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REMOVABILITY OF SINGULARITIES WITH ANISOTROPIC GROWTH

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Dedicated to Prof. Dr.-Ing. Dr. h.c. Wolfgang L. Wendland on the occasion of his 65th birthday

Abstract. With help of suitable anisotropic Minkowski's contents and Hausdorff measures some results are obtained concerning removability of singularities for solutions of partial differential equations with anisotropic growth in the vicinity of the singular set.

Keywords: solutions of partial differential equations, removable singularities, anisotropic metric, Minkowski's contents

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Let $G \subset \mathbb{R}^N$ be an open set. We consider differential operators of the form

$$P(D) = \sum_{\alpha} a_\alpha D^\alpha, \qquad \alpha \in M, \tag{1}$$

acting on distributions in G; M is a finite set of multiindices $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, where the components α_j ($1 \leqslant j \leqslant N$) are nonnegative integers and a_α are infinitely differentiable complex-valued functions on G. We write

$$D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_N^{\alpha_N}, \qquad \text{where } D_j = -\mathrm{i}\partial_j,$$

∂_j is the partial derivative with respect to the j-th variable and i is the imaginary unit. Let us fix $m \in \mathbb{R}^N$ with components $m_1, m_2, \ldots, m_N \in \mathbb{N}$ ($\mathbb{N}$ is the set of all

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positive integers) in such a way that

$$\alpha \in M \Longrightarrow \alpha : m \equiv \sum_{k=1}^{N} \frac{\alpha_k}{m_k} \leqslant 1.$$

(After all, we can assume that

$$M = \{\alpha \in \mathbb{N}_0^N;\ \alpha : m \leqslant 1\},$$

where $\mathbb{N}_0$ is the set of all nonnegative integers.) Put

$$\overline{m} = \max\{m_k;\ 1 \leqslant k \leqslant N\} \tag{2}$$

and define for $x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$ and $y = (y_1, y_2, \ldots, y_N) \in \mathbb{R}^N$

$$\varrho_m(x, y) = \max\{|x_k - y_k|^{m_k/\overline{m}};\ 1 \leqslant k \leqslant N\}. \tag{3}$$

Then ϱ_m is a metric on $\mathbb{R}^N$. If λ_N is the Lebesgue measure in $\mathbb{R}^N$ and

$$B_r(x, \varrho_m) = \{y \in \mathbb{R}^N;\ \varrho_m(x, y) \leqslant r\} \tag{4}$$

is the closed ball centred at $x \in \mathbb{R}^N$ of radius $r \geqslant 0$, then

$$\begin{aligned} B_r(x, \varrho_m) = [x_1 - r^{\overline{m}/m_1}, x_1 + r^{\overline{m}/m_1}] \times [x_2 - r^{\overline{m}/m_2}, x_2 + r^{\overline{m}/m_2}] \\ \times \ldots \times [x_N - r^{\overline{m}/m_N}, x_N + r^{\overline{m}/m_N}] \end{aligned}$$

and

$$\lambda_N\bigl(B_r(x, \varrho_m)\bigr) = 2^N r^{\overline{m}b}, \qquad \text{where} \quad b = \sum_{k=1}^{N} \frac{1}{m_k}. \tag{5}$$

(In what follows b will always have this meaning.)

For $L \subset \mathbb{R}^N$ we denote by

$$\operatorname{diam}(L, \varrho_m) = \sup\{\varrho_m(x, y);\ x, y \in L\} \tag{6}$$

the diameter of the set L and for $\gamma \in \mathbb{R}_+$ ($\mathbb{R}_+$ is the set of all nonnegative numbers in $\mathbb{R}$) we define the outer anisotropic Hausdorff γ-dimensional measure of the set L by setting

$$\mathscr{H}_\gamma(\emptyset, \varrho_m) = 0$$

and for $L \neq \emptyset$

(7)

$$\mathscr{H}_\gamma(L, \varrho_m) = \sup_{\varepsilon>0} \inf \Big\{ \sum_{n\in\mathbb{N}} \operatorname{diam}^\gamma(L_n, \varrho_m);\ L \subset \bigcup_n L_n, 0 \leqslant \operatorname{diam}(L_n, \varrho_m) \leqslant \varepsilon \Big\}.$$

The distance of $x \in \mathbb{R}^N$ from $L \subset \mathbb{R}^N$ with respect to the metric ϱ_m will be denoted by

$$\varrho_m(x, L) = \inf\big\{\varrho_m(x, y);\ y \in L\big\}. \tag{8}$$

For $\varepsilon > 0$ and $L \subset \mathbb{R}^N$ put

$$L_\varepsilon = \big\{x \in \mathbb{R}^N;\ \varrho_m(x, L) < \varepsilon\big\}. \tag{9}$$

For compact $K \subset \mathbb{R}^N$ denote by $\mathcal{N}_\varepsilon(K, \varrho_m)$ the minimal number of balls $B_\varepsilon(x, \varrho_m)$ with $x \in K$ sufficient for covering K. The lower γ-dimensional Minkowski's content of $K \neq \emptyset$ is given by

$$\mathcal{M}_\gamma(K, \varrho_m) = \liminf_{\varepsilon\searrow 0} \mathcal{N}_\varepsilon(K, \varrho_m) \cdot \varepsilon^\gamma, \tag{10}$$

the upper γ-dimensional Minkowski's content of $K \neq \emptyset$ is given by

$$\mathcal{M}^\gamma(K, \varrho_m) = \limsup_{\varepsilon\searrow 0} \mathcal{N}_\varepsilon(K, \varrho_m) \cdot \varepsilon^\gamma. \tag{11}$$

Further put

$$\mathcal{M}_{\varrho_m}\text{-}\dim K = \inf\big\{\gamma > 0;\ \mathcal{M}^\gamma(K, \varrho_m) = 0\big\}, \tag{12}$$

which is the so-called Minkowski's dimension of K corresponding to the metric ϱ_m.

