GRAPHS ISOMORPHIC TO THEIR PATH GRAPHS

Martin Knor[1], Bratislava, Ľudovít Niepel[2], Kuwait

(Received November 15, 2000)

Abstract. We prove that for every number $n \geqslant 1$, the n-iterated P_3-path graph of G is isomorphic to G if and only if G is a collection of cycles, each of length at least 4. Hence, G is isomorphic to $P_3(G)$ if and only if G is a collection of cycles, each of length at least 4. Moreover, for $k \geqslant 4$ we reduce the problem of characterizing graphs G such that $P_k(G) \cong G$ to graphs without cycles of length exceeding k.

Keywords: line graph, path graph, cycles

MSC 2000: 05C38

1. Introduction

Let G be a graph, $k \geqslant 1$, and let $\mathcal{P}_k$ be the set of paths of length k in G. The vertex set of a *path graph* $P_k(G)$ is the set $\mathcal{P}_k$. Two vertices of $P_k(G)$ are joined by an edge if and only if the edges in the intersection of the corresponding paths form a path of length $k-1$ in G, and their union forms either a cycle or a path of length $k+1$. It means that the vertices are adjacent if and only if one can be obtained from the other by "shifting" the corresponding paths in G.

Path graphs were investigated by Broersma and Hoede in [2] as a natural generalization of line graphs, since $P_1(G)$ is the line graph $L(G)$ of G (for further connections to line graphs see [6]). Traversability of P_2-path graphs is studied in [9], and a characterization of P_2-path graphs is given in [2] and [7]. Distance properties of path graphs are studied in [1], [3] and [5], and in [4] graphs with connected P_3-path graphs are characterized.

When a new function on graphs appears, one of the very first problems is to determine the fixed points of the function, i.e., graphs that are isomorphic to their

[1] Supported by VEGA grant 1/6293/99.
[2] Supported by Kuwait University grant #SM 02/00.

images. It is well known (and trivial to prove) that a connected graph G is isomorphic to its line graph $L(G)$ if and only if G is a cycle.

In [2] it is proved that a connected graph G is isomorphic to its P_2-path graph if and only if G is a cycle. We remark that $P_k(G)$ is not necessarily a connected graph if G is connected and $k \geqslant 2$. However, slight modifications of the proof in [2] give rise to the following theorem:

Theorem A. *Let G be a graph isomorphic to $P_2(G)$. Then each component of G is a cycle.*

However, even stronger theorem follows from [5]. Let $P_k^i(G)$ denote the i-iterated P_k-path graph of G, i.e., $P_k^i(G) = P_k(P_k^{i-1}(G))$ if $i > 0$, and $P_k^0(G) = G$. We have

Theorem B. *Let G be a graph and n a number, $n \geqslant 1$, such that G is isomorphic to $P_2^n(G)$. Then each component of G is a cycle.*

By now, only a little is known about P_k-path graphs for $k \geqslant 3$. In [8] Li and Zhao proved the following theorem:

Theorem C. *Let G be a connected graph isomorphic to $P_3(G)$. Then G is a cycle of length greater than or equal to* 4.

In this paper we generalize Theorem C to an analogue of Theorem A for P_3-path graphs:

Theorem 1. *Let G be a graph isomorphic to $P_3(G)$. Then each component of G is a cycle of length greater than or equal to* 4.

In fact, we prove more. We prove an analogue of Theorem B for P_3-path graphs:

Theorem 2. *Let G be a graph and n a number, $n \geqslant 1$, such that G is isomorphic to $P_3^n(G)$. Then each component of G is a cycle of length greater than or equal to* 4.

We remark that the proof of Theorem C in [8] is analogous to the proof of Theorem A (in a weaker form) in [2], and since it is based on some counting arguments, it is not clear how to extend it to the proof of Theorem 2. In fact, our approach to the problem is completely different.

