ON THE CONNECTIVITY OF SKELETONS OF PSEUDOMANIFOLDS WITH BOUNDARY

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Abstract. In this note we show that 1-skeletons and 2-skeletons of n-pseudomanifolds with full boundary are $(n+1)$-connected graphs and n-connected 2-complexes, respectively. This generalizes previous results due to Barnette and Woon.

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Introduction

The classical notion of n-connectedness in graph theory admits an immediate generalization to 2-complexes. Moreover, Woon [5] showed that 2-skeletons of closed combinatorial n-manifolds are examples of n-connected 2-complexes. This result is a partial analogue of a theorem due to Barnette [2] stating that 1-skeletons of closed n-pseudomanifolds are $(n+1)$-connected graphs. It is then a natural question to ask for an extension of Woon's theorem to all closed pseudomanifolds. Here we provide such an extension and, moreover, we show that both the theorems actually hold for pseudomanifolds with full boundaries; see Theorem 2.3 and Theorem 3.4, respectively.

We recall that a *locally finite simplicial complex*, K, is a countable set of simplexes such that:

(a) If $\sigma \in K$ and τ is a face of σ ($\tau < \sigma$, for short) then $\tau \in K$.

(b) If $\sigma, \sigma' \in K$ then $\sigma \cap \sigma'$ is either empty or a common face of σ and σ'.

(c) Any $\sigma \in K$ is a face of only finitely many simplexes of K.

For $\sigma \in K$, the *star* and the *link* of σ in K are the subcomplexes $\mathrm{st}(\sigma; K) = \{\mu;\ \mu, \sigma < \tau \in K\}$, and $\mathrm{lk}(\sigma; K) = \{\mu \in \mathrm{st}(\sigma; K); \sigma \cap \mu = \emptyset\}$, respectively. Here a *subcomplex* L of K is a complex whose simplexes are simplexes of K. Given a subcomplex $L \subseteq K$, the notation $K - L$ will stand for the subcomplex of K, $K - L = \{\tau \in K; \tau < \varrho \text{ and } \varrho \notin L\}$. The *i-skeleton* of K is the subcomplex $\mathrm{sk}^i K \subseteq K$ consisting of all simplexes $\sigma \in K$ with $\dim \sigma \leqslant i$.

A simplicial complex K is said to be *purely n-dimensional* if any simplex $\sigma \in K$ is a face of an n-simplex of K. A purely n-dimensional simplicial complex K is said to be *strongly connected* if, given any two n-simplexes $\gamma, \gamma' \in K$, there exists a chain of n-simplexes connecting them. Here by a *chain* from γ to γ' we mean a sequence $\sigma_0 \ldots \sigma_k$ of n-simplexes such that $\gamma = \sigma_0$, $\gamma' = \sigma_k$, and $\sigma_i \cap \sigma_{i+1}$ is a common $(n-1)$-face. For the sake of simplicity, we will say that K is an *n-complex* when K is a purely n-dimensional locally finite complex. Let σ be an $(n-1)$-simplex of an n-complex K. The *valence* of σ, $\mathrm{val}(\sigma)$, is the number of n-simplexes in $\mathrm{st}(\sigma; K)$. The valence of K is the number

$$\mathrm{val}(K) = \min\{\mathrm{val}(\sigma);\ \dim \sigma = n - 1\}.$$

An $(n-1)$-simplex $\sigma \in K$ is said to be a *boundary simplex* if $\mathrm{val}(\sigma) = 1$. Otherwise we say that σ is an *interior simplex.* The *boundary* of K, ∂K, is the smallest subcomplex of K containing the boundary simplexes. The boundary ∂K is said to be *full* if any simplex in K meets ∂K in a (possibly empty) face.

We recall that an *n-pseudomanifold* M is a strongly connected n-complex such that $\mathrm{val}(\sigma) \leqslant 2$ for any $(n-1)$-simplex $\sigma \in M$. We will denote by $\mathcal{P}$ the class of all pseudomanifolds. An *n-pseudomanifold* M is said to be *normal* if for every k-simplex σ ($k \leqslant n-2$) the link $\mathrm{lk}(\sigma; M)$ is an $(n-k)$-pseudomanifold, and ∂M is also a normal $(n-1)$-pseudomanifold. Let $\mathcal{N}$ stand for the class of all normal pseudomanifolds.

