RESOLVING DOMINATION IN GRAPHS

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(Received June 11, 2001)

Abstract. For an ordered set $W = \{w_1, w_2, \ldots, w_k\}$ of vertices and a vertex v in a connected graph G, the (metric) representation of v with respect to W is the k-vector $r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k))$, where $d(x, y)$ represents the distance between the vertices x and y. The set W is a resolving set for G if distinct vertices of G have distinct representations with respect to W. A resolving set of minimum cardinality is called a minimum resolving set or a basis and the cardinality of a basis for G is its dimension $\dim G$. A set S of vertices in G is a dominating set for G if every vertex of G that is not in S is adjacent to some vertex of S. The minimum cardinality of a dominating set is the domination number $\gamma(G)$. A set of vertices of a graph G that is both resolving and dominating is a resolving dominating set. The minimum cardinality of a resolving dominating set is called the resolving domination number $\gamma_r(G)$. In this paper, we investigate the relationship among these three parameters.

Keywords: resolving dominating set, resolving domination number

MSC 2000: 05C12, 05C69

1. Introduction

Let G be a connected graph of order n and let $W = \{w_1, w_2, \ldots, w_k\}$ be an ordered set of vertices of G. For a vertex v of G, the k-vector

$$r(v|W) = (d(v, w_1), d(v, w_2), \ldots, d(v, w_k)),$$

where $d(x, y)$ represents the distance between the vertices x and y, is called the *representation of v with respect to W*. The set W is a *resolving set* for G if $r(u|W) = r(v|W)$ implies that $u = v$ for every pair u, v of vertices of G. A resolving set of

Research supported in part by the Western Michigan University Research Development Award Program Fund.

minimum cardinality is called a *minimum resolving set* or a *basis* and the cardinality of a basis for G is its *dimension* $\dim G$.

The concepts of resolving set and minimum resolving set have previously appeared in the literature. In [6] and later in [7], Slater introduced these ideas and used locating set for what we have called resolving set. He referred to the cardinality of a minimum resolving set in a graph G as its location number of G. Slater described the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations. Independently, Harary and Melter [5] investigated these concepts as well, but used metric dimension rather than location number, the terminology that we have adopted. We refer to [2] for graph theory notation and terminology not described here.

The following results on dimension appear in the papers [1], [5], [6], [7].

Theorem A. *Let G be a connected graph of order $n \geqslant 2$.*

(a) *Then $\dim(G) = 1$ if and only if $G = P_n$.*

(b) *Then $\dim(G) = n - 1$ if and only if $G = K_n$.*

(c) *For $n \geqslant 4$, $\dim(G) = n - 2$ if and only if $G = K_{r,s}$ $(r, s \geqslant 1)$, $G = K_r + \overline{K}_s$ $(r \geqslant 1, s \geqslant 2)$, or $G = K_r + (K_1 \cup K_s)$ $(r, s \geqslant 1)$.*

We note that when determining whether a given set W of vertices of a graph G is a resolving set for G, we need only investigate the vertices of $V(G) - W$ since $w \in W$ is the only vertex of G whose distance from w is 0. The following lemma is useful. For a vertex v in a graph G, the *open neighborhood* $N(v)$ of v is the set of all vertices that are adjacent to v and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$.

Lemma 1.1. *Let u and v be vertices of a connected graph G. If either (1) u and v are not adjacent and $N(u) = N(v)$ or (2) u and v are adjacent and $N[u] = N[v]$, then every resolving set of G contains at least one of u and v.*

Proof. Assume, to the contrary, that there is a resolving set $W = \{w_1, w_2, \ldots, w_k\}$ containing neither u nor v. For $w_i \in W$, where $1 \leqslant i \leqslant k$, the path $u, x_1, x_2, \ldots, x_k = w_i$ is a $u - w_i$ geodesic (or a shortest $u - w_i$ path) not containing v if and only if $v, x_1, x_2, \ldots, x_k = w_i$ is a $v - w_i$ geodesic not containing u. Since a shortest path from u to $w_i \neq v$ will not contain v and a shortest path from v to $w_i \neq u$ will not contain u, it follows that $d(u, w_i) = d(v, w_i)$ for all i with $1 \leqslant i \leqslant k$. This implies that $r(u|W) = r(v|W)$, a contradiction. □