At present, we are not able to generalize Theorem 1 to P_k-path graphs for $k \geqslant 4$. The problem seems to be too complicated in general; for a solution for graphs in which every component contains a "large" cycle see Lemma 3. Thus, for $k \geqslant 4$ it may be useful to start with trees, see also Corollary 6. However, even for trees the problem appears to be hard. In this connection we pose the following

Problem. *Does there exist a tree T and a number k, $k \geqslant 4$, such that $P_k^i(T)$ is a nonempty forest for every $i \geqslant 0$?*

If such a tree does not exist, then $P_k(G) \ncong G$ for every number k and a forest G. (We remark that the problem is trivial for $k = 1$; for $k = 2$ it is solved negatively in [5]; and for $k = 3$ it is solved negatively by Lemma 7.)

In the next two sections we prove Theorem 2. In Section 2 we present some general results for P_k-path graphs when $k \geqslant 2$, and Section 3 is devoted to P_3-path graphs of trees.

2. General results

We use standard graph-theoretic notation. Let G be a graph. The vertex set and the edge set of G, respectively, are denoted by $V(G)$ and $E(G)$. If v is a vertex of G then $\deg_G(v)$ denotes the degree of v in G. For two subgraphs, H_1 and H_2 of G, we denote by $H_1 \cup H_2$ the union of H_1 and H_2 in G, and by $H_1 \cap H_2$ their intersection. A path and a cycle, respectively, of length l are denoted by P_l and C_l.

For easier handling of paths of length k in G (i.e., the vertices of $P_k(G)$) we make the following agreement. We denote the vertices of $P_k(G)$ (as well as the vertices of G) by small letters $a, b, \ldots$, while the corresponding paths of length k in G are denoted by capital letters $A, B, \ldots$. It means that if A is a path of length k in G and a is a vertex in $P_k(G)$, then a must be the vertex corresponding to the path A.

Throughout this section, the symbol k is used for the length of paths producing the path graph. I.e., we consider here only P_k-path graphs, $k \geqslant 2$. By a *large cycle* we mean a cycle of length greater than k. A cycle whose length does not exceed k is a *small cycle*.

Lemma 3. *Let G be a graph such that $P_k^n(G) \cong G$ for some $n \geqslant 1$ and $k \geqslant 2$. Then every component of G containing a large cycle is isomorphic to a single cycle.*

P r o o f. If C is a large cycle then $P_k(C) \cong C$, and hence $P_k^n(C) \cong C$, too. As $P_k^n(G) \cong G$, all large cycles in $P_k^n(G)$ are images of large cycles in G.

If a and b are adjacent vertices in $P_k(G)$, then A and B share a path of length $k-1$. This implies that if C is a large cycle in G, then $P_k(C)$ does not contain a chord in $P_k(G)$, and hence, $P_k^n(C)$ does not contain a chord in $P_k^n(G)$, either. Since the number of large cycles in G is equal to the number of large cycles in $P_k^n(G)$, no large cycle contains a chord in G.

Suppose that there is a large cycle in G with a vertex incident to an edge outside this cycle. For every large cycle C, let $I_G(C)$ denote the total number of edges outside C that are incident to a vertex of C. As C does not contain a chord,

$I_{P_k(G)}(P_k(C)) = 2 \cdot I_G(C)$, and $I_{P_k^n(G)}(P_k^n(C)) = 2^n \cdot I_G(C)$. Let $I(G)$ denote the maximum value of $I_G(C)$, where C is a large cycle in G. Then $I(P_k^n(G)) = 2^n \cdot I(G)$. Since $I(G) > 0$, $P_k^n(G)$ is not isomorphic to G. $\square$

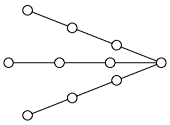
Lemma 4. *Every small cycle in $P_k(G)$ has an even length.*

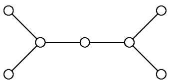
Proof. Let $C = (a_1, a_2, \ldots, a_l)$ be a small cycle of length l in $P_k(G)$, $l \leqslant k$. If u and v are adjacent vertices in $P_k(G)$, then U and V share a path of length $k-1$. Since $l \leqslant k$, $A_1, A_2, \ldots, A_l$ share a path P of length $t \geqslant k-(l-1) \geqslant 1$ in G. Let $P = (p_0, p_1, \ldots, p_t)$.

