CANTOR-BERNSTEIN THEOREM FOR LATTICES

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Abstract. This paper is a continuation of a previous author's article; the result is now extended to the case when the lattice under consideration need not have the least element.

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In the paper [6] a result of Cantor-Bernstein type was proved for lattices which (a) have the least element, (b) are σ-complete, and (c) are infinitely distributive.

In the present paper we modify the method from [6] to obtain a generalization of the mentioned result such that the condition (a) is omitted and the conditions (b), (c) are substantially weakened.

We remark that a theorem of Sikorski [10] (proved independently by Tarski [13], cf. also Sikorski [11]) concerning σ-complete Boolean algebras is a corollary of the result from [6].

1. Preliminaries

We denote by $\mathcal{T}_\sigma^0$ the class of all lattices satisfying the conditions (a), (b) and (c) above.

Let L be a lattice and $x_0 \in L$. An indexed system $(x_i)_{i\in I}$ of elements of L will be called orthogonal over x_0 if (i) $x_i \geqslant x_0$ for each $i \in I$, and (ii) $x_{i(1)} \wedge x_{i(2)} = x_0$ whenever $i(1)$ and $i(2)$ are distinct elements of I. The orthogonality under x_0 is defined dually.

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Let α be an infinite cardinal. Consider the following conditions for L:

(b$'_\alpha$) If $x_0 \in L$ and $(x_i)_{i\in I}$ is an indexed system of elements of L which is orthogonal over x_0 and if $\operatorname{card} I \leqslant \alpha$, then the join $\bigvee_{i\in I} x_i$ exists in L.

(b$''_\alpha$) If the assumption of (b$'_\alpha$) is satisfied and if, moreover, the system $(x_i)_{i\in I}$ is upper bounded in L, then $\bigvee_{i\in I} x_i$ exists in L.

(c$'_\alpha$) If the assumption of (b$'_\alpha$) is satisfied and if, moreover, the join $\bigvee_{i\in I} x_i$ exists in L, then for each element $y \in L$ the relation

$$y \wedge (\bigvee_{i\in I} x_i) = \bigvee_{i\in I} (y \wedge x_i)$$

is valid in L.

We denote by (b$'_{\alpha d}$), (b$''_{\alpha d}$) and (c$'_{\alpha d}$) the conditions which are dual to (b$'_\alpha$), (b$''_\alpha$) or (c$'_\alpha$), respectively.

Let $\mathcal{T}^1_\alpha$ be the class of all lattices which satisfy the conditions (b$''_\alpha$), (c$'_\alpha$), (b$''_{\alpha d}$) and (c$'_{\alpha d}$). Next let $\mathcal{T}^2_\alpha$ be the class of all lattices satisfying (b$'_\alpha$), (c$'_\alpha$), (b$'_{\alpha d}$) and (c$'_{\alpha d}$).

We use the notion of internal direct factor with a given central element of a lattice in the same sense as in [6].

The main results of the present paper are Theorem 2.7 and Theorem 3.8. The first one of these theorems says that if $L \in \mathcal{T}^1_\alpha$ and $x^0 \in L$, then the Boolean algebra of all internal direct factors of L with the central element s^0 is α-complete. Theorem 3.8 is a result of Cantor-Bernstein type for lattices belonging to $\mathcal{T}^2_\alpha$, where $\alpha = \aleph_0$; this result is stronger than Theorem 2 of [6].

We substantially apply the methods from [6].

Some theorems of Cantor-Bernstein type for lattice ordered groups and for MV-algebras were proved in [1]–[5], [7]–[9].

2. Internal direct product decompositions

Let L be a lattice belonging to $\mathcal{T}^1_\alpha$, where α is an infinite cardinal. Further let s^0 be an arbitrary but fixed element of L.

We use the terminology and the notation as in [5]; the reader is assumed to be acquainted with the results of Section 2 of [5].