A vertex v in a graph G is said to *dominate* itself as well as its neighbors. A set S of vertices in G is a *dominating set* for G if every vertex of G is dominated by some vertex of S. The minimum cardinality of a dominating set is the *domination number*

$\gamma(G)$ of G. A dominating set of cardinality $\gamma(G)$ is called a *γ-set* for G. A thorough treatment of domination in graphs can be found in the book by Haynes, Hedetniemi, and Slater [4]. Dominating sets satisfying additional properties have been studied extensively. For example, independent dominating sets require a dominating set to be independent and connected dominating sets require a dominating set to induce a connected subgraph. See [4, Chap. 6] for these and other conditional domination numbers.

A connected graph G ordinarily contains many dominating sets. Indeed, every superset of a dominating set is also a dominating set. The same statement is true for resolving sets. It is the goal of this paper to study those dominating sets that are resolving sets as well. Such sets will be called *resolving dominating sets*. Thus a resolving dominating set S of vertices of G not only dominates all the vertices of G but has the added feature that distinct vertices of G have distinct representations with respect to S. It is natural to seek minimum resolving sets for G. The cardinality of a minimum resolving dominating set is called the *resolving domination number* of G and is denoted by $\gamma_r(G)$. A resolving dominating set of cardinality $\gamma_r(G)$ is called a *γ_r-set* for G. Necessarily, $\dim(G) \leqslant \gamma_r(G)$ and $\gamma(G) \leqslant \gamma_r(G)$. Let S be a γ_r-set and W a basis for a graph G. Since every superset of a dominating set is a dominating set and every superset of a resolving set is a resolving set, it follows that $S \cup W$ is a resolving dominating set. This gives the following lower and upper bounds for $\gamma_r(G)$ in terms of $\gamma(G)$ and $\dim(G)$.

Proposition 1.2. *For every graph G,*

$$\max\{\gamma(G), \dim(G)\} \leqslant \gamma_r(G) \leqslant \gamma(G) + \dim(G).$$

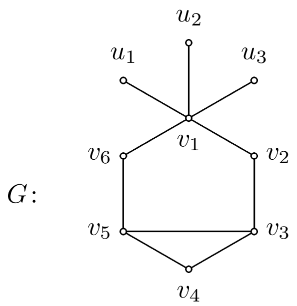


Figure 1. A graph G with $\gamma(G) = 2$, $\dim(G) = 3$, and $\gamma_r(G) = 4$

To illustrate these concepts, consider the graph G of Figure 1. Let $U = \{u_1, u_2, u_3\}$. By Lemma 1.1, every resolving set of G contains at least two vertices from U. Since no 2-element subset of U is a resolving set, $\dim(G) \geqslant 3$. On the other hand, the set $\{u_1, u_2, v_5\}$ is a resolving set for G, implying that $\dim(G) = 3$. The set $\{v_1, v_5\}$ is a γ-set of G and so $\gamma(G) = 2$. To determine the resolving domination number of G, observe that if S is a γ_r-set of G, then either (1) S does not contain v_1 and consequently contains U or (2) S contains v_1 and at least two vertices from U. Since neither U nor a set of vertices of G consisting of v_1 and two vertices from U is a resolving dominating set of G, it follows that $\gamma_r(G) \geqslant 4$. On the other hand, the 4-element set $\{u_1, u_2, v_1, v_5\}$ is a resolving dominating set of G and so $\gamma_r(G) = 4$. Note that, in this case, $\max\{\gamma(G), \dim(G)\} < \gamma_r(G) < \gamma(G) + \dim(G)$.

The resolving domination numbers of some well-known classes of graphs are presented next as additional examples.

Proposition 1.3.