A *combinatorial n-ball* (*n-sphere*) is a simplicial complex which admits a subdivision simplicially isomorphic to a subdivision of the n-simplex Δ^n ($\partial \Delta^{n+1}$, respectively). A *combinatorial n-manifold* M is a purely n-dimensional complex such that for every vertex $v \in M$, $\mathrm{lk}(v; M)$ is a combinatorial $(n-1)$-ball or a combinatorial $(n-1)$-sphere. The class of all combinatorial manifolds will be denoted by $\mathcal{C}$.

More generally, a *homology n-manifold* M is an n-complex such that for every vertex $v \in M$, $\widetilde{H}_*(\mathrm{lk}(v;M))$ is either trivial or isomorphic to $\widetilde{H}_*(S^{n-1})$. Here $\widetilde{H}_*$ denotes reduced simplicial homology. If $\mathcal{H}$ denotes the class of all homology manifolds, we have $\mathcal{C} \subseteq \mathcal{H} \subseteq \mathcal{N} \subseteq \mathcal{P}$. Moreover, it can be shown that all these classes coincide in dimension 1. In dimension 2 one has $\mathcal{N} = \mathcal{H} = \mathcal{C}$ while in dimension 3 the equality $\mathcal{H} = \mathcal{C}$ still holds.

2. Connectivity of 1-skeletons of pseudomanifolds

By a *graph* we mean a connected 1-complex G. A *path* $\alpha\colon a_0 - a_n$ between two vertices $a_0, a_n \in G$ is a finite sequence of vertices $\{a_0, \ldots, a_n\}$ such that $a_i \neq a_j$ $(i \neq j)$, and the segment $\langle a_i,\ a_{i+1}\rangle$ is an edge of G $(0 \leqslant i \leqslant n-1)$.

Two paths $\alpha, \beta\colon a - b$ are said to be *independent* if $\alpha \cap \beta = \{a, b\}$. The dual notion to independent paths is the notion of a *juncture set*. Namely, given a set of vertices $J \subseteq V(G)$, we say that J is a juncture set for $a, b \in V(G)$ if a and b lie in different components of $G - J$.

The classical Menger-Whitney Theorem relating independent paths and juncture sets is the following

Theorem 2.1 ([3]). *Given two vertices a, b in a graph G, the following two statements are equivalent:*

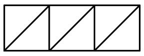
(a) *There is no juncture set for a and b with fewer than n vertices.*

(b) *There exist n independent paths from a to b.*

A graph G is said to be *n-connected* if condition (b), and hence (a), holds for any pair of vertices.

In [2] it is proved that the 1-skeleton $\mathrm{sk}^1 M$ of a finite n-pseudomanifold without boundary is an $(n+1)$-connected graph. The same proof works for infinite pseudomanifolds without boundary. We will extend this result to any n-pseudomanifold $(n \geqslant 2)$ with full boundary (see §1). The following simple example shows that fullness of boundaries is needed. We recall that for any pseudomanifold M the boundary of its first barycentric subdivision is always full.

Example 2.2. Clearly the 1-skeleton of the 2-ball M below is only 2-connected:

$M \equiv$ 

Theorem 2.3. *Let M^n be an n-pseudomanifold with full boundary $(n \geqslant 2)$. Then its 1-skeleton $\mathrm{sk}^1\ M$ is an $(n+1)$-connected graph.*

The proof of (2.3) follows the same pattern as the proof for closed pseudomanifolds in [2]. In particular, we use the following lemma ([2]; §3).

Lemma 2.4. *Let K be a strongly connected n-complex. Then $\mathrm{sk}^1 K$ is n-connected.*

In addition to (2.4) we will also need the following results to start induction in the proof of (2.3).

Lemma 2.5. *Let M be a 2-pseudomanifold with full boundary. Given a vertex $v \in M$, the family $\mathcal{L} = \{L_i\}$ of connected components of $L = \mathrm{lk}(v; M)$ is a disjoint family of cycles and arcs. Moreover, $\mathcal{L}$ possesses the following properties with respect to the family $\{M_j\}$ of strongly connected components of $M' = M - \mathrm{st}(v; M)$:*

(a) *Any 1-simplex in $L - \partial M$ lies in the boundary of exactly one M_j. Moreover, if L_i is an arc then L_i is contained in the boundary of a unique M_j.*

(b) *Given M_j and $M_{j'}$, there exists at least one cycle L_i such that $\dim(M_j \cap L_i) = \dim(M_{j'} \cap L_i) = 1$.*

P r o o f. If a vertex $w \in L$ has valence $\geqslant 3$ in L then the edge $\langle v, w\rangle$ is contained in at least three 2-simplexes, and M is not a 2-pseudomanifold. Hence $\mathrm{val}(w) \leqslant 2$, and $\mathcal{L}$ is a (disjoint) family of cycles and arcs. If v is a vertex in the interior of M then $\mathcal{L}$ only contains cycles.