Let I be a set with $\operatorname{card} I = \alpha$. Assume that for each $i \in I$ we have an internal direct product decomposition

$$L = (s^0)L_i \times L'_i \tag{1}$$

with the central element s^0. We suppose that whenever $i(1)$ and $i(2)$ are distinct elements of I, then

(2) $$L_{i(1)} \cap L_{i(2)} = \{s^0\}.$$

For $x \in L$ and $i \in I$ we denote

$$x_i = x(L_i), \quad x'_i = x(L'_i).$$

Let $x, y \in L$. We put xR_iy if $x(L'_i) = y(L'_i)$. Analogously, we set xR'_iy if $x(L_i) = y(L_i)$. Then R_i and R'_i are permutable congruence relations on L with $R_i \wedge R'_i = R_{\min}$ and $R_i \vee R'_i = R_{\max}$.

For each congruence relation ϱ on L and each $x \in L$ we put

$$x_\varrho = \{y \in L \colon x\varrho y\}.$$

Then we have

(3) $$L_i = s^0_{R'_i}, \quad L'_i = s^0_{R_i}, \quad \{x(L_i)\} = s^0_{R_i} \cap x_{R'_i}, \quad \{x(L'_i)\} = s^0_{R'_i} \cap x_{R_i}.$$

We shall systematically apply the relations (3).

2.1. Lemma. *Let $x^0 \in L$. Then*

$$x^0_{R_{i(1)}} \cap x^0_{R_{i(2)}} = \{x^0\}$$

whenever $i(1)$ and $i(2)$ are distinct elements of I.

Proof. This is an immediate consequence of (3). $\square$

Let $a, b \in L$, $a \leqslant b$. Further let $i \in I$. There exist uniquely determined elements x^i and y^i in L such that

$$(x^i)_i = b_i, \quad (x^i)'_i = a'_i,$$
$$(y^i)_i = a_i, \quad (y^i)'_i = b_i.$$

Then

(4) $$\{x^i\} = a_{R_i} \cap b_{R'_i}, \quad \{y^i\} = a_{R'_i} \cap b_{R_i}.$$

From the definition of x^i and y^i we obtain

(5) $$x^i, y^i \in [a, b] \quad \text{for each } i \in I.$$

2.2. Lemma. *Let $i(1)$ and $i(2)$ be distinct elements of I. Then*

$$x^{i(1)} \wedge x^{i(2)} = a, \quad y^{i(1)} \vee y^{i(2)} = b.$$

P r o o f. Put $x^{i(1)} \wedge x^{i(2)} = t$. In view of (5), $t \geqslant a$. Then $t \in [a, x^{i(1)}]$ and hence according to (4), $t \in a_{R_{i(1)}}$; similarly, $t \in a_{R_{i(2)}}$. Thus 2.1 yields that $t = a$. Therefore $x^{i(1)} \wedge x^{i(2)} = a$. Analogously we obtain $y^{i(1)} \vee y^{i(2)} = b$. □

2.3. Corollary. *Under the notation as above, the indexed system $(x^i)_{i\in I}$ is orthogonal over a, and the indexed system $(y^i)_{i\in I}$ is orthogonal under b.*

Since these systems are bounded, we get

2.4. Corollary. *There exist elements x and y in L such that*

$$x = \bigvee_{i\in I} x^i, \quad y = \bigwedge_{i\in I} y^i.$$

2.5. Lemma. *$x \wedge y = a$ and $x \vee y = b$.*

P r o o f. We apply the same steps as in proving the relations (4) and (5) in [6], Section 4 with the distinction that instead of infinite distributivity we apply 2.3 and the relation $L \in \mathcal{T}^1_\alpha$. □

The assertions of 4.3, 4.4 and 4.5 in [6] remain valid for our case (again, in the proof of 4.3 we have to use Lemma 2.3 above).