(a) *For* $n \geqslant 2$, $\gamma_r(K_n) = \gamma_r(K_{1,n-1}) = n - 1$;

(b) $\gamma_r(P_3) = 2$ *and for* $n \geqslant 4$, $\gamma_r(P_n) = \lceil n/3 \rceil$;

(c) *For* $n \geqslant 3$, $\gamma_r(C_n) = \lceil n/3 \rceil$ *if* $n \neq 6$ *and* $\gamma_r(C_6) = 3$;

(d) *For integers* $2 \leqslant n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k$ *with* $n_1 + n_2 + \ldots + n_k = n$ *and* $k \geqslant 2$, $\gamma_r(K_{n_1, n_2, \ldots, n_k}) = n - k$.

2. Graphs with prescribed resolving domination number

There are only finitely many connected graphs having a fixed resolving domination number. To verify this, we first establish a lower bound for the resolving domination number of a graph. For positive integers d and n with $d < n$, define

$$f(n, d) = \min\left\{k\colon\ k + \sum_{i=1}^{k} \binom{k}{i} (d-1)^{k-i} \geqslant n\right\}.$$

Proposition 2.1. *If G is a connected graph of order $n \geqslant 2$ and diameter d, then*

$$\gamma_r(G) \geqslant f(n, d).$$

Proof. Let $\gamma_r(G) = k$ and let S be a γ_r-set of G. For each $x \in V(G) - S$, the representation $r(x|S)$ of x with respect to S is a k-vector, every coordinate of which is a positive integer not exceeding d, at least one coordinate of which is 1. The number of such distinct k-vectors $r(x|S)$ having exactly i coordinates ($1 \leqslant i \leqslant k$) equal to 1 is at most $\binom{k}{i}(d-1)^{k-i}$. This implies that the number of distinct k-vectors $r(x|S)$, where $x \in V(G) - S$, is at most $\sum\limits_{i=1}^{k} \binom{k}{i}(d-1)^{k-i}$. Since all $n-k$ representations with respect to S are distinct, it follows that

$$n - k \leqslant \sum_{i=1}^{k} \binom{k}{i} (d-1)^{k-i}$$

and the result follows. □

The following results are immediate consequences of Proposition 2.1.

Corollary 2.2. *If G is a connected graph of order $n \geqslant 2$, diameter d, and resolving domination number k, then*

$$n \leqslant k + \sum_{i=1}^{k} \binom{k}{i} (d-1)^{k-i}.$$

Corollary 2.3. *For every positive integer k, there are only finitely many connected graphs G with resolving domination number k.*

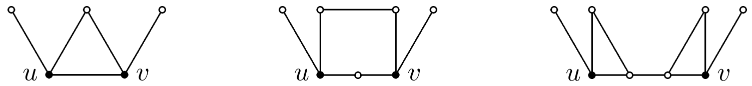
P r o o f. Let G be a connected graph of order $n \geqslant 2$ with $\gamma_r(G) = k$. By Proposition 1.1, $\gamma(G) \leqslant k$ and so the diameter of G is at most $3k-1$. The result follows by Corollary 2.2. □

It is an immediate observation that the only nontrivial graph having resolving domination number 1 is K_2. By Corollary 2.2, the order of any connected graph G with resolving domination number 2 is at most 11. In fact, we can improve upon this statement.

Proposition 2.4. *The order of every connected graph of order n with resolving domination number 2 is at most 8.*

P r o o f. Let G be a connected graph with $\gamma_r(G) = 2$ and let $S = \{u, v\}$ be a γ_r-set for G. Necessarily, $1 \leqslant d(u,v) \leqslant 3$. We consider three cases.

C a s e 1. $d(u,v) = 1$. Since every vertex in $G - S$ is adjacent to at least one of u and v and has distance at most 2 from the other, the only possible representations of a vertex in $V(G) - S$ with respect to S are $(1,1)$, $(1,2)$, and $(2,1)$ and the order of G is at most 5. One such graph of order 5 is shown in Figure 2(a).