Assume that $A_1 = (a_{1,0}, a_{1,1}, \ldots, a_{1,k}), \ldots, A_l = (a_{l,0}, a_{l,1}, \ldots, a_{l,k})$ are denoted so that for every i, $1 \leqslant i \leqslant l$, we have $i_0 < i_t$, where $a_{i,i_0} = p_0$ and $a_{i,i_t} = p_t$. Then i_0 and $(i+1)_0$ have different parity, $1 \leqslant i < l$. As $(a_1, a_2, \ldots, a_l)$ forms a cycle, 1_0 and l_0 have different parity, too, and hence l is even. $\square$

We remark that Lemma 3 and Lemma 4 reduce the examination of graphs G for which $P_k^n(G) \cong G$, to graphs without large cycles and without odd small cycles. Hence, to complete the proof of Theorem 2 it remains to study forests.

Definition. A tree Ψ_t is obtained from a claw $K_{1,3}$ subdividing each edge of $K_{1,3}$ by $t-1$ vertices, see Figure 1 for Ψ_3. A tree $\bowtie_{t_1,t_2}$ is obtained from two paths of length $2t_1$ central vertices of which are joined by a path of length t_2, see Figure 2 for $\bowtie_{1,2}$ and Figure 3 for $\bowtie_{2,1}$.

Ψ_3: 

$\bowtie_{1,2}$: 

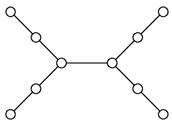
$\bowtie_{2,1}$: 

Figure 1 Figure 2 Figure 3

It is easy to see that $P_k(\Psi_k) = C_{3k}$ and $P_k(\bowtie_{t,k-t})$ contains C_{4t}, $1 \leqslant t < k$. Thus, P_k-path graphs of trees containing a copy of Ψ_k or $\bowtie_{t,k-t}$ contain cycles. The next lemma shows a converse.

Lemma 5. *If T is a tree such that $P_k(T)$ contains a cycle, then T contains either Ψ_k or $\bowtie_{t,k-t}$, $1 \leqslant t < k$.*

Proof. Only for this proof we define the notion of a turning path (see also [1]). Let $(x_0, x_1, \ldots, x_{l-1})$ be a closed walk of length l in $P_k(G)$. Then for every i, $0 \leqslant i < l$, the edges of $X_{i-1} \cap X_i$ form a path of length $k-1$ (the indices are modulo l). A path X_i is a turning path if and only if the edges of $X_{i-1} \cap X_i \cap X_{i+1}$ form a path of length $k-1$, too.

Let T be a tree, and let $C = (a_0, a_1, \ldots, a_{l-1})$ be a cycle of length l in $P_k(T)$. Since T is a tree, there are at least two turning paths among $A_0, A_1, \ldots, A_{l-1}$. Let A_{i_1} and A_{i_2} be turning paths such that $t = [(i_2 - i_1) \bmod l]$ is the smallest possible. Then $A_{i_1} \cup A_{i_1+1} \cup \ldots \cup A_{i_2}$ forms a path of length $k + t$ (the indices are modulo l).

If $t \geqslant k$, then $A_{i_1-t} \cup A_{i_1-t+1} \cup \ldots \cup A_{i_1}$ forms another path of length $t + k$, since $A_{i_1-t+1}, A_{i_1-t+2}, \ldots, A_{i_1-1}$ are not turning paths. As $A_{i_1-1} \neq A_{i_1+1}$, T contains Ψ_k.