Now we can use the same argument as in Section 5 of [6] (instead of Lemma 4.2 of [5] we take Lemma 2.5 above). We apply the definitions of R and R' on L (cf. [5]) and we obtain

2.6. Lemma. *$L = (s^0)s^0_R \times s^0_{R'}$ and the relation*

$$s^0_R = \bigvee_{i\in I} L_i$$

is valid in the Boolean algebra $F(L, s^0)$.

By applying the well-known theorem of Smith and Tarski [12] (cf. also Sikorski [11], Chapter II, Theorem 20.1) we conclude from 2.6 that the following theorem holds.

2.7. Theorem. *Let α be an infinite cardinal and let $L \in \mathcal{T}^1_\alpha$. Then the Boolean algebra $F(L, s^0)$ is α-complete.*

3. On lattices belonging to $\mathcal{T}_\sigma^2$

If $\alpha = \aleph_0$, then instead of $\mathcal{T}_\alpha^2$ we write $\mathcal{T}_\sigma^2$.

Let L be a lattice belonging to $\mathcal{T}_\sigma^2$ and $s^0 \in L$. Suppose that for each $n \in \mathbb{N}$ we have an internal direct product decomposition

$$L = (s^0)L_n \times L'_n \tag{1}$$

such that, whenever $n(1)$ and $n(2)$ are distinct positive integers, then

$$L_{n(1)} \cap L_{n(2)} = \{s^0\}. \tag{2}$$

We use analogous notation as in Section 2 with the distinction that we now have $\mathbb{N}$ instead of I.

In particular, the relation

$$s_R^0 = \bigvee_{n\in\mathbb{N}} L_n \tag{3}$$

is valid in the Boolean algebra $F(L, s^0)$; we have

$$L = (s^0)s_R^0 \times (s_R^0)' \tag{4}$$

and, in view of the duality, (3) yields

$$(s_R^0)' = \bigcap_{n\in\mathbb{N}} L'_n. \tag{5}$$

If a, b, x and y are as in 2.5, then we write

$$x = x(a, b), \quad y = y(a, b).$$

3.1. Lemma. *Let $x^0 \in L$. Then x_R^0 is the set of all elements $z \in L$ such that there exist $u, v \in L$ with $x^0, z \in [u, v]$, $x(x^0, v) = v$ and $y(u, x^0) = u$.*

P r o o f. This is a consequence of the definition of R (cf. [5], Section 5). $\square$

3.2. Lemma. *Let m and n be distinct positive integers. Then $L_m \subsetneqq L'_n$.*

P r o o f. In view of (1) and according to 3.7 in [5] we have

$$L_m = (s^0)(L_m \cap L_n) \times (L_m \cap L'_n).$$

Thus according to (2),

$$L_m = (s^0)\{s^0\} \times (L_m \cap L'_n) = L_m \cap L'_n.$$

$\square$

Since the element s^0 was arbitrarily chosen, we get

3.3. Corollary. *Let m, n be as in 3.2 and $x \in L$. Then*

$$x_{R_m} \subseteq x_{R'_n}.$$

3.4. Lemma. *Let $x^0 \in L$ and suppose that $(x^n)_{n\in\mathbb{N}}$ is an indexed system of elements of L such that* (i) *this system is orthogonal over x^0, and* (ii) *$x^n \in x^0_{R_n}$ for each $n \in \mathbb{N}$. Let $x = \bigvee_{n\in\mathbb{N}} x^n$. Then for each $n \in \mathbb{N}$, $xR'_n x^n$.*

Proof. Let $n \in \mathbb{N}$. Since $L \in \mathcal{T}^2_\sigma$, there exists $t \in L$ with

$$t = \bigvee_{m\in\mathbb{N}\setminus\{n\}} x^m.$$

According to 3.3, all elements x^m under consideration belong to $x^0_{R'_n}$. Thus t belongs to the set $x^0_{R'_n}$ as well. Clearly $x = x^n \vee t$. Then $xR'_n\ (x^n \vee x^0)$, whence $xR'_n x^n$. $\square$

