ON APPLICATION OF SYMMETRICAL FUNCTIONS OF SEVERAL VARIABLES TO SOLVING SOME FUNCTIONAL EQUATIONS

Jerzy Połubiński, Łódź

(Received October 13, 2000)

Abstract. In this work we apply the method of a unique partition of a complex function f of complex variables into symmetrical functions to solving a certain type of functional equations.

Keywords: functional equations, $(j,l|k)$-symmetrical functions

MSC 2000: 39B32

1. Introduction

The main result of part 2 consists in the presentation of Theorem 2.1 about the uniqueness of decomposition of an arbitrary complex multivariable function into the sum of symmetrical functions. Part 3 and 4 are devoted to the presentation of the general method of solving certain functional equations (compare (6)) with an unknown complex multivariable function. This method applies the decomposition introduced in part 2. Part 5 consists of illustrative examples of the solutions to the equations $f(\mathrm{i}x,y) - f(x,\mathrm{i}y) = 0$ and $f(-x,y)f(x,-y) - f^2(x,y) + 2x^2y^2 = 0$ for $(x,y) \in \mathbb{C}^2$.

Let us fix $k \in \mathbb{N}$, $k \geqslant 2$ and let $\varepsilon_k := \exp(2\pi\mathrm{i}/k)$. Let us assume that a set $U \subset \mathbb{C}$ has the following property of *k-symmetry*: for any z, if $z \in U$ then $\varepsilon_k z \in U$. For every number j from the set of all integers $\mathbb{Z}$ and every k-symmetrical set U a function $f\colon U \to \mathbb{C}$ will be called *(j,k)-symmetrical* if $f(\varepsilon_k z) = \varepsilon_k^j f(z)$ for each $z \in U$. In the paper [1] was proved the following lemma

Lemma 1.1. *Let f be a complex function defined on a k-symmetrical set U. Then f can be written as the sum*

$$f = \sum_{j=0}^{k-1} G_k^j f \tag{1}$$

of (j,k)-symmetrical functions $G_k^j f$, where for $z \in U$

$$G_k^j f(z) = \frac{1}{k} \sum_{l=0}^{k-1} \varepsilon_k^{-lj} f(\varepsilon - k^l z), \quad j = 0, 1, \dots, k-1.$$

Moreover, this partition is unique in the following sense: if $f = \sum\limits_{j=0}^{k-1} g_j$, where g_j are (j,k)-symmetrical functions for $j = 0, 1, \dots, k-1$, then $g_j = G_k^j f$.

2. $(j, l|k)$-symmetrical functions

Let us fix $k \in \mathbb{N}$, $k \geqslant 2$ and let $\varepsilon := \varepsilon_k = \exp(2\pi i/k)$. A set $U \subset \mathbb{C}^2$ will be called *k-symmetrical* if for any $(x, y) \in U$ the points $(\varepsilon x, y)$ and $(x, \varepsilon y)$ are also elements of this set.

The class of the nonempty k-symmetrical sets will be denoted by $S_k(\mathbb{C}^2)$ or, shortly, S_k.

Let us fix $U \in S_k$. By $F_k(U)$, or F_k, we shall denote the complex linear space of functions $f\colon U \to \mathbb{C}$.

For any $j, l \in \mathbb{Z}$ a function $f \in F_k(U)$ will be called *$(j, l|k)$-symmetrical* if $f(\varepsilon x, y) = \varepsilon^j f(x, y)$ and $f(x, \varepsilon y) = \varepsilon^l f(x, y)$ for any $(x, y) \in U$. The $(j, l|k)$-symmetrical functions form a linear subspace of the space $F_k(U)$. This subspace will be denoted by $F_k^{j,l}(U)$ or, shortly, $F_k^{j,l}$.

