SUBTRACTION ALGEBRAS AND BCK-ALGEBRAS

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Abstract. In this note we show that a subtraction algebra is equivalent to an implicative BCK-algebra, and a subtraction semigroup is a special case of a BCI-semigroup.

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B. M. Schein ([9]) considered systems of the form $(\Phi; \circ, \backslash)$, where Φ is a set of functions closed under the composition "$\circ$" of functions (and hence $(\Phi; \circ)$ is a function semigroup) and the set theoretic subtraction "$\backslash$" (and hence $(\Phi; \backslash)$ is a subtraction algebra in the sense of [2]). He proved that every subtraction semigroup is isomorphic to a difference semigroup of invertible functions. B. Zelinka ([11]) discussed a problem proposed by B. M. Schein concerning the structure of multiplication in a subtraction semigroup. He solved the problem for subtraction algebras of a special type, called the atomic subtraction algebras. In this note we show that a subtraction algebra is equivalent to an implicative BCK-algebra, and a subtraction semigroup is a special case of a BCI-semigroup which is a generalization of a ring.

By a *BCI-algebra* ([7]) we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms for all $x, y, z \in X$:

(i) $((x * y) * (x * z)) * (z * y) = 0$,

(ii) $(x * (x * y)) * y = 0$,

(iii) $x * x = 0$,

(iv) $x * y = 0$ and $y * x = 0$ imply $x = y$.

A *BCK-algebra* is a BCI-algebra satisfying the axiom:

(v) $0 * x = 0$ for all $x \in X$.

We can define a partial ordering $\leqslant$ on X by $x \leqslant y$ if and only if $x * y = 0$. In any BCI-algebra X, we have

(1) $x * 0 = x$,

(2) $(x * y) * z = (x * z) * y$,

(3) $x \leqslant y$ imply $x * z \leqslant y * z$ and $z * y \leqslant z * x$,

(4) $(x * z) * (y * z) \leqslant x * y$

for any $x, y, z \in X$.

A *subtraction algebra* is a groupoid $(X; -)$ where "$-$" is a binary operation, called a *subtraction*; this subtraction satisfies the following axioms: for any $x, y, z \in X$,

(I) $x - (y - x) = x$;

(II) $x - (x - y) = y - (y - x)$;

(III) $(x - y) - z = (x - z) - y$.

Note that a subtraction algebra is the dual of the implication algebra defined by J. C. Abbott ([1]), by simply exchanging $x - y$ by yx. If to a subtraction algebra $(X; -)$ a semigroup multiplication is added safisfying the distributive laws

$$x \cdot (y - z) = x \cdot y - x \cdot z,$$
$$(y - z) \cdot x = y \cdot x - z \cdot x$$

then the resulting algebra $(X; \cdot, -)$ is called a *subtraction semigroup*. In [9] it is mentioned that in every subtraction algebra $(X; -)$ there exists an element 0 such that $x - x = 0$ for any $x \in X$. The proof is given by J. C. Abbott ([1], Theorem 1). Note that $x - 0 = x$ for any x in a subtraction algebra $(X; -, 0)$. H. Yutani ([10]) obtained equivalent simple axioms for an algebra $(X; -, 0)$ to be a commutative BCK-algebra.

Theorem 1 ([10]). *An algebra $(X; -, 0)$ is a commutative BCK-algebra if and only if it satisfies*

(II) $x - (x - y) = y - (y - x)$;

(III) $(x - y) - z = (x - z) - y$;

(IV) $x - x = 0$;

(V) $x - 0 = x$

for any $x, y, z \in X$.

A BCK-algebra $(X; -, 0)$ is said to be *implicative* if (I) $x - (y - x) = x$ for any $x, y \in X$. Using this concept and comparing the axiom system of the subtraction algebra with the characterizing equalities of the implicative BCK-algebra (by H. Yutani), we summarize to obtain the main result of this paper.

Theorem 2. *A subtraction algebra is equivalent to an implicative BCK-algebra.*

The notion of a BCI-semigroup was introduced by Y. B. Jun et al. ([5]), and studied by many researchers ([3], [4], [6], [8]). A *BCI-semigroup* (or shortly, *IS-algebra*) is a non-empty set X with two binary operations "$-$" and "$\cdot$" and a constant 0 satisfying the axioms (i) $(X; -, 0)$ is a BCI-algebra; (ii) $(X; \cdot)$ is a semigroup; (iii) $x \cdot (y - z) = x \cdot y - x \cdot z$, $(x - y) \cdot z = x \cdot z - y \cdot z$ for all $x, y, z \in X$.

Example 3 ([3]). If we define two binary operations "$*$" and "$\cdot$" on a set $X := \{0, 1, 2, 3\}$ by

$*$	0	1	2	3
0	0	0	2	2
1	1	0	3	2
2	2	2	0	0
3	3	2	1	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

then $(X; *, \cdot, 0)$ is a BCI-semigroup.

Every p-semisimple BCI-algebra turns into an abelian group by defining $x + y := x * (0 * y)$, and hence a p-semisimple BCI-semigroup leads to the ring structure. On the other hand, every ring turns into a BCI-algebra by defining $x * y := x - y$ and hence we can construct a BCI-semigroup. This means that *the category of p-semisimple BCI-semigroups is equivalent to the category of rings.* In Example 3, we can see that $2 + 3 = 0 \neq 1 = 3 + 2$ and $3 + 2 = 1 = 3 + 3$, hence $(X; +)$ is not a group. This means that there exist BCI-semigroups which cannot be derived from rings. Hence *the BCI-semigroup is a generalization of the ring.*

Since an implicative BCK-algebra is a special case of a BCI-algebra, we conclude that *a subtraction semigroup is a special case of a BCI-semigroup.*

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