ALGEBRAIC APPROACH TO LOCALLY FINITE TREES WITH ONE END

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To the memory of Bedřich Pondělíček

Abstract. Let T be an infinite locally finite tree. We say that T has exactly one end, if in T any two one-way infinite paths have a common rest (infinite subpath). The paper describes the structure of such trees and tries to formalize it by algebraic means, namely by means of acyclic monounary algebras or tree semilattices. In these algebraic structures the homomorpisms and direct products are considered and investigated with the aim of showing, whether they give algebras with the required properties. At the end some further assertions on the structure of such trees are stated, without the algebraic formalization.

Keywords: locally finite tree, one-way infinite path, acyclic monounary algebra, tree semilattice

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In this paper we study infinite locally finite trees with one end (shortly $1E$-trees) and their formalization by algebraic structures. We consider infinite undirected trees and directed trees obtained from them.

The concept of an end of an infinite locally finite graph was introduced by R. Halin [1]. Let G be an infinite locally finite graph, let $\mathcal{W}(G)$ be the family of all one-way infinite paths in G. As G is infinite and locally finite, $\mathcal{W}(G) \neq \emptyset$. Let $\mathcal{E}$ be a binary relation on $\mathcal{W}(G)$ defined so that $(W_1, W_2) \in \mathcal{E}$ for $W_1 \in \mathcal{W}(G)$, $W_2 \in \mathcal{W}(G)$ if and only if there exists a one-way infinite path $W_0 \in \mathcal{W}(G)$ such that the numbers of elements of the intersections $W_0 \cap W_1$, $W_0 \cap W_2$ are both infinite. The relation $\mathcal{E}$ is an equivalence relation on the family $\mathcal{W}(G)$; its equivalence classes are called ends of G.

This was the definition of an end of a graph in general. The following theorem shows that the definition may be formulated more simply in the case when G is a tree, i.e. a connected graph without circuits.

Theorem 1. *Let T be an infinite locally finite tree, let $W_1 \in \mathcal{W}(G)$, $W_2 \in \mathcal{W}(T)$. The paths W_1, W_2 belong to the same end of T if and only if the intersection $W_1 \cap W_2$ is a rest of both W_1 and W_2.*

Here we have used the word rest. The rest of a one-way infinite path is its subgraph which is itself a one-way infinite path.

P r o o f. Suppose that $W_1 \cap W_2$ is a rest of both W_1 and W_2. Then we may put $W_0 = W_1 \cap W_2$. We have $W_0 \in \mathcal{W}(T)$, $W_0 \cap W_1 = W_0 \cap W_1 = W_0$ and the number of vertices of W_0 is infinite. Therefore $(W_1, W_2) \in \mathcal{E}$.

Now suppose that W_1, W_2 belong to the same end of T. Let W_0 be a one-way infinite path in T such that $W_0 \cap W_1$ has an infinite number of vertices. Let a be a vertex of $W_0 \cap W_1$, let b_0 (or b_1) be the vertex immediately following a in W_0 (or in W_1 respectively). Let W_0' (or W_1') be the rest of W_0 (or of W_1 respectively) with the initial vertex a. Suppose that $b_0 \neq b_1$. As $W_0 \cap W_1$ has an infinite number of vertices, the same holds for the rests W_0', W_1', i.e. also $W_0' \cap W_1'$ has an infinite number of vertices. Let c be a vertex of $W_0' \cap W_1'$ having the minimum distance from a. Let W_0'' (or W_1'') be the finite path connecting a and c and being a subpath of W_0' (or W_1' respectively). Then the union $W_0'' \cup W_1''$ is a circuit, which is a contradiction with the assumption that T is a tree. Hence $b_0 = b_1$ and also $c = b_0 = b_1$. Therefore, if the intersection $W_0' \cap W_1'$ contains a vertex a, then it contains its successors in both W_0' and W_1' and hence it contains a common rest R_2 of W_0' and W_1'. Analogously $W_0 \cap W_2$ contains a common rest R_2 of W_0' and W_2'. The intersection $R_1 \cap R_2$ is a common rest of W_1 and W_2. □

We can formulate a corollary.