Any 1-simplex $\gamma \subseteq L - \partial M$ must be an interior simplex of M. Therefore $\gamma \subseteq \partial M'$, and there is only one strong component $M_j \subseteq M'$ with $\gamma \subseteq \partial M_j$ since $\dim(M_j \cap M_{j'}) \leqslant 0$.

Assume now $\partial L_i \neq \emptyset$. Then $v \in \partial M$, and for two consecutive 1-simplexes $\gamma, \gamma' \subseteq L_i$ with $\gamma \subseteq \partial M_j$ and $\gamma' \subseteq \partial M_{j'}$, the vertex $w = \gamma \cap \gamma'$ must be an interior point since ∂M is full in M. Let $S \subseteq \mathrm{lk}(w; M)$ be the connected component of $\mathrm{lk}(w; M)$ containing v. It is easy to check that S actually contains two vertices u, u' with $\gamma = \langle u, w\rangle$ and $\gamma' = \langle w, u'\rangle$. As S is a cycle, we can find an arc from u to u' in $S - \{v\}$. Hence there is a chain of 2-simplexes from γ to γ', and so $M_j = M_{j'}$. This proves (a).

Finally, any chain of 2-simplexes connecting M_j and $M_{j'}$ must pass through some component $L_i \subseteq L$. Here we use the fact that the 2-pseudomanifold M is strongly connected. Moreover, L_i is a cycle by (a). □

Proposition 2.6. *Given a 2-pseudomanifold M with full boundary, the 1-skeleton $\mathrm{sk}^1 M$ is 3-connected.*

P r o o f. By (2.4) $\mathrm{sk}^1 M$ is at least 2-connected. Assume $\mathrm{sk}^1 M$ is not 3-connected, and let $J = \{v_1, v_2\}$ be a juncture set for the vertices $a, b \in M$. Let γ_1 and γ_2 be two independent paths from a to b with $\gamma_i \cap J = \{v_i\}$ $(i = 1, 2)$. Let p_1 and p_2 be the vertices in M with $\langle p_1, v_1\rangle \cup \langle v_1, p_2\rangle \subseteq \gamma_1$. In particular, $p_1, p_2 \in \operatorname{lk}(v_1; M)$. Let M_i be the strongly connected components of $M' = M - \operatorname{st}(v_1; M)$ containing p_i $(i = 1, 2)$. By (2.5 (b)) there exists a cycle $S \subseteq \operatorname{lk}(v_1; M)$ such that $\dim(M_i \cap S) = 1$ $(i = 1, 2)$.

By applying (2.4) to M_i we can find a path $\varrho_i \subseteq M_i$ from p_i to $p_i' \in S$ such that $v_2 \notin \varrho_i$ $(i = 1, 2)$. Moreover, we can assume $p_i \neq p_i'$. Now, we can find a new path $\eta \subseteq S$ from p_1' to p_2' with $v_2 \notin \eta$. It is clear that a path $\xi \subseteq \varrho_1 \cup \varrho_2 \cup \eta \cup \gamma_1$ can be defined from a to b avoiding J. This contradicts the fact that J is a juncture set for a, b. □

With the help of (2.6) the proof of (2.3) is the same as Barnette's proof in [2]. We include the proof here because it will be used later in the proof of the 2-dimensional analogue of (2.3) in (3.4) below.