For any $(x, y) \in U$, $f \in F_k(U)$ and $m, n \in \mathbb{Z}$ let

$$G_k^{m,n} f(x, y) := \frac{1}{k^2} \sum_{j=0}^{k-1} \sum_{l=0}^{k-1} \varepsilon^{-mj-nl} f(\varepsilon^j x, \varepsilon^l y).$$

In this way linear operators $G_k^{m,n}$ are defined on F_k. It is easy to see that for any $f \in F_k$ we have $G_k^{m,n} f \in F_k^{m,n}$. Moreover $F_k^{j,l} = F_k^{j+mk, l+nk}$ and $G_k^{j,l} = G_k^{j+mk, l+nk}$ for $j, l, m, n \in \mathbb{Z}$. Therefore the analysis of the spaces $F_k^{j,l}$ and the operators $G_k^{j,l}$ can be restricted to the case when $j, l = 0, 1, \dots, k-1$.

To simplify notation let $K := \{0, 1, \dots, k-1\}$.

Theorem 2.1. *For any $U \in S_k$ and $f \in F_k(U)$ we have*

$$f = \sum_{m,n=0}^{k-1} G_k^{m,n} f. \tag{2}$$

The above representation of f in the form (2) is unique, i.e. if $f = \sum\limits_{m,n=0}^{k-1} g_{m,n}$ for some $g_{m,n} \in F_k^{m,n}$, then $g_{m,n} = G_k^{m,n} f$ for $m, n \in K$.

Proof. For $(x,y) \in U$, $m, n \in K$ and $f \in F_k(U)$ let

$$G_k^{m,-} f(x,y) := \frac{1}{k} \sum_{j=0}^{k-1} \varepsilon^{-mj} f(\varepsilon^j x, y), \; G_k^{-,n} f(x,y) := \frac{1}{k} \sum_{l=0}^{k-1} \varepsilon^{-nl} f(x, \varepsilon^l y).$$

These equalities define linear operators mapping the space $F_k(U)$ into itself. It is easily checked that

$$G_k^{m,n} = G_k^{m,-} \circ G_k^{-,n} = G_k^{-,n} \circ G_k^{m,-} \tag{3}$$

Let us take $x \in \mathbb{C}$ such that $U_x := \{y \in \mathbb{C}\colon\ (x,y) \in U\} \neq \emptyset$. Obviously, if $y \in U_x$ then $\varepsilon y \in U_x$ and therefore U_x is a k-symmetrical set. Due to Lemma 1.1 for any $y \in U_x$ we have

$$f(x,y) = \sum_{n=0}^{k-1} G_k^{-,n} f(x,y). \tag{4}$$

This equality holds for any $(x,y) \in U$ as if $(x,y) \in U$ then $y \in U_x$.

Let us denote $G_k^{-,n} f(x,y) := g_n(x,y)$ for $(x,y) \in U$ and let us take $y \in \mathbb{C}$ such that $W_y := \{x \in \mathbb{C}\colon\ (x,y) \in U\} \neq \emptyset$. If $x \in W_y$ then $\varepsilon x \in W_y$. Hence W_y is a k-symmetrical set, so for a fixed y we can apply Lemma 1.1 to the function g_n on the set W_y and therefore for any $x \in W_y$

$$g_n(x,y) = \sum_{m=0}^{k-1} G_k^{m,-} g_n(x,y).$$

Hence using (4) and (3) for any $(x,y) \in U$ we obtain

$$f(x,y) = \sum_{n=0}^{k-1} g_n(x,y) = \sum_{n=0}^{k-1} \sum_{m=0}^{k-1} G_k^{m,-} (G_k^{-,n} f)(x,y) = \sum_{m,n=0}^{k-1} G_k^{m,n} f(x,y).$$

To prove the uniqueness of the decomposition (2) let us assume that $f = \sum\limits_{m,n=0}^{k-1} h_{m,n}$ for some $h_{m,n} \in F_k^{m,n}$. Then for any $j, l \in K$ we have

$$\begin{aligned} G_k^{j,l} f &= \sum_{m,n=0}^{k-1} G_k^{j,l} h_{m,n} = \sum_{m,n=0}^{k-1} G_k^{j,-}(G_k^{-,l} h_{m,n}) \\ &= \sum_{m=0}^{k-1} G_k^{j,-}\left(\sum_{n=0}^{k-1} G_k^{-,l} h_{m,n}\right) = \sum_{m=0}^{k-1} G_k^{j,-} h_{m,l} = h_{j,l}. \end{aligned}$$

N o t i c e . The above results can be without any difficulty generalized and proved in the case of functions of n variables for $n > 2$.

