IRREDUCIBLE ALGEBRAIC SETS OF MATRICES WITH DOMINANT RESTRICTION OF THE CHARACTERISTIC MAP

Marcin Skrzyński, Kraków

(Received November 12, 2001)

Abstract. We collect certain useful lemmas concerning the characteristic map, $\mathcal{GL}_n$-invariant sets of matrices, and the relative codimension. We provide a characterization of rank varieties in terms of the characteristic map as well as some necessary and some sufficient conditions for linear subspaces to allow the dominant restriction of the characteristic map.

Keywords: characteristic map, dominant map, linear subspace, $\mathcal{GL}_n$-invariant set of matrices, rank variety

MSC 2000: 15A18, 14A10

Preliminaries and introduction

Throughout the present note, we work over an algebraically closed field $\mathbb{F}$ of characteristic zero. We write $\mathbb{F}^*$ instead of $\mathbb{F} \setminus \{0\}$. We denote by $\mathbb{N}$ the set of all non-negative integers.

Given integers m, $k \in \mathbb{N}$, we define $\mathcal{M}_{m\times k}$ to be the set of all $(m \times k)$-matrices whose entries are elements of $\mathbb{F}$. (Obviously, $\mathcal{M}_{m\times k} = \{0\}$ whenever $\min\{m, k\} = 0$.) We write $\mathcal{M}_m$ instead of $\mathcal{M}_{m\times m}$. By $O_{m\times k}$ we denote the zero matrix belonging to $\mathcal{M}_{m\times k}$. We put $O_m = O_{m\times m}$. We define $\mathcal{GL}_m$ to be the full linear group of size m over $\mathbb{F}$, i.e. $\mathcal{GL}_m = \{U \in \mathcal{M}_m;\ U \text{ is invertible}\}$.

We denote by $\mathbb{F}[\mathcal{M}_m]$ (with $m \geqslant 1$) the polynomial ring over $\mathbb{F}$ in m^2 variables $T_{11}, T_{12}, \ldots, T_{mm}$ which are the entries of the "generic matrix" $\top := [T_{jl}]_{j,l=1,\ldots,m}$. We will consider polynomials $\mathrm{s}_m^j \in \mathbb{F}[\mathcal{M}_m]$, $j = 1, \ldots, m$. The polynomial s_m^j is defined to be the sum of all principal minors of size j of the matrix $\top$. (In particular, $\mathrm{s}_m^1 = \mathrm{tr}$ and $\mathrm{s}_m^m = \det$.) Let us notice that $T^m + \sum_{j=1}^{m} (-1)^j \mathrm{s}_m^j(A) T^{m-j} \in \mathbb{F}[T]$ is the characteristic polynomial of a matrix $A \in \mathcal{M}_m$.

For a point $x = (x_1, \dots, x_m) \in \mathbb{F}^m$ $(m \geqslant 1)$ we define $\mathrm{Diag}\,(x) = \mathrm{Diag}\,(x_1, \dots, x_m) \in \mathcal{M}_m$ to be the diagonal matrix whose diagonal entries are equal to the coordinates of x.

The group $\mathcal{GL}_m$ acts on $\mathcal{M}_m$ by the conjugation. The orbit of a matrix $A \in \mathcal{M}_m$ under this action (i.e. the conjugacy class of A) will be denoted by $\mathcal{O}(A)$. A set $\mathcal{E} \subseteq \mathcal{M}_m$ is *$\mathcal{GL}_m$-invariant* if $\mathcal{E} \supseteq \bigcup\limits_{A \in \mathcal{E}} \mathcal{O}(A)$. The set $\mathcal{E}$ is a *cone* if $\mathcal{E} \neq \emptyset$ and $\mathcal{E} \supseteq \mathbb{F}\mathcal{E} := \{\lambda A;\ \lambda \in \mathbb{F},\ A \in \mathcal{E}\}$.