P r o o f o f (2.3). The case $n = 2$ is (2.6). Assume the result for $n-1$, and let $J = \{v_1, \dots, v_m\}$ be a minimal juncture set for $\operatorname{sk}^1 M$ where M is an n-pseudomanifold with full boundary. Suppose for a moment that $J' = \{v_2, \dots, v_m\}$ does not separate any strongly connected component of $L = \operatorname{lk}(v_1; M)$. Then the following n-pseudomanifold M' is constructed: If $\widehat{M} = M - \operatorname{st}(v_1; M)$ and $\{L_1, \dots, L_k\}$ are the strongly connected components of L, we define $M' = \widehat{M} \cup \{c_i * L_i\}_{1 \leqslant i \leqslant k}$ where $c_i * L_i$ is the cone over L_i with vertex c_i. The set J' separates $\operatorname{sk}^1 M'$ since otherwise, given two vertices $a, b \in M'$, there exists a path $\gamma \subseteq M'$ from a to b with $\gamma \cap J' = \emptyset$. As J separates M, then $\gamma \cap L_i \neq \emptyset$ for some i. Let p_i and q_i denote the first and the last vertex of γ in L_i, respectively. As J' does not separate L_i, we can find a path $\eta_i \subseteq L_i$ from p_i to q_i with $\eta_i \cap J' = \emptyset$. It is now easy to find a path $\xi \subseteq \bigcup\{\eta_i;\ \gamma \cap (c_i * L_i) \neq \emptyset\} \cup \gamma$ from a to b in $\operatorname{sk}^1 M - \{v_1\}$ which does not meet J'. Then J' does not separate $\operatorname{sk}^1 M - \{v_1\}$, a contradiction. Therefore J' necessarily separates one of the strongly connected components of L, and hence $m \geqslant n + 1$ by the induction hypothesis. □

We conclude this section by introducing a class of $(n+1)$-connected graphs containing the 1-skeletons of normal pseudomanifolds. By doing that we give an alternative and shorter proof of (2.3) for the class of normal pseudomanifolds, and hence for homology and combinatorial manifolds.

Definition 2.7. A graph G is said to be *relatively n-connected* with respect to a vertex v if $\operatorname{lk}(v; G)$ is contained in an n-connected subgraph which does not contain v.

R e m a r k 2.8. Notice that any n-simplicial graph G in the sense of Larman and Mani ([4]) is relatively n-connected with respect to any vertex $v \in G$.

Proposition 2.9. *Any graph G relatively n-connected with respect to all vertices $v \in G$ is $(n+1)$-connected.*

P r o o f. Assume that for $m \leqslant n$, $J = \{v_1, \dots, v_m\}$ is a minimal juncture set of vertices for G. We find m independent paths $\gamma_1, \dots, \gamma_m$ going between two vertices $a, b \in G - J$ and such that $\gamma_i \cap J = \{v_i\}$ for all i. Hence each intersection $\gamma_i \cap \mathrm{lk}(v_i; G)$ contains at least two vertices. Let $p, q \in \gamma_1 \cap \mathrm{lk}(v_1; G)$ with $p \neq q$. As G is relatively n-connected with respect to v_1, let $G' \subseteq G$ be an n-connected subgraph with $\mathrm{lk}(v_1; G) \subseteq G'$ and $v_1 \notin G'$. Since $p, q \in G'$, we find a path $\eta \subseteq G'$ between p and q such that $\eta \cap J = \emptyset$. Therefore a path can be found in $\gamma_1 \cup \eta$ from a to b avoiding J. This yields a contradiction. □

Proposition 2.10. *The 1-skeleton $\mathrm{sk}^1 M$ of any normal n-pseudomanifold M ($n \geqslant 2$) with full boundary is relatively n-connected for each vertex $v \in M$.*

P r o o f. The case $n = 2$ is obvious since for each vertex v, $\mathrm{lk}(v; M) \subseteq \mathrm{sk}^1 M_v$ where $M_v = M - \mathrm{st}(v; M)$ is a connected 2-manifold and hence (2.4) applies. Notice that M_v is connected since M has a full boundary but M_v need not have full boundary.

Assume (2.10) holds for $n - 1$. Given a normal n-pseudomanifold M, each link $\mathrm{lk}(v; M)$ is a normal $(n-1)$-pseudomanifold with full boundary and by the induction hypothesis and (2.9) $\mathrm{sk}^1 \mathrm{lk}(v; M)$ is n-connected. The inclusion $\mathrm{lk}(v; \mathrm{sk}^1 M) \subseteq \mathrm{sk}^1 \mathrm{lk}(v; M)$ yields that $\mathrm{sk}^1 M$ is relatively n-connected. □

3. Connectivity of 2-skeletons of pseudomanifolds

Given a 2-complex P, then a 2-*path* in P, $\alpha\colon e_0 - e_n$, is a finite sequence of edges and triangles $\{e_0, t_1, e_1, t_2, \dots, t_n, e_n\}$ such that t_i are triangles, e_i are edges and $e_i, e_{i+1} < t_i$ $(1 \leqslant i \leqslant n)$. Two 2-paths $\alpha, \beta\colon e_0 - e_n$ are said to be *independent* if $\alpha \cap \beta = \{e_0, e_n\}$.