3. $(j,l|k)$-symmetrical extensions

Let us fix $k \in N$, $k \geqslant 2$ and $U \in S_k(\mathbb{C}^2)$. For any $m, n \in K$, where $K = \{0, 1, \ldots, k-1\}$, the set

$$\begin{aligned} U^{m,n} := \{(x,y) \in U\colon\ & \operatorname{Arg}(x) \in [m2\pi/k, (m+1)2\pi/k) \\ & \wedge \operatorname{Arg}(y) \in [n2\pi/k, (n+1)2\pi/k)\} \end{aligned}$$

will be called a *sector of the set* U. As usual to represent $\operatorname{Arg}(0)$ we can take an arbitrary real number. Consequently, if $(0,0) \in U$ then $(0,0) \in U^{m,n}$ for arbitrarily chosen $m, n \in K$.

It can be seen that $\bigcup\limits_{m,n=0}^{k-1} U^{m,n} = U$. Nevertheless, sectors are not pairwise disjoint. Putting

$$U^j := \{x \in \mathbb{C}\colon\ \operatorname{Arg}(x) \in [j2\pi/k, (j+1)2\pi/k)\}$$

for $j \in K$ we have $U^{m,n} = (U^m \times U^n) \cap U$. Hence for $j, l, m, n \in K$

$$U^{j,l} \cap U^{m,n} = [(U^j \cap U^m) \times (U^l \cap U^n)] \cap U.$$

Therefore

$$U^{j,l} \cap U^{m,n} = \begin{cases} \{(0,0)\} \cap U & \text{for } j \neq m; l \neq n \\ (U^j \times \{0\}) \cap U & \text{for } j = m; l \neq n \\ (\{0\} \times U^l) \cap U & \text{for } j \neq m; l = n. \end{cases}$$

If $f_{j,l} \in F_k^{j,l}(U)$ for $j, l \in K$, then the function $f_{j,l}$ is determined uniquely by $f_{j,l}|U^{0,0}$.

The following procedure enables us to recover the function $f_{j,l}$ from its behaviour on the sector $U^{0,0}$ and is of fundamental importance for our considerations.

Let us fix a function $h\colon U^{0,0} \to \mathbb{C}$ and $j,l \in K$. A $(j,l|k)$-*symmetrical extension* of the function h onto a set U is a function $[h]^{j,l}\colon U \to \mathbb{C}$ such that
1. if $(x,y) \in U^{p,q}$ for $p,q \in K$ and $xy \neq 0$ then

$$[h]^{j,l}(x,y) := \varepsilon^{jp+lq} h(\varepsilon^{-p}x, \varepsilon^{-q}y),$$

2. if $xy = 0$ then

$$[h]^{j,l}(0,y) = \begin{cases} 0 & \text{for } j \neq 0 \\ \varepsilon^{lq} h(0,\varepsilon^{-q}y) & \text{for } j = 0, y \in U^q, y \neq 0, \end{cases}$$

$$[h]^{j,l}(x,0) = \begin{cases} 0 & \text{for } l = 0 \\ \varepsilon^{jp} h(\varepsilon^{-p}x, 0) & \text{for } l = 0, x \in U^p, x \neq 0, \end{cases}$$

$$[h]^{j,l}(0,0) = \begin{cases} 0 & \text{for } j,l \neq 0 \\ h(0,0) & \text{for } j,l = 0. \end{cases}$$

Lemma 3.1. (a) *For any $j,l \in K$ and $h\colon U^{0,0} \to \mathbb{C}$ we have $[h]^{j,l} \in F_k^{j,l}(U)$.*
(b) *If $f \in F_k^{j,l}(U)$ then there exists a function $h\colon U^{0,0} \to \mathbb{C}$ such that $f = [h]^{j,l}$.*

P r o o f. (a) If $(x,y) \in U^{p,q}$ and $xy \neq 0$ then $(\varepsilon x, y) \in U^{p+1,q}$. Therefore

$$[h]^{j,l}(\varepsilon x, y) = \varepsilon^{j(p+1)+lq} h(\varepsilon^{-p-1}\varepsilon x, \varepsilon^{-q}y) = \varepsilon^j [h]^{j,l}(x,y).$$