(a) (b) (c)

Figure 2. Three graphs with resolving domination number 2

C a s e 2. $d(u,v)=2$. Let $x\in V(G)-S$. Then $r(x|S)=(i,j)$, where at least one of i,j is 1 and $|i-j|\leqslant 2$. Hence the possible values of $r(x|S)$ are $(1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, and $(3,1)$ and the order of G is at most 7. Such a graph of order 7 is shown in Figure 2(b).

C a s e 3. $d(u,v)=3$. Let $x\in V(G)-S$, where $r(x|S)=(i,j)$. At least one of i and j is 1 and $1\leqslant|i-j|\leqslant 3$. This implies that the possible values of $r(x|S)$ are $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, $(3,1)$, $(4,1)$ and that the order of G is at most 8. One such graph of order 8 is shown in Figure 2(c). □

We now consider the other extreme by determining all connected graphs of order $n\geqslant 2$ with $\gamma_r(G)=n-1$. By Proposition 1.3, $\gamma_r(K_n)=\gamma_r(K_{1,n-1})=n-1$. We show, in fact, that these are the only graphs of order n having resolving domination number $n-1$.

Theorem 2.5. *A connected graph G of order $n\geqslant 2$ has resolving domination number $n-1$ if and only if $G=K_n$ or $G=K_{1,n-1}$.*

P r o o f. We have already noted that $\gamma_r(K_n)=\gamma_r(K_{1,n-1})=n-1$, so it remains only to verify the converse. Let G be a connected graph of order $n\geqslant 2$ with $\gamma_r(G)=n-1$. Consequently, no $(n-2)$-subset of $V(G)$ is a resolving dominating set of G. We consider two cases.

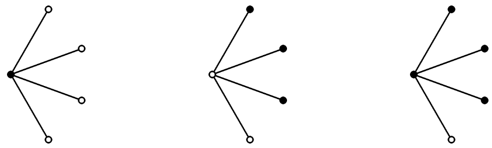
C a s e 1. There exists an $(n-2)$-element subset $S\subseteq V(G)$ that is not a dominating set for G. Let $V(G)-S=\{u,v\}$. Since G is connected, exactly one of u and v is not dominated by S, say u. Then u is adjacent to no vertex in S. Since G is connected, u is an end-vertex adjacent only to v. We claim that G is the star $K_{1,n-1}$ centered at v. First we show that v is adjacent to every vertex in S and so $\deg v=n-1$. Assume, to the contrary, that there exists a vertex $w\in S$ such that v is not adjacent to w. Let $S'=(S-\{w\})\cup\{v\}=V(G)-\{u,w\}$. Since G is connected and w is not adjacent to u, it follows that w is adjacent to some vertex in S. Thus, S' is a dominating set of G. On the other hand, $d(u,v)=1$ and $d(w,v)\geqslant 2$, implying that that S' is resolving dominating set of G. Thus $\gamma_r(G)\leqslant|S'|=n-2$, a contradiction. Hence $\deg v=n-1$. Next we show that S is independent. Suppose that x and y are adjacent vertices of S. The set $S''=(S-\{x\})\cup\{v\}=V(G)-\{u,x\}$ is a dominating set of G. Since $d(x,y)=1$ and $d(u,y)=2$, it follows that S'' is also a resolving set of G and so $\gamma_r(G)\leqslant|S''|=n-2$, again a contradiction.

C a s e 2. Every $(n-2)$-element subset of $V(G)$ is a dominating set of G. Consequently, no $(n-2)$-element subset of $V(G)$ is a resolving set of G. This implies that $\dim(G)=n-1$. It then follows by Theorem A (a) that $G=K_n$, as desired. □