2-paths in P induce a stronger notion of connectedness in any 2-complex P. Namely, P is said to be 2-*path connected* if any two edges e, e' can be joined by a 2-path in P. The definition of a 2-path connected component is now clear. Notice that the term "strongly connected" is equivalent to "2-path connected" for 2-complexes.

For the sake of simplicity, we will say that P is an *admissible* 2-*complex* if P is a 2-path connected 2-complex such that any triangle in P contains at most one boundary edge. Notice that this is the case if ∂P is full in P.

If P is an admissible 2-complex and $\mathcal{E}_{\mathrm{int}}(P)$ denotes the set of interior edges of P (see Sect. 1), then the *bipartite graph of* P, $G(P)$, is defined as follows. Let

$V(G(P)) = E \cup T$ where E is the set consisting of the barycentres $\overline{e}$ with $e \in \mathcal{E}_{\mathrm{int}}(P)$, and T is the set of the barycentres of triangles of P. Now $\overline{e} \in E$ is joined in $G(P)$ to $\bar{t} \in T$ if $e < t$. Clearly $G(P)$ is a subcomplex of the 1-skeleton $\mathrm{sk}^1 P^{(1)}$ of the first barycentric subdivision $P^{(1)}$ of P.

It is obvious that any 2-path in P yields a path in $G(P)$ and viceversa. In particular, P is 2-path connected if and only if $G(P)$ is connected. Moreover, two 2-paths in P are independent if and only if their associated paths in $G(P)$ are independent.

The notion of a *juncture set* in a 2-complex is now clear. Namely, a set J of edges and/or triangles of P is a *juncture set* for the interior edges e and e' if they lie in different 2-path components of $P - J$. Notice that J is a juncture set for P if and only if the set $\overline{J} = \{\bar{\nu}; \nu \in J\}$ is a juncture set for $G(P)$. Then the following theorem is an immediate consequence of (2.1).

Theorem 3.1. *Let P be an admissible 2-complex. If $e, e' \in \mathcal{E}_{\mathrm{int}}(P)$ the following two statements are equivalent:*

(a) *There is no juncture set of edges and/or triangles for e and e' with fewer than n elements.*

(b) *There exist n independent 2-paths from e to e'.*

An admissible 2-complex P is said to be *n-connected* if condition (b), and hence (a), holds for any pair of interior edges.

The next theorem allows us to consider juncture sets containing only edges. See ([5]; Thm. 3) or ([1]; Thm. 2.15) for a proof.

Theorem 3.2. *Let P be an admissible 2-complex. Assume $\mathrm{val}(e) \geqslant n$ for any $e \in \mathcal{E}_{\mathrm{int}}(P)$. Then P is n-connected if and only if there exists no juncture set $J \subseteq \mathcal{E}_{\mathrm{int}}(P)$ with fewer than n edges.*

We are now ready to prove the 2-dimensional analogue of Theorem 2.3 in (3.4) below. For this we start with

Proposition 3.3. *Any 2-pseudomanifold M with full boundary is 2-connected.*

P r o o f. By (3.2) it suffices to show that no interior edge e separates two edges $\alpha, \beta \in \mathcal{E}_{\mathrm{int}}(M)$. As ∂M is full in M, there is at least one vertex v in $e \cap (M - \partial M)$. Therefore $\mathrm{lk}(v; M)$ is a disjoint union of cycles by (2.5). If σ_1 and σ_2 are the two 2-simplexes of M with $\sigma_1 \cap \sigma_2 = e$, let e_1 and e_2 be the opposite faces of v in σ_1 and σ_2, respectively. Clearly $e_1 \cup e_2$ lies in a component $C \subseteq \mathrm{lk}(v; M)$. Moreover, given a 2-path $R\colon\ \alpha - \beta$, the 2-simplexes σ_1 and σ_2 must be contained in R. As C is a cycle the difference $A = C - e_1 \cup e_2$ is an arc in $\mathrm{lk}(v; M)$. Let $\widehat{A} \subseteq \mathrm{st}(v; M)$ be the subcomplex generated by v and A. It is now easy to define 2-paths from α to β in $\widehat{A} \cup R$ avoiding e. $\square$