By analogy $[h]^{j,l}(x,\varepsilon y) = \varepsilon^l [h]^{j,l}(x,y)$. Now let $x = 0$ and $y \in U^q - \{0\}$. Then $\varepsilon y \in U^{q+1}$ and

$$[h]^{j,l}(0,\varepsilon y) = \begin{cases} 0 & \text{for } \ j \neq 0 \\ \varepsilon^{l(q+1)} h(0, \varepsilon^{-q-1}\varepsilon y) & \text{for } j = 0 \end{cases} = \varepsilon^l [h]^{j,l}(0,y).$$

By analogy, when $x \in U^p - \{0\}$, then $[h]^{j,l}(\varepsilon x, 0) = \varepsilon^j [h]^{j,l}(x,0)$. It can also be seen that $[h]^{j,l}(\varepsilon 0, 0) = \varepsilon^j [h]^{j,l}(0,0)$ and $[h]^{j,l}(0,\varepsilon 0) = \varepsilon^l [h]^{j,l}(0,0)$.

To prove (b) it is sufficient to take $h = f|U^{0,0}$.

It follows from Lemma 3.1 that $F_k^{j,l}(U)$ is the set of all $(j,l|k)$-symmetrical extensions of all functions $h\colon U^{0,0} \to \mathbb{C}$ onto U.

Now let us assume that on a set $U \in S_k$ we have defined a certain family of functions

$$g_{m_{0,0},m_{0,1},\ldots,m_{k-1,k-1}}\colon U \to \mathbb{C},$$

where $m_{0,0}, m_{0,1}, \ldots, m_{k-1,k-1} \in \mathbb{N} \cup \{0\}$. Let us consider a functional equation

$$\sum_{m_{0,0},m_{0,1},\ldots,m_{k-1,k-1}} g_{m_{0,0},m_{0,1},\ldots,m_{k-1,k-1}} \prod_{j,l=0}^{k-1} f_{j,l}^{m_{j,l}} = 0 \tag{5}$$

with unknowns $f_{j,l} \in F_k^{j,l}(U)$ for $j,l \in K$.

The equation (5) will be called *homogeneous* if for some $m, n \in K$ all addends of the sum (5) are elements of the same space $F_k^{m,n}(U)$.

Theorem 3.1. *If $U \in S_k$ and if the equation* (5) *is homogeneous then for any $h_{j,l}\colon U^{0,0} \to \mathbb{C}$, $j, l \in K$ the functions $f_{j,l} = [h_{j,l}]^{j,l}$ satisfy the equation* (5) *on U if and only if the functions $h_{j,l}$ satisfy it on $U^{0,0}$.*

P r o o f. It is clear that if $f_{j,l}$ satisfy the equation (5) on U then the functions $h_{j,l} := f_{j,l}|U^{0,0}$ satisfy it on $U^{0,0}$ and $f_{j,l} = [h_{j,l}]^{j,l}$ for any $j, l \in K$.

Conversely, let us assume that the functions $h_{j,l}\colon U^{0,0} \to \mathbb{C}$ satisfy the equation (5) on $U^{0,0}$ and $f_{j,l} = [h_{j,l}]^{j,l}$. Since (5) is a homogeneous equation there exist $m, n \in K$ such that all addends of the sum (5) belong to $F_k^{m,n}(U)$.

Let $s := \sum\limits_j jm_{j,l}$, $t := \sum\limits_l lm_{j,l}$. Then $\prod\limits_{j,l=0}^{k-1} f_{j,l}^{m_{j,l}} \in F_k^{s,t}$ and therefore $g_{m_{0,0},m_{0,1},\ldots,m_{k-1,k-1}} \in F_k^{m-s,n-t}$.