Thus, suppose that $t < k$. Since $t < k$, A_{i_1} and A_{i_2} share a path $P = (p_0, p_1, \ldots, p_{l'})$ of length $l' = k - t$. Assume that $A_{i_1} = (b_t, b_{t-1}, \ldots, b_1, p_0, p_1, \ldots, p_{l'})$ and $A_{i_2} = (p_0, p_1, \ldots, p_{l'}, c_1, c_2, \ldots, c_t)$. Then $A_{i_1} \cup A_{i_1+1} \cup \ldots \cup A_{i_2} = (b_t, b_{t-1}, \ldots, b_1, p_0, p_1, \ldots, p_{l'}, c_1, c_2, \ldots, c_t)$. Since t is the smallest possible distance between vertices of C corresponding to turning paths, we have $A_{i_1-t} \cup A_{i_1-t+1} \cup \ldots \cup A_{i_1} = (b_t, b_{t-1}, \ldots, b_1, p_0, p_1, \ldots, p_{l'}, d_1, d_2, \ldots, d_t)$, where $d_1 \neq c_1$, and $A_{i_2} \cup A_{i_2+1} \cup \ldots \cup A_{i_2+t} = (e_t, e_{t-1}, \ldots, e_1, p_0, p_1, \ldots, p_{l'}, c_1, c_2, \ldots, c_t)$, where $e_1 \neq b_1$. As T is a tree and $l' = k - t$, T contains $\bowtie_{t,k-t}$. $\square$

By Lemma 5 and the note before it, we have

Corollary 6. *Let T be a tree. Then $P_k(T)$ contains a cycle if and only if T contains either Ψ_k or $\bowtie_{t,k-t}$, $1 \leqslant t < k$.*

3. P_3-path graphs

Definition. A *caterpillar* is a tree T with a path $P = (v_0, v_2, \ldots, v_l)$, such that the eccentricity of v_i is at most 2 in the component of $T - E(P)$ containing v_i, $0 \leqslant i \leqslant l$. A 3-*caterpillar* T is a caterpillar with a path $P = (v_0, v_2, \ldots, v_{3r})$ such that $\deg_T(v_i) = 2$ if $0 < i < 3r$ and $i \neq 3j$. The vertices v_{3j} are called the *basic vertices* of a 3-caterpillar, $0 \leqslant j \leqslant r$.

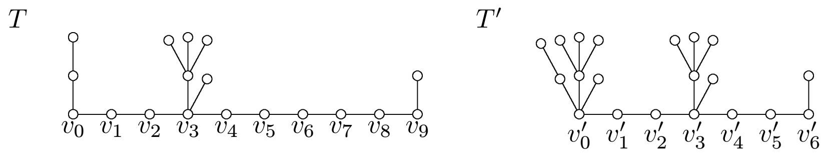


Figure 4

In Figure 4 we have a 3-caterpillar T with 4 basic vertices v_0, v_3, v_6 and v_9. We remark that usually, the term caterpillar is used for a bit different tree. However, in this paper we use this notion only for the graph defined in the preceding definition.

Let G be a forest such that $P_3^n(G) \cong G$ for some $n \geqslant 1$. By Lemma 4, every cycle in $P_3(G)$ is a large cycle. Thus, G does not contain Ψ_3, $\bowtie_{1,2}$ or $\bowtie_{2,1}$ by Corollary 6, as otherwise $P_3^n(G)$ contains a large cycle. In particular, since G does not contain $\mathbb{Y}_3$, it is a disjoint union of caterpillars.

Lemma 7. *Let T be a caterpillar such that $P_3^i(T)$ is a forest for every $i \geqslant 0$. Then there is j such that $P_3^j(T)$ is an empty graph.*

P r o o f. Let T be a caterpillar such that $P_3^i(T)$ does not contain a cycle for every $i \geqslant 0$. If the diameter of T is at most 4, then $P_3(T)$ does not contain a path of length 3 (recall that T does not contain $\mathsf{X}_{1,2}$), so that $P_3^2(T)$ is an empty graph. Hence, assume that the diameter of T is at least 5.

If G is a tree, then at most one nontrivial component of $P_3(G)$ is different from a complete bipartite graph, see [4, Corollary 5]. Since the diameter of a complete bipartite graph is at most two, at most one nontrivial component of $P_3(T)$ is a caterpillar containing a path of length 3. Hence, in what follows it is enough to consider this unique "large" caterpillar of $P_3(G)$.