Corollary 1. *Let T be an infinite locally finite tree with exactly one end $\mathcal{E}$. Let W_0 be an arbitrary one-way infinite path in T. Then every one-way infinite path in T has a common rest with W_0.*

An infinite locally finite tree with exactly one end will be called a one-end tree, shortly $1E$-tree. Let us describe intuitive how such a tree looks out. We may choose one one-way infinite path W_0 in T. Then we choose an arbitrary vertex a of T. Let b be the vertex of W_0 whose distance from a is minimum. Let $W(a)$ be the one-way infinite path which is the union of the rest of W_0 with the initial vertex b of the finite path $P(a, b)$ in T connecting a and b. As T is a tree, it is evident that there is a one-to-one correspondence between the vertices a of T and the one-way infinite paths $W(a)$ in T. Each vertex a of T is the initial vertex of exactly one one-way infinite path $W(a)$ in T and each one-way infinite path in T is $W(a)$ for exactly one vertex a of T. From the construction it is clear that $W(a)$ does not depend on the choice of W_0, because the rest of W_0 with the initial vertex a is determined uniquely.

The following assertion is evident.

Theorem 2. *A one-end tree cannot contain a two-way infinite path.*

2. Algebraic formalization

We shall try to formalize algebraically the structure of a $1E$-tree. We introduce an orientation of T which will be called canonical.

We have yet defined W_0 and $W(a)$ for each $a \in V(T)$. Orient edges of W_0 in such a way that W_0 becomes a directed one-way infinite path with the orientation towards the infinity. Further, orient edges of $P(a,b)$ in such a way that $P(a,b)$ becomes a directed finite path from a to b. The orientation thus obtained is the orientation of $W(a)$. In any case, each edge becomes oriented from its end vertex farther from W_0 to its end vertex nearer to W_0. If we orient all one-way infinite paths in T in this way, we obtain the orientation of the whole tree T. We will call it the canonical orientation of T.

As with finite trees, we speak about branches of T. Let $u \in v(T)$. If the degree of u is 1, then there exists exactly one branch of T and u, namely T itself. If the degree of u is at least 2, then on $V(T) - \{u\}$ we may introduce a binary relation $\pi(u)$ in such a way that $(a,b) \in \pi(u)$ if and only if the finite path in T connecting a and b does not contain u. This relation is an equivalence relation on $V(T) - \{u\}$. A branch of T at u is then a subtree of T induced by $C \cup \{u\}$, where C is an equivalence class of $\pi(u)$.

Theorem 3. *Let T be a $1E$-tree. Then T contains at least one vertex of degree 1 (pendant vertex).*

P r o o f. Let $u \in V(T)$. Then exactly one branch of T at u contains one-way infinite paths; all others are finite. If no finite branches of T at u exist, then u has degree 1 and the assertion holds. Otherwise let B a finite branch of T at u. We walk along edges of B, starting at u. As T is a tree, this walk is a simple path. As B is finite, this path must end at some vertex from which it cannot continue. And this is a vertex of degree 1. □

Corollary 2. *Let T be a $1E$-tree. In the canonical orientation of T there is no sink, but there is at least one source.*

Obviously there is exactly one course if and only if T consists of exactly one one-way infinite path.

The following proposition is evident.

Proposition 1. *Let T be a $1E$-tree and consider T as directed by its canonical orientation. Then the outdegree of each vertex of T is* 1.

It follows that the canonical orientation of T induces a partial ordering on $V(T)$. If $u \in V(T)$, $v \in V(T)$, then we have $u \leqslant v$ if and only if there exists a (finite) path $P(u, v)$ in T connecting u and v and directed from u to v in the canonical orientation. This enables two algebraic formalizations of T. The first of them is the formalization by means of a connected monounary algebra.

A total monounary algebra is a directed pair (A, f), where A is a set and f is a mapping of A into itself.