3.5. Lemma. *Let $(y^n)_{n\in\mathbb{N}}$ be an indexed system of elements of L such that for each $n \in \mathbb{N}$, $y^n \in L_n$. Then there exists $p \in s^0_R$ such that for each $n \in \mathbb{N}$, $p(L_n) = y^n$.*

Proof. For each $n \in \mathbb{N}$ we denote

$$y^n \vee s^0 = x^n, \quad y^n \wedge s^0 = z^n.$$

Then in view of (2), the indexed system $(x^n)_{n\in\mathbb{N}}$ is orthogonal over s^0, and $(z^n)_{n\in\mathbb{N}}$ is orthogonal under s^0. Hence there exist elements

$$x = \bigvee_{n\in\mathbb{N}} x^n, \quad z = \bigwedge_{n\in\mathbb{N}} z^n$$

in L. Thus xRs^0Rz, whence

$$[z, x] \subseteq s^0_R.$$

Also, $y^n \in s^0_R$ for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$. There exists a uniquely determined element t^n in L such that

$$\{t^n\} = x^n_{R'_n} \cap z_{R_n}.$$

Then from the relation $z \leqslant x^n$ we obtain that t^n belongs to the interval $[z, x^n]$ and hence $t^n \in s^0_R$. Put $p^n = t^n \wedge y^n$. We have $p^n \in [z, t^n]$, thus

$$p^n R_n z. \tag{6}$$

Therefore 3.2 yields that the indexed system $(p^n)_{n\in\mathbb{N}}$ is orthogonal over z. Hence there exists

$$p = \bigvee_{n\in\mathbb{N}} p^n$$

in L. Clearly $p \in [z,x] \subseteq s^0_R$. In view of 3.4, for each $n \in \mathbb{N}$ we have

$$pR'_n p^n. \tag{7}$$

Since $x^n R'_n t^n$ we get

$$(x^n \wedge y^n) R'_n (t^n \wedge y^n),$$

thus $y^n R'_n p^n$. Hence in view of (7), $y^n R'_n p$. But $y^n \in L_n$ and hence $p(L_n) = y^n$. $\square$

3.6. Lemma. *Let $x, y \in s^0_R$. Suppose that $x(L_n) = y(L_n)$ for each $n \in \mathbb{N}$. Then $x = y$.*

Proof. Denote $a = x \wedge y$, $b = x \vee y$. Then $a(L_n) = b(L_n) = x(L_n)$ for each $n \in \mathbb{N}$. It suffices to show that $a = b$.

Let $n \in \mathbb{N}$. Put $a(L_n) = t$. Then $\{t\} = L_n \cap a_{R'_n}$. Hence $aR'_n t$ and similarly $bR'_n t$, which implies that $aR'_n b$.

We have $a, b \in s^0_R$. Then there exists an indexed system $(x^n)_{n\in\mathbb{N}}$ which is orthogonal over a such that $aR_n x^n$ for each $n \in \mathbb{N}$ and $\bigvee\limits_{n\in\mathbb{N}} x^n = b$ (cf. the definition of R in [6]).

From the relations $a \leqslant x^n \leqslant b$ and $aR'_n b$ we obtain $aR'_n x^n$, whence $a = x^n$ for each $n \in \mathbb{N}$. Thus $b = a$. $\square$

Consider the mapping

$$\varphi\colon\ s^0_R \to \prod_{n\in\mathbb{N}} L_n$$

defined by

$$\varphi(x) = (x(L_n))_{n\in\mathbb{N}}$$

for each $x \in s^0_R$.