Remark 1. There exist constants $c_N > 0$, $d_N > 0$ (depending only on N) such that for each compact $K \subset \mathbb{R}^N$

$$\begin{aligned}
c_N \mathcal{M}_\gamma(K, \varrho_m) &\leqslant \liminf_{\varepsilon\searrow 0} \frac{\lambda_N(K_\varepsilon)}{\varepsilon^{\overline{m}b-\gamma}} \leqslant d_N \mathcal{M}_\gamma(K, \varrho_m),\\
c_N \mathcal{M}^\gamma(K, \varrho_m) &\leqslant \limsup_{\varepsilon\searrow 0} \frac{\lambda_N(K_\varepsilon)}{\varepsilon^{\overline{m}b-\gamma}} \leqslant d_N \mathcal{M}^\gamma(K, \varrho_m),\\
\mathcal{M}_{\varrho_m}\text{-}\dim K &\leqslant \overline{m}b.
\end{aligned}$$

Proof. Given $\varepsilon > 0$, observe that $\mathcal{N}_\varepsilon(K, \varrho_m)$ closed balls of radius 2ε suffice to cover K_ε. Hence

$$\begin{aligned}
\liminf_{\varepsilon\searrow 0} \lambda_N(K_\varepsilon)/\varepsilon^{\overline{m}b-\gamma} &\leqslant \mathcal{M}_\gamma(K, \varrho_m) \cdot 2^{\overline{m}b+N},\\
\limsup_{\varepsilon\searrow 0} \lambda_N(K_\varepsilon)/\varepsilon^{\overline{m}b-\gamma} &\leqslant \mathcal{M}^\gamma(K, \varrho_m) \cdot 2^{\overline{m}b+N};
\end{aligned}$$

we see that it is sufficient to set $d_N = 2^{\overline{m}b+N}$. Consider the map $\pi_\varepsilon \colon \mathbb{R}^N \to \mathbb{R}^N$ given by

$$\pi_\varepsilon(x_1, x_2, \ldots, x_N) = \bigl(x_1 \cdot \varepsilon^{1-\overline{m}/m_1}, x_2 \cdot \varepsilon^{1-\overline{m}/m_2}, \ldots, x_N \cdot \varepsilon^{1-\overline{m}/m_N}\bigr).$$

For fixed $z \in \mathbb{R}^N$,

$$\pi_\varepsilon\bigl(B_\varepsilon(z, \varrho_m)\bigr)$$

is the cube centred at $\pi_\varepsilon(z)$ of side length $2 \cdot \varepsilon^{\overline{m}/m_1} \cdot \varepsilon^{1-\overline{m}/m_1} = 2\varepsilon$ and for any compact $K \subset \mathbb{R}^N$

$$\pi_\varepsilon(K) \subset \pi_\varepsilon(K_\varepsilon) \subset \bigcup_{z \in K} \pi_\varepsilon\bigl(B_\varepsilon(z, \varrho_m)\bigr).$$

Next, we use the following consequence of the Morse-Besicovitch covering theorem (cf. [2], § 1, chap. 1, Th. 1.1):

There exists a natural number $\gamma(N)$ with the following property:

If A is a bounded set in $\mathbb{R}^N$ and if with each $x \in A$ associate a closed euclidean cube $B(x)$ centred at x, then there are finite sets

$$A_1, A_2, \ldots, A_{\gamma(N)} \subset A$$

such that

$$A \subset \bigcup_{i=1}^{\gamma(N)} \bigcup_{x \in A_i} B(x),$$

where for each fixed $i \in \{1, 2, \ldots, \gamma(N)\}$ we have

$$B(x_1) \cap B(x_2) = \emptyset \qquad \text{whenever } x_1, x_2 \in A_i,\ x_1 \neq x_2.$$

Put $A = \pi_\varepsilon(K)$. There exist $A_1, A_2, \ldots, A_{\gamma(N)} \subset \pi_\varepsilon(K)$ such that

$$\pi_\varepsilon(K) \subset \bigcup_{i=1}^{\gamma(N)} \bigcup_{\pi_\varepsilon(z) \in A_i} \pi_\varepsilon\bigl(B_\varepsilon(z, \varrho_m)\bigr)$$

[so $K \subset \bigcup_{i=1}^{\gamma(N)} \bigcup_{z \in S_i} B_\varepsilon(z, \varrho_m)$ for $S_i = \pi_\varepsilon^{-1}(A_i)$] and $B_\varepsilon(z_1, \varrho_m) \cap B_\varepsilon(z_2, \varrho_m) = \emptyset$ whenever $z_1, z_2 \in S_i$, $z_1 \neq z_2$. If $\operatorname{card}(M)$ means the number of elements in M, then

$$\mathcal{N}_\varepsilon(K, \varrho_m) \leqslant \operatorname{card}\biggl(\bigcup_{i=1}^{\gamma(N)} S_i\biggr).$$

If B° denotes the interior of $B \subset \mathbb{R}^N$, then we have

$$2^N \varepsilon^{\overline{m}b} \mathcal{N}_\varepsilon(K, \varrho_m) \leqslant 2^N \varepsilon^{\overline{m}b} \operatorname{card}\Big(\bigcup_{i=1}^{\gamma(N)} S_i\Big) \leqslant \sum_{i=1}^{\gamma(N)} \sum_{z \in S_i} \lambda_N\big(B_\varepsilon^\circ(z, \varrho_m)\big)$$
$$= \sum_{i=1}^{\gamma(N)} \lambda_N\Big(\bigcup_{z \in S_i} B_\varepsilon^\circ(z, \varrho_m)\Big) \leqslant \sum_{i=1}^{\gamma(N)} \lambda_N(K_\varepsilon) = \gamma(N)\lambda_N(K_\varepsilon).$$

Hence

$$\varepsilon^\gamma \mathcal{N}_\varepsilon(K, \varrho_m) \leqslant \varepsilon^\gamma \cdot \gamma(N) \cdot 2^{-N} \cdot \varepsilon^{-\overline{m}b} \lambda_N(K_\varepsilon) = \gamma(N) \cdot 2^{-N} \lambda_N(K_\varepsilon)/\varepsilon^{\overline{m}b-\gamma}.$$

It is sufficient to put $c_N = 2^N/\gamma(N)$ to obtain

$$c_N \mathcal{M}_\gamma(K, \varrho_m) \leqslant \liminf_{\varepsilon \searrow 0} \lambda_N(K_\varepsilon)/\varepsilon^{\overline{m}b-\gamma},$$
$$c_N \mathcal{M}^\gamma(K, \varrho_m) \leqslant \limsup_{\varepsilon \searrow 0} \lambda_N(K_\varepsilon)/\varepsilon^{\overline{m}b-\gamma}.$$

We conclude that $\mathcal{M}^\gamma(K, \varrho_m) = 0$ for $\gamma > \overline{m}b$, whence $\mathcal{M}_{\varrho_m}\text{-}\dim K \leqslant \overline{m}b$, which proves Remark 1. □

Lemma 1. *Suppose that* $K \subset \mathbb{R}^N$ *is compact and* $0 \leqslant q < \overline{m}b$. *If*

$$\mathcal{M}_{\varrho_m}\text{-}\dim K < \overline{m}b - q,$$

then

$$\int_{K_\varepsilon} \varrho_m(x, K)^{-q} \, \mathrm{d}x < +\infty$$

for each $\varepsilon > 0$.