The set A is the support of (A, f) and f is the unary operation on (A, f). For f and a positive integer n we may define the n-th iteration f^n of f. We define it recurrently by putting $f^1(x) = f(x)$, $f^n(x) = f(f^{n-1}(x))$ for $n \geqslant 2$.

We will consider only total monounary algebras, although there exist also partial ones in which f need not be defined for all elements of A.

A monounary algebra (A, f) is connected, if for any two elements x, y of A there exist positive integers p, q such that $f^p(x) = f^q(y)$. It is called acyclic, if for each $x \in A$ and positive integers p, q the inequality $p \neq q$ implies $f^p(x) \neq f^q(x)$.

Now let τ be a reflexive and symmetric binary relation on A. If $(x, y) \in \tau \Rightarrow (f(x), f(y)) \in \tau$ for any two elements x, y of A, then the relation τ is called a tolerance on (A, f). If moreover τ is transitive, then τ is called a congruence on (A, f). Now let $\mathcal{A} = (A, f)$, $\mathcal{B} = (B, g)$ be two monounary algebras. Their direct product $\mathcal{A} \times \mathcal{B}$ is the algebra $\mathcal{C} = (C, h)$ such that $C = A \times B$ and $h((a, b)) = (f(a), g(b))$ for $a \in A$, $b \in B$.

From this definition the following proposition is evident.

Proposition 2. *Let $\mathcal{A} = (A, f)$, $\mathcal{B} = (B, g)$ be two connected acyclic monounary algebras. Then the direct product $\mathcal{A} \times \mathcal{B}$ is also a connected acyclic monounary algebra.*

Now we prove another theorem.

Theorem 4. *Let T be a $1E$-tree. Then there exists a connected acyclic monounary algebra $\mathcal{A}(T) = (V(T), f)$ such that for each $u \in V(T)$ the element $f(u)$ is the terminal vertex of the arc outgoing from u in the canonical orientation on T.*

P r o o f. As the outdegree of each vertex of T in its canonical orientation is 1, the vertex $f(u)$ is uniquely determined for each $u \in V(T)$ and thus $(V(T), f)$ is a total monounary algebra. As T is a tree, it is acyclic. As every finite directed path goes towards some vertex b of W_0, the algebra $\mathcal{A}(T)$ is connected. $\square$

Also the inverse assertion holds.

Theorem 5. *Let $\mathcal{A} = (A, f)$ be an acyclic connected monounary algebra. Then there exists an infinite locally finite tree T_α with exactly one end and such that $\mathcal{A} = \mathcal{A}(T_a)$.*

P r o o f. Obviously T_a is obtained from $\mathcal{A}$ in such a way that its vertex set $V(T_a)$ is A and an arc (in the canonical orientation) from u to v exists if and only if $f(u) = v$. For each vertex u the vertices $u, f(u), f^2(u), \ldots$ must form a one-way infinite path because $\mathcal{A}$ is acyclic. And, as $\mathcal{A}$ is connected, any two of these paths $W(u)$ have a common rest. The union of all such paths $W(u)$ is the tree T_a. $\square$

Therefore, by constructing direct products of monounary algebras we may obtain new trees with exactly one end from the given ones. Now we would like to know whether this is possible also by taking homomorphic images of monounary algebras determined by congruences on them.

Proposition 3. *There exists an acyclic connected monounary algebra $\mathcal{A} = (A, f)$ and a congruence τ on $\mathcal{A}$ such that the factor algebra $\mathcal{A}/\tau$ is not acyclic.*