Let us assume that $(x, y) \in U^{p,q}$ and $xy \neq 0$ for $p, q \in K$. Then, writing $m_{0,0}, m_{0,1}, \ldots, m_{k-1,k-1} := M$ we have

$$\begin{aligned}
&\sum_M g_M(x,y) \prod_{j,l=0}^{k-1} f_{j,l}^{m_{j,l}}(x,y)\\
&\quad= \sum_M \varepsilon^{p(m-s)+q(n-t)} g_M(\varepsilon^{-p}x, \varepsilon^{-q}y) \prod_{j,l=0}^{k-1} \varepsilon^{jpm_{jl}+lqm_{jl}} h_{j,l}^{m_{j,l}}(\varepsilon^{-p}x, \varepsilon^{-q}y)\\
&\quad= \sum_M \varepsilon^{p(m-s)+q(n-t)} \varepsilon^{sp+tq} g_M(\varepsilon^{-p}x, \varepsilon^{-q}y) \prod_{j,l=0}^{k-1} h_{j,l}^{m_{j,l}}(\varepsilon^{-p}x, \varepsilon^{-q}y)\\
&\quad= \varepsilon^{pm+qn} \sum_M g_M(x_0, y_0) \prod_{j,l=0}^{k-1} h_{j,l}^{m_{j,l}}(x_0, y_0) = 0
\end{aligned}$$

where $(x_0, y_0) := (\varepsilon^{-p}x, \varepsilon^{-q}y) \in U^{0,0}$, as $h_{j,l}$ satisfy the eqation (5) on $U^{0,0}$. In the same way we check that $f_{j,l}$ satisfy the equation (5) at the points $(x, 0)$ and $(0, y)$ of the set U.

4. Application

In what follows we assume that k is a fixed natural number, $k \geqslant 2$ and $\varepsilon = \exp(2\pi \mathrm{i}/k)$.

Let us take a set $U \in S_k(\mathbb{C}^2)$ and consider a functional equation

$$W(x, y, f(x, y), f(\varepsilon x, y), \ldots, f(\varepsilon^j x, \varepsilon^l y), \ldots, f(\varepsilon^{k-1}x, \varepsilon^{k-1}y)) = 0 \tag{6}$$

for $(x, y) \in U$, $j, l \in K$, where f is an unknown complex function defined on U, while $W(x, y, p_1, \ldots, p_{k^2})$, for fixed $(x, y) \in U$, is a polynomial with variables $p_1, \ldots, p_{k^2}$.

Note that if the unknown function f is represented in the form (2), i.e.

$$f(x,y) = \sum_{j,l=0}^{k-1} f_{j,l}(x,y)$$

for $(x, y) \in U$, where $f_{j,l} = G_k^{j,l} f$, then after substituting it into (6) and rearranging it the equation takes the form

$$\sum_{m,n=0}^{k-1} \Bigg[\sum_{m_{0,0},\ldots,m_{k-1,k-1}} g_{m_{0,0},\ldots,m_{k-1,k-1}} \prod_{j,l=0}^{k-1} f_{j,l}^{m_{j,l}} \Bigg] = 0 \tag{7}$$

where for certain $m_{0,0}, \ldots, m_{k-1,k-1} \in \mathbb{N} \cup \{0\}$, $g_{m_{0,0},\ldots,m_{k-1,k-1}}$ are functions defined on U such that the addends in the square bracket in (7) belong to $F_k^{m,n}(U)$. Due to the uniqueness of the decomposition (2) the equation (7) is equivalent to a system of k^2 homogeneous equations on U:

$$\sum_{m_{0,0},\ldots,m_{k-1,k-1}} g_{m_{0,0},\ldots,m_{k-1,k-1}} \prod_{j,l=0}^{k-1} f_{j,l}^{m_{j,l}} = 0 \tag{8}$$

with unknowns $f_{j,l}$; $j, l \in K$.

In order to find solutions $f_{j,l}$ of the system (8) it is sufficient—due to Theorem 3.1—to find all solutions $h_{j,l}$ on the sector $U^{0,0}$; then the functions $f_{j,l} = [h_{j,l}]^{j,l}$ form a complete set of solutions of the system (8) on U. Consequently, the functions $f = \sum\limits_{j,l=0}^{k-1} f_{j,l}$ will form the complete set of solutions of the equation (6).