Proof. (A similar reasoning occurs, e.g., in [5].) As

$$\mathcal{M}_{\varrho_m}\text{-}\dim K < \overline{m}b - q,$$

there exists a $\kappa \in [0, \overline{m}b - q[$ such that $\mathcal{M}^\kappa(K, \varrho_m) = 0$. For all sufficiently small $\varepsilon > 0$ we have

$$\lambda_N(K_\varepsilon) \leqslant c \cdot \varepsilon^{\overline{m}b-\kappa}$$

for a suitable $c \in]0, +\infty[$. Since $\overline{m}b - \kappa > q \geqslant 0$, it follows that $\lim_{\varepsilon \searrow 0} \varepsilon^{\overline{m}b-\kappa} = 0$ and so

$$\lambda_N(K) \leqslant \lim_{\varepsilon \searrow 0} \lambda_N(K_\varepsilon) = 0.$$

Hence $\lambda_N(K)=0$. Setting

$$U_j=\bigl\{x\in\mathbb{R}^N;\ 2^{-j-1}\varepsilon\leqslant\varrho_m(x,K)<2^{-j}\varepsilon\bigr\},$$

we get

$$K_\varepsilon\setminus K=\bigcup_{j=0}^{+\infty}U_j.$$

Consequently,

$$\int_{K_\varepsilon}\varrho_m(x,K)^{-q}\,\mathrm{d}x=\sum_{j=0}^{+\infty}\int_{U_j}\varrho_m(x,K)^{-q}\,\mathrm{d}x,$$

$$\begin{aligned}\int_{U_j}\varrho_m(x,K)^{-q}\,\mathrm{d}x&\leqslant\varepsilon^{-q}\cdot2^{(j+1)q}\cdot\lambda_N(K_{2^{-j}\varepsilon})\leqslant\varepsilon^{-q}\cdot2^{(j+1)q}\cdot c\cdot\bigl(2^{-j}\varepsilon\bigr)^{\overline{m}b-\kappa}\\&=c\cdot\varepsilon^{\overline{m}b-\kappa-q}\cdot2^{(j+1)q-j\overline{m}b+j\kappa}=c\cdot\varepsilon^{\overline{m}b-\kappa-q}\cdot2^q\cdot2^{j(q-\overline{m}b+\kappa)}.\end{aligned}$$

As $\kappa-\overline{m}b+q<0$, we see that the series

$$\sum_{j=0}^{+\infty}\int_{U_j}\varrho_m(x,K)^{-q}\,\mathrm{d}x$$

converges, whence $\int_{K_\varepsilon}\varrho_m(x,K)^{-q}\,\mathrm{d}x<+\infty$. □

A simple compactness argument yields the next lemma.

Lemma 2. *Suppose that $G\subset\mathbb{R}^N$ is open, $\emptyset\neq F\subset G$ is relatively closed in G, $g\colon\ ]0,+\infty[\ \to\ ]0,+\infty[$ and $u\colon\ G\setminus F\to\mathbb{C}$ are functions such that*

$$u(x)=O\bigl(g(\varrho_m(x,F))\bigr)\qquad\text{as } x\to z,\ x\in G\setminus F,$$

for all $z\in F\cap\overline{G\setminus F}$. Then for each compact $Q\subset G$ there exist positive constants c,ε such that for all $x\in Q\setminus F$ with $\varrho_m(x,F)<\varepsilon$ we have

$$\bigl|u(x)\bigr|\leqslant cg\bigl(\varrho_m(x,F)\bigr).$$

Proof. With each $z\in Q\cap\overline{G\setminus F}\cap F$ we associate constants $r_z>0$ and $c_z>0$ such that

$$\bigl|u(x)\bigr|\leqslant c_zg\bigl(\varrho_m(x,F)\bigr)$$

for all $x\in G\setminus F$ with $\varrho_m(x,z)<r_z$. We may clearly suppose that r_z has been chosen small enough to get $B_{r_z}(z,\varrho_m)\subset G$. The open cover

$$\bigl\{B^\circ_{r_z}(z,\varrho_m)\bigr\}_{z\in Q\cap\overline{G\setminus F}\cap F}$$

of the compact $Q \cap \overline{G \setminus F} \cap F$ contains a finite subcover $\{B^{\circ}_{r_{z_i}}(z_i, \varrho_m)\}_{i=1}^n$:

$$F \cap \overline{G \setminus F} \cap Q \subset \bigcup_{i=1}^{n} B^{\circ}_{r_{z_i}}(z_i, \varrho_m).$$

We assert that there is an $\varepsilon > 0$ such that

$$\big(x \in Q, 0 < \varrho_m(x, F) < \varepsilon\big) \Longrightarrow x \in \bigcup_{i=1}^{n} B^{\circ}_{r_{z_i}}(z_i, \varrho_m). \tag{13}$$

Admitting the contrary we would get, for each choice of $\varepsilon = 1/k$ ($k \in \mathbb{N}$), a point $x_k \in G \setminus F$ with

$$\varrho_m(x_k, F) < \frac{1}{k}, \qquad x_k \in Q \setminus \bigcup_{i=1}^{n} B^{\circ}_{r_{z_i}}(z_i, \varrho_m).$$

Passing to a convergent subsequence we would get $\lim\limits_{k \to +\infty} x_k = z$. Obviously $z \in Q \cap F \cap \overline{G \setminus F}$, consequently $z \in B^{\circ}_{r_{z_i}}(z_i, \varrho_m)$ for a suitable $i \in \{1, 2, \dots, n\}$ and so $x_k \in B^{\circ}_{r_{z_i}}(z_i, \varrho_m)$ for all sufficiently large k, which contradicts $x_k \in Q \setminus \bigcup\limits_{i=1}^{n} B^{\circ}_{r_{z_i}}(z_i, \varrho_m)$. Thus (13) is established for a suitable $\varepsilon > 0$.

We may clearly suppose that $\varepsilon \leqslant \min\{r_{z_i}\}_{i=1}^n$. Put $c = \max\{c_{z_i}\}_{i=1}^n$. If $x \in Q \setminus F$ with $\varrho_m(x, F) < \varepsilon$, then (13) tells us that, for a suitable $i \in \{1, 2, \dots, n\}$,

$$\big|u(x)\big| \leqslant c_{z_i} \cdot g\big(\varrho_m(x, F)\big) \leqslant c \cdot g\big(\varrho_m(x, F)\big),$$

which proves the lemma. $\square$

Theorem 1. *Let $G \subset \mathbb{R}^N$ be an open set, $F \subset G$ a relatively closed subset in G, $u\colon G \setminus F \to \mathbb{C}$ a locally integrable function. Suppose that*

$$\mathcal{M}_{\varrho_m}\text{-}\dim Q < \overline{m}b - q \tag{14}$$

for each compact $Q \subset F$ and

$$u(x) = O\big(\varrho_m(x, F)^{-q}\big) \qquad \text{as } x \to z,\ x \in G \setminus F,$$

for each $z \in F$, where $0 \leqslant q < \overline{m}b$ [cf. (5)]. Then u is defined a.e. in G [i.e. $\lambda_N(F) = 0$] and it is locally integrable in G. If, moreover,

$$\gamma \equiv \overline{m}(b-1) - q \geqslant 0$$

and

$$\mathcal{M}_\gamma(Q, \varrho_m) = 0$$

for each compact $Q \subset F$, then the validity of the equation

$$P(D)u = 0 \qquad \text{in } G \setminus F$$

(in the sense of distributions) implies $P(D)u = 0$ in the whole G.