Let T be a 3-caterpillar with a path $P = (v_0, v_1, \ldots, v_{3r})$ denoted as in the definition above. Then T has exactly $r+1$ basic vertices. Let $V_i' = (v_i, v_{i+1}, v_{i+2}, v_{i+3})$, $0 \leqslant i \leqslant 3r-3$. Then the (large) caterpillar T' of $P_3(T)$ is a 3-caterpillar with a path $P' = (v_1', v_2', \ldots, v_{3r-3}')$ with exactly r basic vertices, see Figure 4. Hence, the caterpillar of $P_3^r(T)$ is a 3-caterpillar with a unique basic vertex, so that $P_3^{r+2}(T)$ is an empty graph.

To prove the lemma it is enough to concentrate on caterpillars that are not 3-caterpillars. These caterpillars T^* contain a pair of vertices u_1^* and u_2^* at a distance either $3j+1$ or $3j+2$ such that $\deg_{T^*}(u_1^*) \geqslant 3$ and $\deg_{T^*}(u_2^*) \geqslant 3$. Moreover, if the distance from u_1^* to u_2^* is $3j+1$ and Q^* is a path joining u_1^* with u_2^*, then the eccentricity of u_i^* in the component of $T^* - E(Q^*)$ containing u_i^* is at least 2, $i \in \{1,2\}$. In what follows consider the caterpillar T of $P_3^j(T^*)$. By our assumption, T is not a 3-caterpillar and it contains two vertices u_1 and u_2 at a distance either 1 or 2 such that $\deg_T(u_1) \geqslant 3$ and $\deg_T(u_2) \geqslant 3$. Moreover, if $u_1u_2 \in E(G)$ then the eccentricity of u_i in the component of $T - \{u_1u_2\}$ containing u_i is at least 2, $i \in \{1,2\}$.

If u_1 and u_2 have distance 2 then T contains $\mathsf{X}_{1,2}$, so that $P_3(T)$ contains a large cycle.

Now consider a caterpillar T with $u_1u_2 \in E(T)$. Let $\mathrm{Co}(u_1)$ and $\mathrm{Co}(u_2)$ be components of $T - \{u_1u_2\}$ containing u_1 and u_2, respectively. As mentioned above, the eccentricities of u_1 and u_2 in these components are at least 2. However, if both of them exceed 2, then $P_3(T)$ contains $\mathsf{X}_{1,2}$.

Thus, suppose that the eccentricity of u_1 in $\mathrm{Co}(u_1)$ is exactly 2. Then $\mathrm{Co}(u_1)$ consists of at most 2 paths of length 2 rooted in u_1 and of some edges incident with u_1. Otherwise T contains $\mathsf{X}_{1,2}$ or $P_3(T)$ contains V_3. Moreover, as iterated P_3-path graphs of T do not contain $\mathsf{X}_{1,2}$, the distance from u_2 to u is $3j$ if $u \in V(\mathrm{Co}(u_2))$ and $\deg_T(u) \geqslant 3$.

We distinguish two cases.

C a s e 1: There are exactly two paths of length 2 rooted in u_1.

If the eccentricity of u_2 is at least 5 in $\mathrm{Co}(u_2)$, then $P_3(T)$ contains V_3. On the other hand, if the eccentricity of u_2 is at most 4 in $\mathrm{Co}(u_2)$, then the caterpillar T' of $P_3(T)$ is a 3-caterpillar, or $P_3(T)$ contains $\mathsf{X}_{1,2}$, see Figure 5. (We remark that $U' = (u_1, u_2, u_3, u_4)$. Extra edges that are possibly in T or in T', are represented by halfedges in Figure 5.)

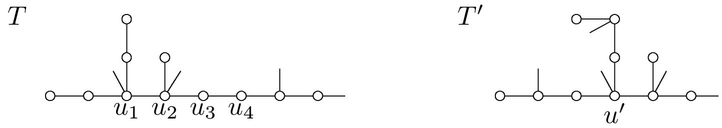


Figure 5

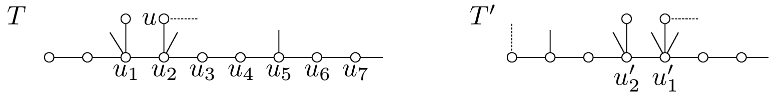


Figure 6

C a s e 2: There is exactly one path of length 2 rooted in u_1.