In order to obtain all solutions $h_{j,l}$ of the system (8) on $U^{0,0}$ we can apply algebraic methods.

5. Examples

Example 5.1. Let us consider the equation

$$f(\mathrm{i}x, y) - f(x, \mathrm{i}y) = 0. \tag{9}$$

It is an equation of the form (6) if we put $k = 4$, $\varepsilon = \mathrm{i} = \exp(2\pi\mathrm{i}/4)$.

Let us take $U = \mathbb{C}^2$. Representing the unknown function f on U in the form (2) and putting $G_k^{j,l} f := f_{j,l}$ we have $f = \sum\limits_{j,l=0}^{3} f_{j,l}$. After substituting it into (9) in the way described in part 4, we obtain the equation

$$\sum_{j,l=0}^{3} (\mathrm{i}^j - \mathrm{i}^l) f_{j,l} = 0$$

which is equivalent on U to the system of equations $f_{j,l} = 0$ for $j, l = 0, 1, 2, 3$; $j \neq l$. Hence we get all solutions of the equation (9) on $\mathbb{C}$: $f = \sum\limits_{j=0}^{3} f_{j,j}$, where $f_{j,j}$ are arbitrary elements of the space $F_k^{j,j}(\mathbb{C}^2)$ for $j = 0, 1, 2, 3$.

E x a m p l e 5.2. Let us consider the equation

$$f(-x, y) f(x, -y) - f^2(x, y) + 2x^2 y^2 = 0 \tag{10}$$

on the set $U = \{(x, y) \in \mathbb{C}^2 : \ xy \neq 0\}$.

It is an equation of the form (6) with $k = 2$ and $\varepsilon = -1$. Representing the function f on U in the form (2): $f = \sum\limits_{j,l=0}^{1} f_{j,l}$ where $f_{j,l} \in F_2^{j,l}(U)$ for $j, l = 0, 1$, substituting it into (10) and following the procedure described in part 4 we obtain a system of equations equivalent to the equation (10)

$$f_{0,1}^2 + f_{1,0}^2 = F, \ f_{0,0} f_{1,1} = 0, \ f_{0,0} f_{1,0} + f_{0,1} f_{1,1} = 0, \ f_{0,0} f_{0,1} + f_{1,0} f_{1,1} = 0, \tag{11}$$

where $F(x, y) := x^2 y^2$ for $(x, y) \in U$.

Due to Lemma 3.2 it is sufficient to find all solutions $h_{j,l} := f_{j,l} | U^{0,0}$ of this system on the sector $U^{0,0}$. Then the functions $f = \sum\limits_{j,l=0}^{1} [h_{j,l}]^{j,l}$ will represent all solutions of the equation (10) on U.

On $U^{0,0}$ the system (11) is equivalent to the system

$$h_{0,1}^2(x, y) + h_{1,0}^2(x, y) = x^2 y^2, \quad h_{0,0}(x, y) = 0, \quad h_{1,1}(x, y) = 0.$$

All its solutions can be obtained by putting for $(x, y) \in U^{0,0}$

$$h_{0,0}(x, y) = 0, \quad h_{1,1}(x, y) = 0, \quad h_{0,1}(x, y) = \sqrt{x^2 y^2 - h_{1,0}^2(x, y)},$$

and $h_{1,0}(x, y)$ arbitrary, where $\sqrt{x^2 y^2 - h_{1,0}^2(x, y)}$ is an arbitrarily chosen square root of $x^2 y^2 - h_{1,0}^2(x, y)$. Hence each solution f of the equation (10) is of the form

$$f = f_{1,0} + \left[\sqrt{(F - f_{1,0}^2) | U^{0,0}} \right]^{0,1}$$

where $f_{1,0}$ is an arbitrary element of the space $F_2^{1,0}(U)$.

References

[1] *P. Liczberski, J. Połubiński*: On (j,k)-symmetrical functions. Math. Bohem. *120* (1995), 13–28.

Author's address: *Jerzy Połubiński*, Institute of Mathematics, Technical University of Łódź, Al. Politechniki 11, 90-924, Łódź, Poland, e-mail: jupolu@ck-sg.p.lodz.pl.