Proof. In the proof of Lemma 1 we have seen that (14) implies $\lambda_N(Q) = 0$. Since this is assumed for each compact $Q \subset F$, we have $\lambda_N(F) = 0$ so that u is defined λ_N-a.e. in G. We have to verify that the growth estimate of u implies that u is integrable in a neighbourhood of any $z \in G$. It is sufficient to consider $z \in F$. According to Lemma 2, for each ball $B_r(z, \varrho_m) \subset G$ there exist constants $c > 0$, $\varepsilon \in]0, r]$ such that

$$\bigl(x \in B_r(z, \varrho_m) \setminus F, \varrho_m(x, F) < \varepsilon\bigr) \Longrightarrow \bigl|u(x)\bigr| \leqslant c \cdot \varrho_m(x, F)^{-q}.$$

We are going to show that Lemma 1 implies integrability of the function

$$x \mapsto \varrho_m(x, F)^{-q}$$

on a neighbourhood of z. Choosing $\delta \in]0, r/2[$, we have

$$\bigl(y \in B_r(z, \varrho_m) \setminus F, \varrho_m(y, z) \leqslant \delta\bigr) \Longrightarrow \varrho_m\bigl(y, B_{2\delta}(z, \varrho_m) \cap F\bigr) = \varrho_m(y, F).$$

For these y and

$$Q = B_{2\delta}(z, \varrho_m) \cap F$$

we get

$$\varrho_m(y, Q)^{-q} = \varrho_m(y, F)^{-q}.$$

According to Lemma 1, the function $y \mapsto \varrho_m(y, Q)^{-q}$ is integrable on a neighbourhood of z. Thus the integrability of the function $x \mapsto \varrho_m(x, F)^{-q}$ as well as of $u(x)$ on a neighbourhood of z is verified.

Suppose now that $P(D)u = 0$ on $G \setminus F$. $\mathscr{D}(G)$ will denote the class of all infinitely differentiable functions with a compact support in G. Choose an arbitrary $\varphi \in \mathscr{D}(G)$ and denote its support $\operatorname{spt} \varphi$ by K. We are going to show that

$$\bigl\langle P(D)u, \varphi\bigr\rangle = 0. \tag{15}$$

This is obvious if $F \cap K = \emptyset$. So let $F \cap K \neq \emptyset$. Fix an infinitely differentiable function ψ on $\mathbb{R}$ such that

$$\operatorname{spt} \psi \subset [-1,1], \qquad \int_{\mathbb{R}} \psi(t)\, \mathrm{d}t = 1.$$

For $L \subset \mathbb{R}^N$, χ_L will denote the characteristic function of the set L, L_ε for $\varepsilon > 0$ [as defined by (9)] is clearly an open set. Using the notation from (2), (5) we put for $x \in \mathbb{R}^N$ and $\varepsilon > 0$

$$\psi_\varepsilon(x) = \varepsilon^{-\overline{m}b} \int_{\mathbb{R}^N} \chi_{F_{2\varepsilon}}(y) \prod_{j=1}^{N} \psi\big((x_j - y_j)\varepsilon^{-\overline{m}/m_j}\big)\, \mathrm{d}y_1 \ldots \mathrm{d}y_N. \tag{16}$$

Then ψ_ε is an infinitely differentiable function on $\mathbb{R}^N$ (compare Chap. I, Section 5 in [3]). After substitution

$$(x_j - y_j)\varepsilon^{-\overline{m}/m_j} = z_j \qquad (j = 1, 2, \ldots, N)$$

we get $y_j = x_j - \varepsilon^{\overline{m}/m_j} z_j$, so that

$$\begin{aligned} \psi_\varepsilon(x) &= \int_{\mathbb{R}^N} \chi_{F_{2\varepsilon}}(\ldots, x_j - \varepsilon^{\overline{m}/m_j} z_j, \ldots) \prod_{j=1}^{N} \psi(z_j)\, \mathrm{d}z_1 \ldots \mathrm{d}z_N \\ &= \int_{\{z \in \mathbb{R}^N;\ z_j \in [-1,1], 1 \leqslant j \leqslant N\}} \chi_{F_{2\varepsilon}}(\ldots, x_j - \varepsilon^{\overline{m}/m_j} z_j, \ldots) \prod_{j=1}^{N} \psi(z_j)\, \mathrm{d}z_1 \ldots \mathrm{d}z_N. \end{aligned}$$

Hence

$$\big|\psi_\varepsilon(x)\big| \leqslant \int_{[-1,1]^N} \prod_{j=1}^{N} \big|\psi(z_j)\big|\, \mathrm{d}z_1 \ldots \mathrm{d}z_n \equiv C_0, \qquad x \in \mathbb{R}^N. \tag{17}$$

If $x \in F_\varepsilon$ then $(\ldots, x_j - \varepsilon^{\overline{m}/m_j} z_j, \ldots) \in F_{2\varepsilon}$ for $z \in [-1,1]^N$ so that

$$\psi_\varepsilon(x) = \int_{[-1,1]^N} 1 \cdot \prod_{j=1}^{N} \psi(z_j)\, \mathrm{d}z_1 \ldots \mathrm{d}z_N = \prod_{j=1}^{N} \int_{-1}^{1} \psi(z_j)\, \mathrm{d}z_j = 1, \qquad \forall x \in F_\varepsilon. \tag{18}$$

If $x \in \mathbb{R}^N \setminus F_{3\varepsilon}$ then $(\ldots, x_j - \varepsilon^{\overline{m}/m_j} z_j, \ldots) \in \mathbb{R}^N \setminus F_{2\varepsilon}$ so that

$$\chi_{F_{2\varepsilon}}(\ldots, x_j - \varepsilon^{\overline{m}/m_j} z_j, \ldots) = 0, \qquad \forall z \in [-1,1]^N.$$

Hence

$$\psi_\varepsilon(x) = 0 \qquad \text{for each } x \in \mathbb{R}^N \setminus F_{3\varepsilon}. \tag{19}$$