3. Comparing the resolving domination number with dimension and the domination number

We saw in Proposition 1.2 that $\gamma_r(G) \leqslant \gamma(G)+\dim(G)$ for every graph G. To show that equality can hold, consider the graph $G = K_{1,4}$ shown in Figure 3. Certainly, the central vertex of G dominates all vertices of G and so $\gamma(G) = 1$. The set consisting of any three of the four end-vertices of G is a basis for G, so $\dim(G) = 3$. On the other hand, the union of any basis and the set consisting of the central vertex produces a γ_r-set. Thus $\gamma_r(G) = 4$. Note that, in this case, $\gamma_r(G) = \gamma(G) + \dim(G)$. In general, if $\gamma_r(G) = \gamma(G) + \dim(G)$, then every γ-set of G and every basis of G are disjoint. The converse is not true, however. For example, consider the path $P_5\colon\ v_1, v_2, \ldots, v_5$ of order 5. It is known that $\dim(P_5) = 1$ and $\gamma(P_5) = 2$. Since $\{v_1\}$ and $\{v_5\}$ are the only bases for P_5 and $\{v_2, v_4\}$ is the only γ-set for P_5, every γ-set of P_5 and every basis of P_5 are disjoint. On the other hand, $\{v_2, v_4\}$ is also a γ_r-set for P_5 and so $\gamma_r(P_5) = 2$. Therefore, $\gamma_r(P_5) \neq \dim(P_5) + \gamma(P_5)$.

$G = K_{1,4}$:



$\gamma(G) = 1$ $\quad$ $\dim(G) = 3$ $\quad$ $\gamma_r(G) = 4$

Figure 3. A graph G with $\gamma_r(G) = \gamma(G) + \dim(G)$

By Theorem A(a), the path P_n of order $n \geqslant 2$ is the only connected graph of order n having dimension 1. Let $P_n\colon\ v_1, v_2, \ldots, v_n$, where $n \geqslant 2$. If $n = 2$, then each vertex of P_2 forms a basis as well as a γ-set of P_2, implying that $\dim(P_2) = \gamma(P_2) = \gamma_r(P_2) = 1$. If $n = 3$, then each end-vertex of P_3 forms a basis and $\{v_2\}$ is the only γ-set for P_3, so $\dim(P_3) = \gamma(P_3) = 1$. Since $\{v_1, v_2\}$ is a γ_r-set for P_3, it follows that $\gamma_r(P_3) = 2$. If $n \geqslant 4$, then every γ-set of P_n is also a γ_r-set and so

$$\gamma_r(P_n) = \gamma(P_n) = \lceil n/3 \rceil.$$

This implies that if G is a connected graph with dimension 1, then $\gamma_r(G) = \gamma(G) + \dim(G)$ if and only if $G = P_3$. On the other hand, for connected graphs with dimension at least 2, we have the following.

Proposition 3.1. *For every triple a, b, c of integers with $a \geqslant 1$, $b \geqslant 2$, and $c = a + b$, there exists a connected graph G with $\gamma(G) = a$, $\dim(G) = b$, and $\gamma_r(G) = c$.*

P r o o f. For each integer i with $1 \leqslant i \leqslant b$, let F_i be a copy of the path $P_2\colon x_i, y_i$. Let G be the graph obtained from the path $P_{3a-1}\colon v_1, v_2, \ldots, v_{3a-1}$ of order $3a-1$ and the graphs F_i $(1 \leqslant i \leqslant b)$ by joining each vertex of F_i to the vertex v_{3a-1} in P_{3a-1}. For $a = b = 2$, the graph G is shown in Figure 4.