Theorem 3.4. *Let M be an n-pseudomanifold ($n \geqslant 2$) with full boundary. Then* $\mathrm{sk}^2 M$ *is n-connected.*

Proof. We proceed inductively. For $n = 2$ the result is (3.3). Assume that Theorem 3.4 holds for $n-1$. Since ∂M is full, any edge $e \subseteq M$ belongs to at least two n-simplexes of M. Hence $\mathrm{val}(e) \geqslant n$ in $\mathrm{sk}^2 M$ and, by (3.2), there are minimal juncture sets consisting of edges. Let $J = \{e_1, \ldots, e_k\}$ be such a juncture set. Let $\widetilde{M}$ be the subdivision of M such that $V(\widetilde{M}) - V(M)$ is the one-point set consisting of the barycentre $b(e_1)$ of e_1. Let e_1' and e_1'' be the two edges in $\widetilde{M}$ with $e_1 = e_1' \cup e_1''$.

As J is a juncture set for $\mathrm{sk}^2 M$, Lemma 3.5 below shows that $\widetilde{J} = \{e_1', e_1'', e_2, \ldots, e_k\}$ is a juncture set for $\mathrm{sk}^2 \widetilde{M}$. By replacing M by $\widetilde{M}$ and $\mathrm{st}(v_1; M)$ by $\mathrm{st}(b(e_1); \widetilde{M})$, the proof of (2.3) can be mimicked here with the obvious changes (edges for vertices, paths for 2-paths, etc.) to show that $k \geqslant n+1$. This completes the proof. $\square$

Lemma 3.5. *The set $\widetilde{J}$ above is a juncture set for* $\mathrm{sk}^2 \widetilde{M}$.

Proof. Assume J is a juncture set for interior edges α and β in $\mathrm{sk}^2 M$, while $\widetilde{J}$ is not a juncture set for the same edges in $\mathrm{sk}^2 \widetilde{M}$. Then there exists a 2-path $R = \{\alpha, t_0, \alpha_1, t_1, \ldots, \alpha_m, t_m, \beta\}$ in $\mathrm{sk}^2 M - \widetilde{J}$ connecting α and β. Below we will reduce R to a new 2-path R' with fewer triangles in $\mathrm{sk}^2 \widetilde{M} - \mathrm{sk}^2 M$. After a finite number of reductions we will have constructed a 2-path between α and β in $\mathrm{sk}^2 M - J$, which will yield a contradiction.

Let v_0, v_1 be the vertices of α and let us assume that $t_0 \notin \mathrm{sk}^2 M$ (otherwise we choose the first triangle in R with this property). Necessarily $t_0 = \langle v_0, v_1, b(e_1) \rangle$, and so α_1 is either the edge $\langle v_1, b(e_1) \rangle$ or $\langle v_0, b(e_1) \rangle$. We will assume $\alpha_1 = \langle v_1, b(e_1) \rangle$, the case $\alpha_1 = \langle v_0, b(e_1) \rangle$ being similar. If $t_1 = \langle v_1, b(e_1), v_2 \rangle$ with $e_1 = \langle v_0, v_2 \rangle$, then $\alpha_2 = \langle v_1, v_2 \rangle$ and we replace R by $R' = \{\alpha, t, \alpha_2, t_2, \ldots\}$ where $t = \langle v_0, v_1, v_2 \rangle$. If $t_1 \neq \langle v_1, b(e_1), v_2 \rangle$ we necessarily have $t_1 = \langle v_1, b(e_1), w \rangle$ where $\sigma = \langle v_0, v_1, v_2, w \rangle$ is a 3-simplex of M. Moreover, α_2 is either the edge $\langle v_1, w \rangle$ or the edge $\langle b(e_1), w \rangle$. In the first case we can reduce R to the 2-path $R' = \{\alpha, t', \alpha_2, t_2, \ldots, \alpha_m, \beta\}$ where $t' = \langle v_0, v_1, w \rangle$. In the other case the edge α_3 is necessarily an edge of σ, and we can find a 2-path $R_0 \subseteq \mathrm{sk}^2 \sigma$ from α to α_3 such that R can be reduced to the 2-path $R' = R_0 \cup \{\alpha_3, t_3, \ldots, \alpha_m, \beta\}$. $\square$