For any multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ we get by differentiating under the sign of the integral

$$\begin{aligned} &D^\alpha \psi_\varepsilon(x) \\ &\quad = \frac{1}{\mathrm{i}^{|\alpha|}} \cdot \varepsilon^{-\overline{m}(\alpha:m)} \int_{[-1,1]^N} \chi_{F_{2\varepsilon}}(\ldots, x_j - \varepsilon^{\overline{m}/m_j} z_j, \ldots) \prod_{j=1}^{N} \psi^{(\alpha_j)}(z_j)\,\mathrm{d}z_1 \ldots \mathrm{d}z_N, \end{aligned}$$

$$\tag{20} \begin{aligned} \big|D^\alpha \psi_\varepsilon(x)\big| &\leqslant \varepsilon^{-\overline{m}(\alpha:m)} \int_{[-1,1]^N} \prod_{j=1}^{N} \big|\psi^{(\alpha_j)}(z_j)\big|\,\mathrm{d}z_1 \ldots \mathrm{d}z_N \\ &= C_\alpha \varepsilon^{-\overline{m}(\alpha:m)}, \qquad \forall x \in \mathbb{R}^N. \end{aligned}$$

If T is a distribution on G we denote by $\operatorname{spt} T$ its support. As $\psi_\varepsilon = 1$ in a neighbourhood of $K \cap \operatorname{spt} P(D)u \subset F$ we have

$$\big\langle P(D)u, \varphi \big\rangle = \big\langle P(D)u, \psi_\varepsilon \varphi \big\rangle = \big\langle u, {}^t\!P(D)(\psi_\varepsilon \varphi)\big\rangle,$$

where ${}^t\!P(D)$ is the formal adjoint of $P(D)$ (cf. p. 40 in [4]);

$${}^t\!P(D)(\psi_\varepsilon \varphi) = \sum_{\alpha:m \leqslant 1} \varphi_\alpha D^\alpha \psi_\varepsilon,$$

where φ_α are certain functions in $\mathscr{D}(G)$ independent of ε which depend on φ and the coefficients a_ω of the operator $P(D)$ only; they have the form of a finite sum of the type

$$\varphi_\alpha = \sum c^\alpha_{\beta\delta\omega} D^\beta a_\omega D^\delta \varphi,$$

where $c^\alpha_{\beta\delta\omega}$ are constants so that $\operatorname{spt} \varphi_\alpha \subset \operatorname{spt} \varphi \equiv K$. We have

$$\tag{21} \big\langle P(D)u, \varphi \big\rangle = \sum_{\alpha:m \leqslant 1} \langle u, \varphi_\alpha D^\alpha \psi_\varepsilon \rangle.$$

Fix an $\varepsilon_0 \in\,]0,1[$ small enough to have $\overline{K}_{\varepsilon_0} \subset G$. We will consider $\varepsilon \in\,]0, \varepsilon_0]$. For $\alpha = 0$ ($\in \mathbb{R}^N$) we have

$$\tag{22} \begin{aligned} \big|\langle u, \varphi_0 D^0 \psi_\varepsilon \rangle\big| &\leqslant \int_{F_{3\varepsilon} \cap G} |u| \cdot |\varphi_0| \cdot |\psi_\varepsilon|\,\mathrm{d}\lambda_N \\ &\leqslant \sup_K |\varphi_0| \cdot \int_{K \cap F_{3\varepsilon}} |u|\,\mathrm{d}\lambda_N \to 0 \qquad \text{as } \varepsilon \searrow 0, \end{aligned}$$

because u is locally integrable in G, $K \cap F_{3\varepsilon} \searrow K \cap F$ as $\varepsilon \to 0$ and $\lambda_N(K \cap F) = 0$.

For $|\alpha| > 0$ we make use of the equality

$$D^\alpha \psi_\varepsilon = 0 \qquad \text{on } F_\varepsilon$$

to get the estimate

$$\left|\langle u, \varphi_\alpha D^\alpha \psi_\varepsilon\rangle\right| \leqslant \sup_K |\varphi_\alpha| \cdot \int_{(K\setminus F_\varepsilon)\cap F_{3\varepsilon}} \left|u(x)\right| \mathrm{d}x \cdot C_\alpha \varepsilon^{-\overline{m}(\alpha:m)};$$

according to Lemma 2 there are constants c_K and $\varepsilon_1 > 0$ such that

$$\left|u(x)\right| \leqslant c_K \varrho_m(x, F)^{-q} \leqslant c_K \varepsilon^{-q}, \qquad \forall x \in (K \setminus F_\varepsilon) \cap F_{\varepsilon_1},$$

so that for $3\varepsilon < \varepsilon_1$ we get

$$\begin{aligned}\left|\langle u, \varphi_\alpha D^\alpha \psi_\varepsilon\rangle\right| &\leqslant \sup_K |\varphi_\alpha| \cdot C_\alpha \varepsilon^{-\overline{m}(\alpha:m)} \int_{F_{3\varepsilon}\cap K\setminus F_\varepsilon} c_K \varrho_m(x, F)^{-q} \, \mathrm{d}x \\ &\leqslant \sup_K |\varphi_\alpha| \cdot C_\alpha \varepsilon^{-\overline{m}(\alpha:m)} \cdot c_K \cdot \varepsilon^{-q} \lambda_N(K \cap F_{3\varepsilon}).\end{aligned}$$

For $\varepsilon < \varepsilon_0/3$ we have

$$K \cap F_{3\varepsilon} \subset \left(F \cap K_{3\varepsilon}\right)_{3\varepsilon} \subset \left(F \cap \overline{K}_{3\varepsilon}\right)_{3\varepsilon}.$$

Indeed, if $y \in K \cap F_{3\varepsilon}$, then there exists a $z \in F$ such that $\varrho_m(y, z) < 3\varepsilon$, whence $z \in F \cap K_{3\varepsilon}$ and, consequently,

$$y \in \left(F \cap K_{3\varepsilon}\right)_{3\varepsilon} \subset \left(F \cap \overline{K}_{\varepsilon_0}\right)_{3\varepsilon}.$$

Assuming

$$0 < \varepsilon < \frac{\min(\varepsilon_0, \varepsilon_1)}{3} < 1$$

we arrive at

$$\begin{aligned}\left|\langle u, \varphi_\alpha D^\alpha \psi_\varepsilon\rangle\right| &\leqslant \sup_K |\varphi_\alpha| \cdot C_\alpha \cdot c_K \cdot \varepsilon^{-\overline{m}(\alpha:m)-q} \cdot \lambda_N\left((F \cap \overline{K}_{\varepsilon_0})_{3\varepsilon}\right) \\ &\leqslant L(\alpha) \cdot \varepsilon^{-\overline{m}-q} \cdot \frac{\lambda_N((F \cap \overline{K}_{\varepsilon_0})_{3\varepsilon})}{(3\varepsilon)^{\overline{m}b-\gamma}} \cdot \varepsilon^{\overline{m}b-\gamma} \\ &= L(\alpha) \cdot \varepsilon^{\overline{m}(b-1)-q-\overline{m}(b-1)+q} \cdot \frac{\lambda_N(Q_{3\varepsilon})}{(3\varepsilon)^{\overline{m}b-\gamma}},\end{aligned}$$

where $Q = F \cap \overline{K}_{\varepsilon_0} \subset F$ is compact and $L(\alpha) = C_\alpha c_K \sup_K |\varphi_\alpha| \cdot 3^{\overline{m}b-\gamma}$. Hence

$$\left|\langle u, \varphi_\alpha D^\alpha \psi_\varepsilon\rangle\right| \leqslant L(\alpha) \frac{\lambda_N(Q_{3\varepsilon})}{(3\varepsilon)^{\overline{m}b-\gamma}} \qquad \text{for } 0 < \alpha : m \leqslant 1.$$