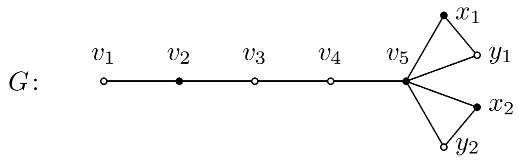


Figure 4. A graph G with $\gamma(G) = 2$, $\dim(G) = 2$, and $\gamma_r(G) = 4$

Let $V = \{v_1, v_2, \ldots, v_{3a-1}\}$, $X = \{x_1, x_2, \ldots, x_b\}$, and $Y = \{y_1, y_2, \ldots, y_b\}$. Since $\{v_2, v_5, \ldots, v_{3a-1}\}$ is a γ-set and X is a basis for G, it follows that $\gamma(G) = a$ and $\dim(G) = b$. Next we show that $\gamma_r(G) = a + b$. By Lemma 1.1, every resolving set of G contains at least one vertex from each set $\{x_i, y_i\}$ for all i with $1 \leqslant i \leqslant b$. These vertices dominate v_{3a-1} and all vertices x_i and y_i for $1 \leqslant i \leqslant b$. To dominate $V - \{v_{3a-1}\}$, at least $\lceil (3a-2)/3 \rceil = a$ vertices of $V - \{v_{3a-1}\}$ are needed. Consequently, $\gamma_r(G) \geqslant a + b$. On the other hand, $S = \{v_2, v_5, v_8, \ldots, v_{3a-1}\} \cup X$ is a resolving dominating set for G (see Figure 4), implying that $\gamma_r(G) \leqslant |S| = a + b$. Therefore, $\gamma_r(G) = a + b$. □

We noted in Proposition 1.2 that $\max\{\gamma(G), \dim(G)\} \leqslant \gamma_r(G)$ for every graph G. The following corollary is an immediate consequence of Proposition 3.1, which shows that it is possible that no two of the numbers $\gamma(G)$, $\dim(G)$, and $\gamma_r(G)$ need be close in value.

Corollary 3.2. *For every positive integer k,*
(a) *there exists a connected graph F_k such that*

$$\gamma_r(F_k) - \dim(F_k) \geqslant k \text{ and } \dim(F_k) - \gamma(F_k) \geqslant k, \text{ and}$$

(b) *there exists a connected graph H_k such that*

$$\gamma_r(H_k) - \gamma(H_k) \geqslant k \text{ and } \gamma(H_k) - \dim(H_k) \geqslant k.$$

At the other extreme, it is possible for all three of these parameters to have the same prescribed value.

Theorem 3.3. *For every positive integer k, there exists a connected graph G_k such that $\gamma(G_k) = \dim(G_k) = \gamma_r(G_k) = k$.*

P r o o f. The result is true for $k=1$ and $k=2$ since $G_1=K_2$ and $G_2=C_4$ have the desired properties. Hence we may assume that $k \geqslant 3$. Here we define the vertex set of G_k to consist of three pairwise disjoint sets U, V, and W, where $U=\{u_1,u_2,\ldots,u_k\}$, $V=\{v_1,v_2,\ldots,v_k\}$, and $W=\{w_T;\ T\subseteq\{1,2,\ldots,k\},|T|\geqslant 2\}$. Next we describe the edge set of G_k. For each i ($1\leqslant i\leqslant k$), $u_iv_i\in E(G_k)$. Furthermore, $\langle V\rangle$ and $\langle W\rangle$ are complete. Lastly, $v_iw_T\in E(G_k)$ if and only if $i\in T$. The graph G_3 is shown in Figure 5.

G_3:

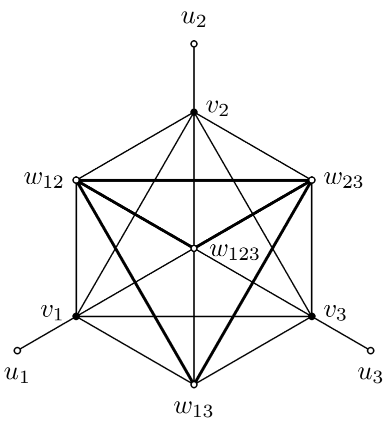


Figure 5. The graph G_3

Clearly, V is the unique γ-set. Also, V is a resolving set since u_i is the only vertex having 1 in the ith coordinate of its representation and 2 elsewhere, while w_T the only vertex having 1 in the ith coordinate for each $i\in T$ and 2 elsewhere. We show in fact that V is a basis, which will complete the proof. To verify this, we show that no $(k-1)$-subset of $V(G_k)$ is a resolving set. Let $S\subseteq V(G_k)$ with $|S|=k-1$.

Certainly, $S\not\subseteq W$, for otherwise any two of the $2^k-2k\geqslant 2$ vertices of $W-S$ have the same representation with respect to S, namely, the k-vector $(1,1,\ldots,1)$. Thus $S\cap(U\cup V)\neq\emptyset$. Let $|S\cap U|=\ell_1$, $|S\cap V|=\ell_2$, and $|S\cap W|=\ell_3$, where then $\ell_1+\ell_2+\ell_3=k-1$ and $\ell_3\leqslant k-2$.