Let $(u_2, u_3, \ldots, u_k)$ be a longest path of $\mathrm{Co}(u_2)$ rooted in u_2. Denote by T' the caterpillar of $P_3(T)$, $U_2' = (u_1, u_2, u_3, u_4)$ and $U_1' = (u_2, u_3, u_4, u_5)$ (if u_5 exists), see Figure 6. Let $\mathrm{Co}(u_1')$ and $\mathrm{Co}(u_2')$ be defined analogously to $\mathrm{Co}(u_1)$ and $\mathrm{Co}(u_2)$ above.

If the eccentricity of u_2 is at least 6 in $\mathrm{Co}(u_2)$ (i.e., if $k \geqslant 8$), then the eccentricities of u_1' and u_2' are greater than 2 in $\mathrm{Co}(u_1')$ and $\mathrm{Co}(u_2')$, respectively. Hence, $P_3^2(T)$ contains $\mathsf{X}_{1,2}$, a contradiction.

If the eccentricity of u_2 is at most 2 in $\mathrm{Co}(u_2)$ (i.e., if $k = 4$), then T' is a 3-caterpillar (recall that T' does not contain $\mathsf{X}_{1,2}$).

Let u be a vertex adjacent to u_2, $u \neq u_3$, such that $\deg_T(u) = 2$, see Figure 6. If $k = 7$ then T' is of the type already solved in Case 1, and if $5 \leqslant k \leqslant 6$ then T' is of the type from Case 2 and all (but one) neighbours of u_2' have degree 1 in $\mathrm{Co}(u_2')$. Hence, we may assume that all neighbours of u_2 (except u_3) have degree 1 in $\mathrm{Co}(u_2)$.

Now if $k = 7$ then T' is of the type from Case 2 with the eccentricity of u_2' exactly 3 in $\mathrm{Co}(u_2')$, and if $5 \leqslant k \leqslant 6$ then T' is a 3-caterpillar. $\square$

By Lemma 7, for every caterpillar T there is a number t such that $P_3^t(T)$ is an empty graph or $P_3^t(G)$ contains a large cycle. Thus, for every n, $n \geqslant 1$, we have $P_3^n(G) \ncong G$ if G is a forest, which completes the proof of Theorem 2.

References

[1] *Belan A., Jurica P.*: Diameter in path graphs. Acta Math. Univ. Comenian. New Ser. *68* (1999), 111–126.

[2] *Broersma H. J., Hoede C.*: Path graphs. J. Graph Theory *13* (1989), 427–444.

[3] *Knor M., Niepel Ľ.*: Centers in path graphs. J. Comb. Inf. Syst. Sci. *24* (1999), 79–86.

[4] *Knor M., Niepel Ľ.*: Connectivity of path graphs. Discuss. Math., Graph Theory *20* (2000), 181–195.

[5] *Knor M., Niepel Ľ.*: Distances in iterated path graphs. To appear in Discrete Math.

[6] *Knor M., Niepel Ľ.*: Path, trail and walk graphs. Acta Math. Univ. Comenian. New Ser. *68* (1999), 253–256.

[7] *Li H., Lin Y.*: On the characterization of path graphs. J. Graph Theory *17* (1993), 463–466.

[8] *Li X., Zhao B.*: Isomorphisms of P_4-graphs. Australas. J. Comb. *15* (1997), 135–143.

[9] *Yu X.*: Trees and unicyclic graphs with Hamiltonian path graphs. J. Graph Theory *14* (1990), 705–708.

Author's address: *Martin Knor*, Slovak University of Technology, Faculty of Civil Engineering, Department of Mathematics, Radlinského 11, 813 68 Bratislava, Slovakia, e-mail: `knor@vox.svf.stuba.sk`; *Ľudovít Niepel*, Kuwait University, Faculty of Science, Department of Mathematics & Computer Science, P.O. Box 5969, Safat 13060, Kuwait, e-mail: `niepel@math-1.sci.kuniv.edu.kw`.