P r o o f. Let τ be a binary relation on A which is the least equivalence relation containing all paris $(x, f^3(x))$ for $x \in A$. If $a \in A$, then the equivalence class $[a]_\tau$ of ϱ containing a is the set consisting of a and of all elements $f^p(a)$, where $p \equiv 0 \pmod 3$. Similarly $[f(a)]_\varepsilon$ (or $[f^2(a)]_\tau$) consists of all elements $f^p(a)$, where $p \equiv 1 \pmod 3$ (or $p \equiv 2 \pmod 3$ respectively). Now consider the factor-algebra $\mathcal{A}/\tau$. Its elements are classes $[x]_\tau$ for all $x \in A$ and the operation is $\hat{f}$ such that $\hat{f}([x]_\tau) = [f(x)]_\tau$. Evidently $\hat{f}([a]_\tau) = [f(a)]_\tau$, $\hat{f}^2([a]_\tau) = [f^2(a)]_\tau$, $\hat{f}^3([a]_\tau) = [a]_\tau$ and the algebra $\mathcal{A}/\tau$ is not acyclic. $\square$

We have seen that constructing factor-algebras of $\mathcal{A}(T)$ need not lead to finding new $1E$-trees.

We have already mentioned that the canonical orientation of T induces a partial ordering on $V(T)$. As the monounary algebra $\mathcal{A}(T)$ is connected, this ordering is the ordering of a semilattice. We have a semilattice $S(T) = (V(T), \vee)$; we denote it as a join semilattice. The support of $S(T)$ is $V(T)$. For $u \in V(T)$, $v \in V(T)$ the element $u \vee v$ is the initial vertex of the common rest of $W(u)$ and $W(v)$. Moreover, $S(T)$ is a tree semilattice, i.e. a semilattice in which every interval is a chain. (We can express this condition in the following way: $a \leqslant c \leqslant b$ & $a \leqslant d \leqslant b \Rightarrow c \leqslant d \wedge d \leqslant c$.)

A tolerance ϱ on a semillatice has the property that $(a, b) \in \varrho$ implies $(x, a \vee b) \in \varrho$ for each x such that $a \leqslant x \leqslant a \vee b$ or $b \leqslant x \leqslant a \vee b$. If moreover ϱ is a congruence, then $(x, y) \in \varrho$ for any elements x, y lying between a and $a \vee b$ or between b and $a \vee b$.

Let ϱ be a congruence on a tree semilattice $(S, \vee)$ and suppose that the factor-semilattice S/ϱ is not a tree semilattice. The elements of S/ϱ are equivalence classes of ϱ on S. If $u \in S$, $v \in S$, then $[u]_\varrho \leqslant [v]_\varrho$ if and only if $x \leqslant y$ for each $x \in [u]_\varrho$ and $y \in [v]_\varrho$. Suppose that there are elements a, b, c, d of S such that $[a]_\varrho \leqslant [c]_\varrho \leqslant [b]_\varrho$, $[a]_\varrho \leqslant [d]_\varrho \leqslant [b]_\varrho$ and $[c]_\varrho \parallel [d]_\varrho$. In S this implies $a \leqslant c \leqslant d$, $a \leqslant d \leqslant b$, $c \parallel d$ and this is a contradiction with the assumption that S is a tree semilattice. We have proved a proposition.

Proposition 4. *Any factor-semilattice of a tree semilattice by a congruence on it is again a tree semilattice.*

Therefore, constructing congruences leads to new $1E$-trees.

Propositin 5. *A direct product of two tree semilattices need not be a tree semilattice.*

Proof. As an example we use the simplest case. Let a semilattice S_1 consist of two elements a, b such that $a \leqslant b$ and let S_2 consist of two elements c, d such that $c \leqslant d$. The direct product $S_1 \times S_2$ has the elements (a, c), (a, d), (b, c), (b, d). We have $(a, c) \leqslant (a, d) \leqslant (b, d)$, $(a, c) \leqslant (b, c) \leqslant (b, d)$, $(a, d) \parallel (b, c)$ and $S_1 \times S_2$ is not a tree semilattice. □

By a similar argument it is possible to prove that the usual Cartesian product of trees as undirected graphs need not be a tree, but it may contain circuits.

Both the algebraic approaches discussed are related in a certain way to the algebraic approaches of L. Nebeský [2], [3] to finite trees. The tree algebra [2] is a ternary algebra whose support is the vertex set $V(T)$ of a finite tree T. For any three elements x, y, z of $V(T)$ the ternary operation of the algebra determines the (unique) vertex which lies simultaneously on the paths of T connecting the pair $\{x, y\}$, the pair $\{y, z\}$ and the pair $\{x, z\}$.

