline{\mathbb{F}\mathcal{O}(A)}$. Then either $r_B(n) = q$ or $B$ is a nilpotent matrix. In particular, if a non-negative integer $l$ is such that $\mathrm{s}_n^l(A) \neq 0$, then*

$$\{C \in \overline{\mathbb{F}\mathcal{O}(A)}\,;\ C \text{ is nilpotent}\} = \{C \in \overline{\mathbb{F}\mathcal{O}(A)}\,;\ \mathrm{s}_n^l(C) = 0\}.$$

Proof. It is evident that $r_B(n) \leqslant q$. Let us assume that $r_B(n) < q$. Then $\mathrm{s}_n^j(B) = 0$ for all $j \geqslant q$. Furthermore, $\mathrm{s}_n^j(B) = 0$ for all $j < q$ such that $\mathrm{s}_n^j(A) = 0$ (because $\mathrm{s}_n^j\colon \mathcal{M}_n \longrightarrow \mathbb{F}$ is a homogeneous regular function). Let $j_0 \in \mathbb{N}$ be such that $1 \leqslant j_0 < q$ and $\mathrm{s}_n^{j_0}(A) \neq 0$. Then

$$(\star) \qquad \overline{\mathbb{F}\mathcal{O}(A)} \subseteq \{D \in \mathcal{M}_n\,;\ [\mathrm{s}_n^{j_0}(D)]^q = \kappa[\mathrm{s}_n^q(D)]^{j_0}\},$$

where $\kappa = [\mathrm{s}_n^{j_0}(A)]^q[\mathrm{s}_n^q(A)]^{-j_0}$ (because $[\mathrm{s}_n^{j_0}(\lambda U^{-1}AU)]^q[\mathrm{s}_n^q(\lambda U^{-1}AU)]^{-j_0} = \kappa$ for all $\lambda \in \mathbb{F}^*$ and for all $U \in \mathcal{GL}_n$). Since $\mathrm{s}_n^q(B) = 0$, it turns out that $\mathrm{s}_n^{j_0}(B) = 0$. Therefore, all the coefficients of the characteristic polynomial of $B$ are equal to 0. This means that the matrix is nilpotent.

To prove the "in particular" part, it suffices to observe that if $C \in \overline{\mathbb{F}\mathcal{O}(A)}$ is not nilpotent, then $\mathrm{s}_n^q(C) \neq 0$ which implies $\mathrm{s}_n^l(C) \neq 0$ (due to the inclusion $(\star)$ with $j_0 = l$, in the case of $l < q$). The proof is complete. $\square$

The main result of the section provides an upper bound for the linear capacity of the cone over the conjugacy class of an arbitrary matrix whose trace is different from 0.

**Theorem 3.2.** *Let $A \in \mathcal{M}_n$ be such that $\mathrm{tr}(A) \neq 0$. Then*

$$\Lambda(\mathbb{F}^*\mathcal{O}(A)) \leqslant 1 + \frac{1}{2}\left(n^2 - q - \sum_{j=1}^{\infty} p_j^2\right),$$

*where $q = r_A(n)$ and $(p_j)_{j=1}^{\infty}$ is the conjugate of the Jordan partition of $A$.*

P r o o f. Let us note that $q \geqslant 1$ and define $\mathcal{V} = \overline{\mathbb{F}\mathcal{O}(A)}$. Let $\mathcal{L} \subseteq \mathcal{V}$ be a linear subspace of $\mathcal{M}_n$.

If there is a non-nilpotent matrix $B_1 \in \mathcal{L}$, then by Lemma 3.1 and by the assumption $\operatorname{tr}(A) \neq 0$, we get $\operatorname{tr}(B_1) \neq 0$ and

$$\{B \in \mathcal{L};\ \operatorname{tr}(B) = 0\} = \{B \in \mathcal{L};\ B \text{ is nilpotent}\}.$$

Consequently, the set $\{B \in \mathcal{L};\ B \text{ is nilpotent}\}$ is a linear subspace and $\dim \mathcal{L} = 1 + \dim\{B \in \mathcal{L};\ B \text{ is nilpotent}\}$. Let $A_0 \in \mathcal{M}_n$ be a nilpotent matrix such that

$$r_{A_0}(j) = \begin{cases} r_A(j), & \text{if} j \leqslant j_0, \\ q + j_0 - j, & \text{if} j_0 < j \leqslant j_0 + q, \\ 0, & \text{if } j > j_0 + q, \end{cases}$$

where $j_0 = \min\{j \in \mathbb{N};\ r_A(j) = q\}$. (Such an $A_0$ actually exists because the above function is weakly decreasing and satisfies the convexity condition.) By Gerstenhaber's Theorem 0.1, we have $\{B \in \mathcal{V};\ B \text{ is nilpotent}\} \subseteq \overline{\mathcal{O}(A_0)}$. Now we apply Gerstenhaber's Theorem 1.2 and the formula ($\bullet$) of Preliminaries to obtain the following sequence of inequalities.