By virtue of (21) we get

$$\big|\langle P(D)u,\varphi\rangle\big| \leqslant \big|\langle u,\varphi_0 D^0\psi_\varepsilon\rangle\big| + \frac{\lambda_N(Q_{3\varepsilon})}{(3\varepsilon)^{\overline{m}b-\gamma}} \sum_{0<\alpha:m\leqslant 1} L(\alpha).$$

Passing to $\liminf\limits_{\varepsilon\searrow 0}$ we obtain in view of (22)

$$\big|\langle P(D)u,\varphi\rangle\big| \leqslant 0 + \mathcal{M}_\gamma(Q) \sum_{0<\alpha:m\leqslant 1} L(\alpha) = 0.$$

Thus (15) is verified and the proof is complete. □

The following simple lemma plays the role similar to that of Lemma 2.

Lemma 2. *Let $G\subset\mathbb{R}^N$ be open and $\emptyset\neq F\subset G$ relatively closed in G. Suppose that $u\colon G\setminus F\to\mathbb{C}$ and $g\colon\ ]0,+\infty[\ \to\ ]0,+\infty[$ are functions satisfying*

$$u(x) = o\big(g(\varrho_m(x,F))\big) \qquad \text{as } x\to z,\ x\in G\setminus F \tag{23}$$

for each $z\in F\cap\overline{G\setminus F}$. Then for each compact $H\subset F\cap\overline{G\setminus F}$ there exist $\delta>0$ and a nondecreasing function $f\colon\ ]0,+\infty[\ \to\ ]0,+\infty[$ such that, in the notation of (9),

$$\begin{gathered}\overline{H}_\delta\subset G,\\ x\in H_\delta\setminus F \Longrightarrow \big|u(x)\big| \leqslant f\big(\varrho_m(x,F)\big)\cdot g\big(\varrho_m(x,F)\big),\\ f(0+)\equiv\lim_{t\searrow 0} f(t) = 0.\end{gathered} \tag{24}$$

Proof. Choose an $\varepsilon_0>0$ small enough to have $\overline{H}_{\varepsilon_0}\subset G$. According to Lemma 2, given a compact $Q=\overline{H}_{\varepsilon_0}$, there exist $c>0$ and $\varepsilon>0$ such that

$$\big(x\in Q\setminus F, \varrho_m(x,F)<\varepsilon\big) \Longrightarrow \frac{|u(x)|}{g(\varrho_m(x,F))} \leqslant c. \tag{25}$$

Fix a $\delta\in\ ]0,\varepsilon[$ and define for $t>0$

$$f(t) = \sup\Big\{\frac{|u(x)|}{g(\varrho_m(x,F))};\ x\in Q\setminus F, \varrho_m(x,F)\leqslant\min(t,\delta)\Big\}. \tag{26}$$

Then $f(t)\leqslant c$ for $t\in\ ]0,+\infty[$ by virtue of (25). Clearly, f is nondecreasing, for $x\in H_\delta\setminus F\subset Q\setminus F$ we have setting $\tau=\varrho_m(x,F)$

$$\frac{|u(x)|}{g(\varrho_m(x,F))} \leqslant f(\tau) = f\big(\varrho_m(x,F)\big)$$

so that

$$\left|u(x)\right| \leqslant f\bigl(\varrho_m(x,F)\bigr) g\bigl(\varrho_m(x,F)\bigr).$$

We are going to verify that $\lim\limits_{t\searrow 0} f(t) = 0$. Choose a decreasing sequence $t_k > 0$ with $\lim\limits_{k\to\infty} t_k = 0$. Then

$$\lim_{k\to\infty} f(t_k) = \inf_{t>0} f(t) \equiv \alpha.$$

If $\alpha > 0$ then $f(t_k) > \alpha/2$ for each k and the definition of f would imply the existence of $x_k \in Q \setminus F$ with $\varrho_m(x_k, F) \leqslant t_k$ such that

$$\frac{|u(x_k)|}{g(\varrho_m(x_k,F))} > \frac{\alpha}{2}. \tag{27}$$

Passing to a convergent subsequence we could achieve the existence of the limit $\lim\limits_{k\to\infty} x_k = z$; obviously, $z \in F \cap \overline{G \setminus F}$ and, in view of the assumption (23),

$$\frac{|u(x_k)|}{g(\varrho_m(x_k,F))} \to 0,$$

which contradicts (27). Thus (24) is verified. □

The following theorem is similar to Theorem 1.

Theorem 2. *Let $G \subset \mathbb{R}^N$ be an open set and let $\emptyset \neq F \subset G$ be relatively closed in G. Suppose that $u\colon G \setminus F \to \mathbb{C}$ is a locally integrable function satisfying*

$$u(x) = o\bigl(\varrho_m(x,F)^{-q}\bigr) \qquad \text{as } x \to z,\ x \in G \setminus F$$

for each $z \in F$, where $0 \leqslant q < \overline{m}b$ [cf. (5)]. If

$$\mathcal{M}_{\varrho_m}\text{-dim}\, Q < \overline{m}b - q$$

for each compact $Q \subset F$, then u is defined a.e. in G and it is locally integrable in G. If, moreover, $\gamma \equiv \overline{m}(b-1) - q \geqslant 0$ and

$$\mathcal{M}_\gamma(Q, \varrho_m) < +\infty$$

for each compact $Q \subset F$, then the validity of the equation

$$P(D)u = 0 \qquad \text{in } G \setminus F$$

(in the sense of distributions) implies

$$P(D)u = 0 \qquad \text{in the whole } G.$$

P r o o f. We have seen in the proof of Theorem 1 that u is locally integrable in G. Choose an arbitrary $\varphi \in \mathscr{D}(G)$. We are going to verify (15). Put $K = \operatorname{spt} \varphi$, which is a compact set. It is again sufficient to consider the case when $K \cap F \neq \emptyset$. As in the proof of Theorem 1 we construct for each $\varepsilon > 0$ an auxiliary function ψ_ε. We have

$$\left|D^\alpha \psi_\varepsilon(x)\right| \leqslant C_\alpha \varepsilon^{-\overline{m}(\alpha:m)}$$

for each multiindex α. Besides that,

$$\begin{aligned} \psi_\varepsilon(x) &= 1, \qquad \forall x \in F_\varepsilon, \\ \psi_\varepsilon(x) &= 0, \qquad \forall x \in \mathbb{R}^N \setminus F_{3\varepsilon}. \end{aligned}$$