First, we consider the case where $\ell_3=0$. Then $\ell_1+\ell_2=k-1$. We may assume, without loss of generality, that neither u_k nor v_k belongs to S. Let $T'=\{1,2,\ldots,k-1\}$ and $T''=\{1,2,\ldots,k\}$. Then $w_{T'}$ and $w_{T''}$ have the same representation with respect to S and S is not a resolving set. This leaves the case $1\leqslant\ell_3\leqslant k-2$. Recall that $\ell_1\neq 0$ or $\ell_2\neq 0$. There are two cases, according to whether $\ell_2=0$ or $\ell_2\neq 0$.

C a s e 1. $\ell_2=0$. Then $\ell_1\neq 0$ and $\ell_1+\ell_3=k-1$. Assume, without loss of generality, that $\{u_1,u_2,\ldots,u_{\ell_1}\}\subseteq S$. Then $u_t\notin S$ for all t with $\ell_1+1\leqslant t\leqslant k$. For each integer i with $1\leqslant i\leqslant\ell_3+1$, define $T_i=\{1,2,\ldots,\ell_1\}\cup\{\ell_1+i\}$ and

$T_{\ell_3+2} = \bigcup_{i=1}^{\ell_3+1} T_i$. Then at least two vertices in $\{w_{T_1};\ 1 \leqslant 1 \leqslant \ell_3+2\}$ do not belong to S, say w_{T_r} and w_{T_s}. The distance from each of these vertices to any vertex in $S \cap W$ is 1; while their distance to any vertex in $S \cap U$ is 2. Thus, w_{T_r} and w_{T_s} have the same representation with respect to S and so S is not a resolving set.

Case 2. $\ell_2 \neq 0$. Assume, without loss of generality, that $\{v_1, v_2, \ldots, v_{\ell_2}\} \subseteq S$. Then there exist t integers $i_1, i_2, \ldots, i_t$ such that $\ell_2 + 1 \leqslant i_j \leqslant k$ for all j $(1 \leqslant j \leqslant t)$ and $u_{i_j} \notin S$. Since the set $U' = \{u_{\ell_2+1}, u_{\ell_2+2}, \ldots, u_k\}$ contains $k - \ell_2 = \ell_1 + \ell_3 + 1$ vertices and at most ℓ_1 vertices of U' belong to S, it follows that at least $\ell_3 + 1$ vertices of U' do not belong to S, that is, $t \geqslant \ell_3 + 1$. For $1 \leqslant j \leqslant t$, define $T_j = \{1, 2, \ldots, \ell_2\} \cup \{i_j\}$ and $T_{t+1} = \bigcup_{j=1}^{t} T_j$. Since $t + 1 \geqslant \ell_3 + 2$, it follows that at least two vertices in $\{w_{T_j};\ 1 \leqslant j \leqslant t+1\}$ do not belong to S, say w_{T_r} and w_{T_s}. The distance from each of these vertices to any vertex in $S \cap V$ or $S \cap W$ is 1; while their distance to a vertex u_a in $S \cap U$ is 2 if $1 \leqslant a \leqslant \ell_2$ and 3 otherwise. Thus, w_{T_r} and w_{T_s} have the same representation with respect to S and so S is not a resolving set.

Thus, as claimed, V is a basis. □

4. Comparing the resolving domination number and dimension with other parameters

A set S of vertices in a graph G is a *k-dominating set* if every vertex of G is within distance k of at least one vertex in S. The cardinality of a minimum k-dominating set is the *k-domination number* $\gamma_k(G)$ of G. A k-dominating set of cardinality $\gamma_k(G)$ is a *γ_k-set* of G (see [4, Chap. 7]). It is known for a connected graph G of order $n \geqslant k+1$ and diameter $d \geqslant k$ that there is a γ_k-set S of G such that for each vertex $v \in S$ there is a private k-neighbor of v at distance exactly k from v (see [4, pp. 203]).