$$\dim \mathcal{L} \leqslant 1 + \dim\{B \in \mathcal{L};\ B \text{ is nilpotent}\} \leqslant 1 + \Lambda(\mathcal{O}(A_0)) = 1 + \frac{1}{2}\left(n^2 - q - \sum_{j=1}^{\infty} p_j^2\right).$$

The above inequalities evidently hold also in the case where $\mathcal{L}$ consists of nilpotent matrices only. Therefore, we get

$$\Lambda(\mathcal{V}) \leqslant 1 + \frac{1}{2}\left(n^2 - q - \sum_{j=1}^{\infty} p_j^2\right).$$

The proof is complete. $\square$

The announced formula for the linear capacity of the cone over the conjugacy class of a "generic" non-nilpotent matrix is the following.

**Theorem 3.3.** *Let $A \in \mathcal{M}_n$ be such that $\operatorname{tr}(A) \neq 0$ and $\#(\sigma(A) \setminus \{0\}) = q$. Then*

$$\Lambda(\mathbb{F}^* \mathcal{O}(A)) = 1 + \frac{1}{2}\left(n^2 - q - \sum_{j=1}^{\infty} p_j^2\right),$$

*where all the notations are as in Theorem* 3.2.

P r o o f. It suffices to find a linear subspace $\widehat{\mathcal{L}} \subseteq \overline{\mathbb{F}\mathcal{O}(A)}$ such that

$$\dim \widehat{\mathcal{L}} = 1 + \frac{1}{2}\left(n^2 - q - \sum_{j=1}^{\infty} p_j^2\right).$$

Let $(\lambda_j)_{j=1}^q$ be the sequence of all the non-zero eigenvalues of the matrix $A$ and let $\mathcal{K}$ be a linear subspace of $\mathcal{M}_{n-q}$ satisfying conditions $\mathcal{K} \subseteq \overline{\mathcal{O}(A_0)}$ and

$$\dim \mathcal{K} = \Lambda(\mathcal{O}(A_0)) = \frac{1}{2}\left((n-q)^2 - \sum_{j=1}^{\infty} p_j^2\right),$$

where $A_0 \in \mathcal{M}_{n-q}$ is a nilpotent matrix such that $A_0 \oplus \bigoplus_{j=1}^{q} \lambda_j \in \mathcal{O}(A)$. (The existence of $\mathcal{K}$ is guaranteed by Gerstenhaber's Theorem 1.2; if $n = q$, we define $\mathcal{K}$ to be the singleton of the "zero matrix of size 0".) We set

$$\widehat{\mathcal{L}} = \left\{ \begin{bmatrix} A & B \\ O_{q\times(n-q)} & \left(\alpha \bigoplus_{j=1}^{q} \lambda_j\right) + C \end{bmatrix} ;\ A \in \mathcal{K},\ B \in \mathcal{M}_{(n-q)\times q},\ \alpha \in \mathbb{F},\ C \in \mathcal{T}_q^0 \right\},$$

where $\mathcal{T}_q^0 = \{C \in \mathcal{M}_q;\ C$ is upper triangular and nilpotent$\}$. Then $\widehat{\mathcal{L}}$ is a linear subspace of $\mathcal{M}_n$, $\dim \widehat{\mathcal{L}} = 1 + \frac{1}{2}(n^2 - q - \sum_{j=1}^{\infty} p_j^2)$, and $\widehat{\mathcal{L}} \subseteq \overline{\mathbb{F}\mathcal{O}(A)}$. (To see the inclusion it is enough to consider the canonical forms of $A$ and of a matrix belonging to $\widehat{\mathcal{L}}$ and having $\alpha = 1$, and apply Gerstenhaber's Theorem 0.1.) The proof is complete. □

The proofs of Theorems 3.2 and 3.3 are slight modifications of the proof of [12, Theorem 3.1]; we present them for the sake of completeness.

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