As $\psi_\varepsilon = 1$ on a neighbourhood of $K \cap F \supset \operatorname{spt} \varphi \cap \operatorname{spt} P(D)u$ we get

$$\left\langle P(D)u, \varphi \right\rangle = \left\langle P(D)u, \psi_\varepsilon \varphi \right\rangle = \left\langle u, {}^t\!P(D)(\psi_\varepsilon \varphi) \right\rangle,$$

which implies (21) with $\varphi_\alpha \in \mathscr{D}(G)$ independent of ε with $\operatorname{spt} \varphi_\alpha \subset K$. For $\alpha = 0$ ($\in \mathbb{R}^N$) we obtain as in the proof of Theorem 1

$$(28) \qquad \left|\langle u, \varphi_0 D^0 \psi_\varepsilon \rangle\right| \leqslant \sup_K |\varphi_0| \int_{K \cap F_{3\varepsilon}} |u| \, \mathrm{d}\lambda_N \to \int_{K \cap F} |u| \, \mathrm{d}\lambda_N = 0 \qquad \text{as } \varepsilon \to 0.$$

We will consider $\varepsilon \in \,]0, \varepsilon_0]$, where $\varepsilon_0 \in \,]0, 1[$ has been chosen small enough to satisfy $\overline{K}_{\varepsilon_0} \subset G$. We have seen that

$$(29) \qquad K \cap F_{3\varepsilon} \subset \left(F \cap \overline{K}_{\varepsilon_0}\right)_{3\varepsilon} \qquad \text{for } 0 < \varepsilon < \varepsilon_0/3.$$

For $0 < \alpha : m \leqslant 1$ we get from (18)

$$D^\alpha \psi_\varepsilon = 0 \qquad \text{on } F_\varepsilon$$

and, in view of (19), (20),

$$\left|\langle u, \varphi_\alpha D^\alpha \psi_\varepsilon \rangle\right| \leqslant \sup_K |\varphi_\alpha| \cdot C_\alpha \cdot \varepsilon^{-\overline{m}(\alpha:m)} \int_{F_{3\varepsilon} \cap K \setminus F_\varepsilon} |u| \, \mathrm{d}\lambda_N.$$

Putting $H = F \cap \overline{K}_{\varepsilon_0}$ we get from (29) for $\varepsilon \in \,]0, \varepsilon_0/3[$

$$\int_{F_{3\varepsilon} \cap K \setminus F_\varepsilon} |u| \, \mathrm{d}\lambda_N \leqslant \int_{H_{3\varepsilon} \setminus F_\varepsilon} |u| \, \mathrm{d}\lambda_N.$$

According to Lemma 3 there exist a $\delta > 0$ and a nondecreasing function

$$f\colon \,]0, +\infty[\, \to \,]0, +\infty[$$

such that $\overline{H}_\delta \subset G$ and (24) holds together with

$$x \in H_\delta \setminus F \Longrightarrow \bigl|u(x)\bigr| \leqslant f\bigl(\varrho_m(x,F)\bigr)\varrho_m(x,F)^{-q}.$$

Assuming $3\varepsilon < \delta$ we have for $x \in H_{3\varepsilon}$ estimates $\varrho_m(x,F) \leqslant 3\varepsilon$ and

$$\int_{H_{3\varepsilon}\setminus F_\varepsilon} |u|\,\mathrm{d}\lambda_N \leqslant f(3\varepsilon)\int_{H_{3\varepsilon}\setminus F_\varepsilon} \varrho_m(x,F)^{-q}\,\mathrm{d}\lambda_N(x).$$

We see that for sufficiently small $\varepsilon \in \,]0,1[$

$$\begin{aligned}\bigl|\langle u,\varphi_\alpha D^\alpha\psi_\varepsilon\rangle\bigr| &\leqslant \sup_K|\varphi_\alpha|C_\alpha\cdot\varepsilon^{-\overline{m}}\cdot f(3\varepsilon)\int_{H_{3\varepsilon}\setminus F_\varepsilon}\varrho_m(x,F)^{-q}\,\mathrm{d}\lambda_N(x)\\ &\leqslant \sup_K|\varphi_\alpha|C_\alpha\cdot\varepsilon^{-\overline{m}-q}f(3\varepsilon)\lambda_N(H_{3\varepsilon})\\ &= \frac{\lambda_N(H_{3\varepsilon})}{(3\varepsilon)^{\overline{m}b-\gamma}}\cdot f(3\varepsilon)E_\alpha,\end{aligned}$$

where

$$E_\alpha = C_\alpha\sup_K|\varphi_\alpha|\cdot 3^{\overline{m}b-\gamma},$$

so that

$$\sum_{0<\alpha:m\leqslant 1}\bigl|\langle u,\varphi_\alpha D^\alpha\psi_\varepsilon\rangle\bigr| \leqslant f(3\varepsilon)\cdot\frac{\lambda_N(F_{3\varepsilon})}{(3\varepsilon)^{\overline{m}b-\gamma}}\sum_{0<\alpha:m\leqslant 1}E_\alpha.$$

As $\mathcal{M}_\gamma(H,\varrho_m) < +\infty$ we obtain from (24), (28)

$$\liminf_{\varepsilon\searrow 0}\sum_{\alpha:m\leqslant 1}\bigl|\langle u,\varphi_\alpha D^\alpha\psi_\varepsilon\rangle\bigr| \leqslant f(0+)\mathcal{M}_\gamma(H,\varrho_m)\sum_{0<\alpha:m\leqslant 1}E_\alpha = 0,$$

whence

$$\bigl|\langle P(D)u,\varphi\rangle\bigr| \leqslant \liminf_{\varepsilon\searrow 0}\Bigl|\sum_{\alpha:m\leqslant 1}\langle u,\varphi_\alpha D^\alpha\psi_\varepsilon\rangle\Bigr| = 0$$

and the proof is complete. □

Remark 2. Let $G \subset \mathbb{R}^N$ be an open set, $F \subset G$ a relatively closed subset in G. We have seen that the assumption

$$\mathcal{M}_{\varrho_m}\text{-}\dim Q < \overline{m}b - q$$

($0 \leqslant q < \overline{m}b$) for each compact $Q \subset F$ guarantees that $\lambda_N(F) = 0$ and each function u locally integrable in $G \setminus F$ satisfying

(30) $$u(x) = O\bigl(\varrho_m(x,F)^{-q}\bigr) \qquad \text{as } x \to z,\ x \in G\setminus F,\quad \forall z \in F,$$

is locally integrable in G. We say that F is $P(D)$-removable for functions u locally integrable in $G \setminus F$ satisfying (30), if each such function u fulfilling $P(D)u = 0$ in $G \setminus F$ (in the sense of distributions) fulfils $P(D)u = 0$ in the whole G. The previous theorems describe in terms of Minkowski's contents sufficient conditions for the $P(D)$-removability of F for functions u satisfying (30) or the growth estimate

$$u(x) = o\big(\varrho_m(x, F)^{-q}\big) \qquad \text{as } x \to z,\ x \in G \setminus F, \quad \forall z \in F. \tag{31}$$

It turns out that for certain operators $P(D)$ it is possible to use Hausdorff measures for obtaining necessary conditions for the $P(D)$-removability of F. Now we are going to consider operators of the form

$$P(D) = \sum_{\alpha : m \leqslant 1} a_\alpha D^\alpha, \tag{32}$$

whose coefficients a_α are complex constants. We associate with such an operator (32) the polynomial

$$P_m(\xi) = \sum_{\alpha : m = 1} a_\alpha \xi^\alpha$$

in N real variables $\xi_1, \xi_2, \dots, \xi_N$.