In our approach by means of a semilattice one of the vertices x, y, z is substituted by the one-way infinite path. The tree groupoid [3] is a groupoid whose support is again $V(T)$ for a finite tree T and in which the result of the binary operation performed with distinct elements x, y is the vertex which is adjacent to x and lies on the path connecting x and y in T. If y is substituted by a one-way infinite path w_0, we have our expression of T by means of a monounary algebra.

However, we must mention also some difference between these approaches. Algebras studied by L. Nebeský are determined by the assumption of the vertex set being finite and by a finite set of axioms in the language of the first order logic.

3. Structure of the $1E$-tree

Let again T be a $1E$-tree. Let $u \in V(T)$ and consider branches of T at u. Exactly one of them contains a one-way infinite path; this is the infinite branch $B_{\text{inf}}(u)$. Any other branch of T and u has all degrees of vertices finite and contains no infinite path, hence it is finite. Therefore u is an articulation of T which separates the infinite branch $B_{\text{inf}}(u)$ from the finite tree $T(u)$ which is the union of all other branches of T at u. Choose a vertex a from the vertices of $T(u)$ whose distance from u in $T(u)$ is maximum. By D denote the union of the finite path $P(a,u)$ connecting a and u in $T(u)$ and the path $W(u)$. The subtree D is the one-way infinite path $W(a)$; we call it a quasi-diametral path in T.

Note that D need not be determined uniquely, but in the case of a diametrical path of a finite tree the situation is similar.

Now we denote the vertices of D. We put $a = x_0$ and then we go along D from x_0 towards infinity, denoting the vertices by x_1, x_2, $x_3 \ldots$. Obviously we exhaust all non-negative integers. For every non-negative integer i let $B_T(i)$ be the union of all branches of T at x_i which do not contain edges of D. The graph $B_T(i)$ is a subtree of T; we take it as a rooted tree with the root x_i. If we depict this tree, it is advantageous to depict it as the directed one in the canonical orientation; then the root x_i is the unique sink in $B_T(i)$. Then the tree T is uniquely determined by the infinite sequence $B_T(0)$, $B_T(1)$, $B_T(2)$, $B_T(3), \ldots$.

Further, if $i < j$, then $T(i,j)$ denotes the subtree of T formed by the subpath $P(x_i, x_j)$ of D connecting x_i and x_j and by all branches of T at vertices of $P(x_i, x_j)$.

Remember that a meromorphism of a graph G is isomorphic mapping of G onto a subgraph of G. If that subgraph is a proper subgraph of G, then the meromorphism is also called proper.

Proposition 6. *Let there exist a positive integer k such that for any non-negative integer j there exists an isomorphic mapping g_j of $T(jk,(j+1)k)$ onto $T((j+1)k, (j+2)k)$ with $g_j(x_{jk}) = x_{(j+1)k}$. Then there exists a proper meromorphism h of T onto the tree obtained from T by deleting the set of vertices $V(T(0,jk)) - \{x_{jk}\}$.*

P r o o f. The mapping h is such that for $u \in V(T(jk,(j+1)k)$ we have $h(u) = g_j(u)$ for each non-negative integer j.

In such a case, T may be determined by its finite subtree $T(0,k)$. The described meromorphism h generates an infinite monogeneous semigroup.

At the end we say something about caterpillars. A finite caterpillar is usually defined as a tree which becomes a simple path after removing all its pendant edges. For an infinite caterpillar some authors have used a definition by means of the embeddings into the plane. We will use the following definition: A caterpillar with exactly one end is an infinity locally finite tree T which contains a one-way infinite path W with the property that each edge of T either belongs to W, or is pendant (incident with a vertex of degree 1).

If we know that T is a caterpillar, it suffices to determine any $B_T(i)$ by the number of pendant edges at x_i; therefore such a caterpillar is given by an infinite sequence of non-negative integers. $\square$

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