Remark 3.6. In general, for any strongly connected n-complex P the 2-dimensional skeleton $\mathrm{sk}^2 P$ is $(n-1)$-connected. Compare (2.4). Indeed, for $n = 2$ the result is trivial since 1-connectivity is just strong connectedness as required. Moreover, for $n \geqslant 3$, let C be the n-complex consisting of two n-simplexes sharing a common $(n-1)$-face. Then one shows as in ([5]; Prop. 4) that $\mathrm{sk}^2 C$ is $(n-1)$-connected. From this one can easily derive that for any chain $C \subseteq P$ of n-simplexes

$\sigma_0, \ldots, \sigma_k$, the 2-skeleton $\mathrm{sk}^2 C$ is $(n-1)$-connected. Now the result is immediate. Compare ([5]; Thm. 5).

We next introduce a class of n-connected 2-complexes containing the 2-skeletons of normal n-pseudomanifolds. This yields an alternative proof of Theorem 3.4 for the class $\mathcal{N}$ of normal pseudomanifolds, and so for homology and combinatorial manifolds.

Definition 3.7. An admissible 2-complex P is said to be *relatively n-connected* with respect to the vertex $v \in P$ if $\mathrm{lk}(v; P)$ is contained in an n-connected 2-subcomplex which does not contain v.

Remark 3.8. Notice that any n-radial 2-complex P in the sense of Woon is relatively $(n-1)$-connected with respect to any vertex $v \in P$ ([5]; Prop. 4.2).

Proposition 3.9. *Let P be an admissible 2-complex relatively $(n-1)$-connected with respect to all vertices $v \in P$. If $\mathrm{val}(e) \geqslant n$ for all $e \in \mathcal{E}_{\mathrm{int}}(P)$ then P is n-connected.*

Proof. Let J be a minimal juncture set for the interior edges α, β in P. Since $\mathrm{val}(v) \geqslant n$ for all $e \in \mathcal{E}_{\mathrm{int}}(P)$ we can use (3.2) to assume that $J = \{e_1, \ldots, e_k\} \subseteq \mathcal{E}_{\mathrm{int}}(P)$. Assume also that $k \leqslant n-1$, and let $R_1, \ldots, R_k$ be 2-paths such that $R_i \cap J = \{e_i\}$ for $1 \leqslant i \leqslant k$.

Let R_1 be a sequence of edges and triangles $R_1 = \{a_0, t_0, a_1, t_1, \ldots, a_m, t_m, a_{m+1}\}$ with $a_0 = \alpha, a_{m+1} = \beta$ and $a_j = e_1 = \langle v, w\rangle$. If $a_{j-1} \cap a_{j+1} = \{w\}$, then $a_{j-1}, a_{j+1} \in \mathrm{lk}(v; P)$ and by a similar argument as in the proof of (2.9) we find a 2-path R_1' from α to β with $R_1' \cap J = \emptyset$, which is a contradiction. Hence $k \geqslant n$.

Assume now $a_{j-1} \cap a_{j+1} = \emptyset$, and let a_{j-1} be an edge of t_{j-1} distinct from a_{j-1} and e_1. As $\mathrm{val}(a_{j+1}) \geqslant n$ and $k \leqslant n-1$ we can find a triangle t with $t \cap t_j = a_{j+1}$ and such that its three edges are not in J. Now let w be the common vertex of a'_{j-1} and a_{j+i}. Then one edge $b < t$ as well as a_{j-1} belong to $\mathrm{lk}(w; P)$, and we conclude as above. $\square$

Proposition 3.10. *The 2-skeleton $\mathrm{sk}^2 M$ of any normal n-pseudomanifold M $(n \geqslant 2)$ with full boundary is relatively $(n-1)$-connected for each vertex $v \in M$.*

Proof. If $n = 2$ we have $\mathrm{lk}(v; M) \subseteq M_v = M - \mathrm{st}(v; M)$, and the result follows by applying (3.6) to M_v. Notice that M_v is connected, and hence strongly connected since M has full boundary. However, one cannot guarantee that M_v has full boundary. We now proceed inductively as in the proof of (2.10) by using the inclusion $\mathrm{lk}(v; \mathrm{sk}^2 M) \subseteq \mathrm{sk}^2\,\mathrm{lk}(v; M)$ and (3.9). $\square$

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