$P(D)$ is termed semielliptic, if

$$\big(\xi \in \mathbb{R}^N, P_m(\xi) = 0\big) \Longrightarrow \xi = 0 \quad (\in \mathbb{R}^N),$$

which means that $P_m(\xi)$ has no nontrivial zeros in $\mathbb{R}^N$. (The corresponding m is then uniquely determined—cf. [14].)

Theorem 3. *Suppose that G, F have the meaning described above. Let $P(D)$ be a semielliptic operator. Assume that* [*cf.* (5)]

$$b > 1, \qquad q \leqslant \overline{m}(b-1), \qquad \gamma \equiv \overline{m}(b-1) - q \geqslant 0.$$

For F to be $P(D)$-removable for all locally integrable functions u in $G \setminus F$ satisfying (30) *it is necessary that*

$$\mathscr{H}_\gamma(F, \varrho_m) = 0 \tag{$33_{\mathscr{H}}$}$$

and sufficient that

$$\mathcal{M}_\gamma(Q, \varrho_m) = 0 \tag{$33_{\mathcal{M}}$}$$

for each compact $Q \subset F$.

For the $P(D)$-removability of F for all locally integrable functions u satisfying the growth condition (31) *it is necessary that F have a σ-finite $\mathscr{H}_\gamma$-measure, and sufficient that $\mathcal{M}_\gamma(Q, \varrho_m) < +\infty$ for each compact $Q \subset F$.*

The proof follows from a combination of Theorem 1, Theorem 2 above and Theorem 5 in [6].

We should point out that $\mathcal{M}_\gamma$ denotes the upper Minkowski's content in [6] so that this Theorem 3 is more general than the corresponding result in [6].

An important example is given by the heat conduction operator

$$\frown = \partial_1^2 + \ldots + \partial_n^2 - \partial_{n+1}$$

in $\mathbb{R}^{n+1}$. This operator is semielliptic with the choice $m_j = 2$ for $1 \leqslant j \leqslant n$, $m_{n+1} = 1$ so that $m = (2, \ldots, 2, 1) \in \mathbb{R}^{n+1}$, $\overline{m} = 2$ and the corresponding metric $\varrho \equiv \varrho_m$ is given by

$$\varrho(x, y) = \max\bigl\{|x_1 - y_1|, \ldots, |x_n - y_n|, |x_{n+1} - y_{n+1}|^{1/2}\bigr\}$$

for $x = (x_1, \ldots, x_{n+1})$, $y = (y_1, \ldots, y_{n+1})$.

Now we will consider an open set $G \subset \mathbb{R}^{n+1}$ and a relatively closed subset $F \subset G$, where

$$\mathcal{M}_\varrho\text{-}\dim Q < n + 2 - q, \qquad \gamma = n - q \tag{34}$$

for each compact $Q \subset F$.

A function u, which is of the class $\mathscr{C}^{(2)}$ on an open set $U \subset \mathbb{R}^{n+1}$, is termed caloric on U, if it satisfies

$$\frown u = 0 \qquad \text{on } U.$$

Theorem 4. *In order that each caloric function u on $G \setminus F$ satisfying*

$$u(x) = O\bigl(\varrho(x, F)^{-q}\bigr) \qquad \text{as } x \to z,\ x \in G \setminus F, \quad \forall z \in F \tag{35}$$

be extendable to a caloric function on the whole G it is necessary that

$$\mathscr{H}_\gamma(F, \varrho) = 0 \tag{36}$$

and it is sufficient that

$$\mathcal{M}_\gamma(Q, \varrho) = 0 \tag{37}$$

for each compact $Q \subset F$.

If we replace (35) *by*

$$u(x) = o\bigl(\varrho(x,F)^{-q}\bigr) \qquad \text{as } x \to z,\ x \in G \setminus F, \quad \forall z \in F, \tag{38}$$

then the assertion remains valid with (36) *replaced by the requirement of σ-finiteness of $\mathscr{H}_\gamma(F,\varrho)$ and the assumption* (37) *replaced by the requirement $\mathcal{M}_\gamma(Q,\varrho) < +\infty$ for each compact $Q \subset F$.*

The proof follows from Theorem 3. Indeed, if u is a caloric function on $G \setminus F$ satisfying (35) then, according to Theorem 3, under the condition (34), u is defined a.e. in G, it is locally integrable and satisfies $\frown u = 0$ in the sense of distributions. Since the operator $\frown$ is semielliptic, there exists $\tilde{u} \in \mathscr{C}^\infty(G)$ satisfying $\frown \tilde{u} = 0$ on G in the classical sense such that $\tilde{u} = u$ a.e. on G (cf. [14]), whence $\tilde{u} = u$ on $G \setminus F$ and $\tilde{u}$ is the required extension.

R e m a r k s. Sufficient conditions for removability of singularities of caloric functions with anisotropic growth were described in terms of the upper anisotropic Minkowski's contents by H. Zlonická in [15].

Another proof in [7] (based on anisotropic Whitney's decomposition) showed that the lower Minkowski's content is sufficient for this purpose.

Application of the upper Minkowski's content (derived from the euclidean metric) in this context goes back to Bochner (compare [1], [4]).

Later Polking pointed out in [11] that a modification of Bochner's proof makes it possible to replace the upper Minkowski's content by the lower Minkowski's content; his method is also described in [13].

Theorems of the Bochner type for operators on manifolds, where the singular set is formed by a submanifold of a suitable dimension, have been studied by Sugimoto and Uchida in [12].

Various applications of anisotropic metrics and anisotropic Minkowski's contents in connection with removability of singularities are described by Littman in [8].

In his thesis [10] M. Píštěk described (in terms of the upper anisotropic Minkowski's content) removable singularities of caloric functions which are locally integrable in power p with a weight depending on the distance from the singular set measured by the caloric metric. It is natural to ask whether in his investigation the upper Minkowski's content could also be replaced by the lower content of the same dimension.

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