Proposition 4.1. *Let k be a positive integer and G a connected graph of order $n \geqslant k+1$ and diameter $d \geqslant k$. Then*

(a) $\dim(G) \leqslant n - k\gamma_k(G)$ *and*

(b) $\gamma_r(G) \leqslant n - [(2k+1)/3]\,\gamma_k(G)$.

Proof. We first verify (a). Let $\gamma_k(G) = p$ and let $S = \{v_1, v_2, \ldots, v_p\}$ be a γ_k-set of G. For each i with $1 \leqslant i \leqslant p$, let x_{ik} be a guaranteed private k-neighbor of v_i and let a $v_i - x_{ik}$-geodesic be $P_i\colon\ v_i, x_{i1}, x_{i2}, \ldots, x_{ik}$. Then $d(v_i, x_{ij}) = j$ for $1 \leqslant j \leqslant k$ and $d(v_t, x_{ij}) > j$ for $t \neq j$. Thus S is a resolving set for the $k\gamma_k(G)$ vertices x_{ij} on the paths P_i. This implies that the set $W = V(G) - \Big[\bigcup_{i=1}^{k} (V(P_i) - \{v_i\})\Big]$ is a resolving set in G. Thus, $\dim(G) \leqslant |W| = n - k\gamma_k(G)$ and so (a) holds.

To verify (b), let $P_i' = P_i - v_i - x_{i1}\colon\ x_{i2}, x_{i3}, \ldots, x_{ik}$ and let S_i be a γ-set of P_i' for all i with $1 \leqslant i \leqslant p$. Then $|S_i| \leqslant \lceil (k-1)/3 \rceil$ for all i. Since $S' = W \cup S_1 \cup S_2 \cup \ldots \cup S_p$ is a resolving dominating set of G, it follows that $\gamma_r(G) \leqslant |S'| \leqslant n - [(2k+1)/3]\gamma_k(G)$, as desired. □

The *diameter* of G is the maximum distance between any two vertices of G. It was shown in [1], [3] that if G is a connected graph of order n, diameter d, and maximum degree Δ, then

$$\lceil \log_3(\Delta + 1) \rceil \leqslant \dim(G) \leqslant n - d$$

and both upper and lower bounds are sharp. We now present new upper and lower bounds for the dimension of a graph in terms of other parameters. Let $\alpha_0(G)$ denote the vertex cover number of G (the minimum number of vertices that cover all edges of G). Next we show that $\dim(G) \leqslant \alpha_0(G)$ for certain graphs G with some additional property.

Proposition 4.2. *Let G be a connected graph with the property that $N(u) \neq N(v)$ for every pair u, v of distinct vertices of G. Then*

$$\dim(G) \leqslant \alpha_0(G).$$

P r o o f. Let X be a minimum vertex cover of G. Then $V(G) - X$ is an independent set of vertices, that is, no two are adjacent. For every pair u, v of distinct vertices in $V(G) - X$, since $N(u) \neq N(v)$, one vertex in the pair u, v has a neighbor in X that is not a neighbor of the second vertex in the pair. Thus, X is a resolving set and $\dim(G) \leqslant |X| = \alpha_0(G)$. □

The *clique number* of a graph is the maximum order among the complete subgraphs of the graph. We now present a lower bound for the dimension of a graph in terms of its clique number.

Proposition 4.3. *If a graph G has clique number ω, then*

$$\dim(G) \geqslant \lceil \log_2 \omega \rceil.$$

P r o o f. Suppose that a minimum resolving set of G contains no vertices of a clique of order ω. Then, for any vertex v in the resolving set, at least half of those vertices in the clique have the same distance to v. At least half of these have the same distances to two vertices in the resolving set. In general, for any set of k vertices of the resolving set, at least $\omega/2^k$ vertices of the clique have the same set of distances to the k specified vertices in the resolving set. Thus, if $k = \dim(G)$, then $\omega/2^{\dim(G)} \leqslant 1$ and so the result follows. □

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