From the definition of φ we immediately obtain that φ is a homomorphism. In view of 3.5, φ is an epimorphism. According to 3.6, φ is a monomorphism. Hence φ is an isomorphism of s^0_R onto $\prod\limits_{n\in\mathbb{N}} L_n$. All L_n are sublattices of s^0_R containing the element s^0. If $x \in L_n$ for some $n \in \mathbb{N}$, then $(\varphi(x))_n = x$ and $(\varphi(x))_m = s^0$ for $m \neq n$. Hence in view of (3) we have

3.7. Lemma. *Let* (1) *and* (3) *be valid. Then*

$$\bigvee_{n\in\mathbb{N}} L_n = (s^0) \prod_{n\in\mathbb{N}} L_n.$$

3.8. Theorem. *Let L_1 and L_2 be lattices belonging to $\mathcal{T}_\sigma^2$. Suppose that*

(i) *L_1 is isomorphic to some direct factor of L_2;*

(ii) *L_2 is isomorphic to some direct factor of L_1.*

Then L_1 is isomorphic to L_2.

Proof. It suffices to apply the same argument as in proving Theorem 2 of [6] (Section 6) with the distinction that instead of Lemma 6.3 from [6] we now use Lemma 3.7. □

Theorem 3.8 generalizes Theorem 2 of [6].

4. Examples

4.1. Let $\mathbb{N}$ be the set of all positive integers with the usual linear order and let A be a two-element lattice. Put $B = A \times \mathbb{N}$, $L = B \cup \{\omega\}$, where $b < \omega$ for each $b \in B$. Then $L \in \mathcal{T}_\alpha^1 \cap \mathcal{T}_\alpha^2$ for each infinite cardinal α, but L fails to be infinitely distributive.

4.2. Let L be as in 4.1 and let L_1 be a sublattice of L such that $L_1 = L \setminus \{\omega\}$. Then $L_1 \in \mathcal{T}_\alpha^1 \cap \mathcal{T}_\alpha^2$, L_1 is infinitely distributive and fails to be σ-complete.

Now let us return to the conditions (b'_α), ($\mathrm{b}'_{\alpha d}$), (b''_α), ($\mathrm{b}''_{\alpha d}$), (c'_α) and ($\mathrm{c}'_{\alpha d}$). We denote the system of these condition by S. Let α be an arbitrary infinite cardinal.

It is obvious that (b'_α) $\Rightarrow$ (b''_α) and ($\mathrm{b}'_{\alpha d}$) $\Rightarrow$ ($\mathrm{b}''_{\alpha d}$).

4.3. Let F be the system of finite subsets of the set $\mathbb{N}$; the system F is partially ordered by the set-theoretic inclusion. Then F satisfies all the conditions from S except (b'_α).

4.4. Let F be as in 4.3 and let F_1 be a lattice which is dual to F. Let α be an arbitrary infinite cardinal. Then F_1 satisfies all the conditions from S except the condition ($\mathrm{b}'_{\alpha d}$).

4.5. Let F be as in 4.3 and let $\mathbb{N}$ be the set of all positive integers with the natural linear order. The lattice dual to $\mathbb{N}$ will be denoted by $\mathbb{N}'$. We may assume that $F \cap \mathbb{N}' = \emptyset$. Put $L = F \cup \mathbb{N}'$. The partial order in L is defined as follows: for each $x \in F$ and each $y \in \mathbb{N}'$ we put $x < y$. If $x, y \in F$ or $x, y \in \mathbb{N}'$, then the relation $x \geqslant y$ has its original meaning (deduced from F or from $\mathbb{N}'$, respectively). The lattice L satisfies all conditions from S except (b'_α) and (b''_α).

4.6. Let L be as in 4.5 and let L_1 be a lattice dual to L. Then L_1 satisfies all conditions from S except ($\mathrm{b}'_{\alpha d}$) and ($\mathrm{b}''_{\alpha d}$).

4.7. Let L be as in 4.5. We denote by ω the greatest element of L. Further, let L_1 be the sublattice of L with the underlying set $F \cup \{\omega\}$. Then L_1 satisfies all the conditions of the system S except (c'_{α}).

4.8. Let L_1 be as in 4.7 and let L_2 be a lattice dual to L_1. Then L_2 satisfies all the conditions of S except ($\mathrm{c}'_{\alpha d}$).

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