Throughout the text, we consider $\mathcal{M}_m$, $\mathbb{F}^m$ and their subsets as topological spaces equipped with the Zariski topology induced by the polynomial rings $\mathbb{F}[\mathcal{M}_m]$ and $\mathbb{F}[T_1, \dots, T_m]$. In particular, bars denote Zariski closures. We write “algebraic set” instead of “Zariski closed set”. A subset of $\mathcal{M}_m$ (and, analogously, of $\mathbb{F}^m$) is said to be *quasi-algebraic* if it is locally closed in the Zariski topology. A map $\Phi = (\varphi_j)_{j=1}^k \colon Q \longrightarrow \mathbb{F}^k$, where $k \geqslant 1$ and Q is an irreducible quasi-algebraic set (contained either in $\mathcal{M}_m$ or in $\mathbb{F}^m$), is *regular* if each coordinate function $\varphi_j \colon Q \longrightarrow \mathbb{F}$ is locally the quotient of two polynomials. The map Φ is *dominant* if it is regular and the range $\Phi(Q)$ is dense in $\mathbb{F}^k$.

We refer to [2], [4], [5], [6] for all the notions and facts needed of Matrix Theory, Algebra, and Algebraic Geometry.

A function $\varrho\colon \mathbb{N} \longrightarrow \mathbb{N}$ is a *rank function* if it is weakly decreasing and satisfies the convexity condition $\varrho(j) + \varrho(j+2) \geqslant 2\varrho(j+1)$ for all $j \in \mathbb{N}$. The set of all rank functions is partially ordered by the natural relation $\leqslant$. Namely,

$$\varrho_1 \leqslant \varrho_2 \Leftrightarrow \varrho_1(j) \leqslant \varrho_2(j) \text{ for all } j \in \mathbb{N},$$

where ϱ_1 and ϱ_2 are rank functions. The ordering $\leqslant$ is the only one we will consider on the sets of rank functions.

For a matrix $A \in \mathcal{M}_m$ and an integer $j \in \mathbb{N}$ we define $r_A(j) = \mathrm{rank}\,(A^j)$. (In particular, $r_A(0) = m$.) It is easy to check that $r_A\colon \mathbb{N} \longrightarrow \mathbb{N}$ is a rank function. It is remarkable though not difficult to prove that there is a greatest element in the set $\{r_A;\ A \in \mathcal{E}\}$ whenever $\mathcal{E} \subseteq \mathcal{M}_m$ is such that the Zariski closure $\overline{\mathcal{E}}$ is irreducible (cf. [7, Corollary 5.2]).

For an arbitrary rank function ϱ we define $\mathcal{X}_\varrho = \{A \in \mathcal{M}_{\varrho(0)};\ r_A \leqslant \varrho\}$. The set $\mathcal{X}_\varrho$ is called a *rank variety* (associated with the function ϱ). One can prove that the rank varieties are irreducible algebraic sets (cf., for instance, [8, Theorem 1.1]). The following formula for the dimension of $\mathcal{X}_\varrho$ will be useful in the sequel:

$$(\bullet) \qquad \dim\, \mathcal{X}_\varrho = [\varrho(0)]^2 - \sum_{j=1}^{\infty} [\varrho(j) - \varrho(j-1)]^2.$$

For further information about rank varieties and rank functions we refer to the fundamental papers [1], [10] as well as to [7], [8]. Finally, for an irreducible algebraic set $\mathcal{V} \subseteq \mathcal{M}_m$ we define its *relative codimension*, $\operatorname{r.codim} \mathcal{V}$, by $\operatorname{r.codim} \mathcal{V} = \dim \mathcal{X}_\mu - \dim \mathcal{V}$, where $\mu = \max\limits_{A \in \mathcal{V}} r_A$ (cf. [9]). Let us notice that $\mathcal{X}_\mu$ is the smallest (in the sense of inclusion) rank variety in which $\mathcal{V}$ is contained and that $\mathcal{V}$ is a rank variety if and only if $\operatorname{r.codim} \mathcal{V} = 0$.

From now on, the letter n stands for an integer not smaller than 2.

In the note we deal with the characteristic map in the following sense. Let $A \in \mathcal{M}_n$ and let q be an integer not smaller than 1. By the *characteristic map* we mean

$$\chi_A^q\colon\ \mathcal{M}_n \ni B \mapsto (\mathrm{s}_n^j(A+B))_{j=1}^q \in \mathbb{F}^q.$$

We will focus our attention on $\chi_{O_n}^q =: \chi^q$. In [3], it is shown (over the field of complex numbers) that if $\mathcal{L} \subseteq \mathcal{M}_n$ is a linear subspace such that $\dim \mathcal{L} \geqslant n$ and tr does not identically vanish on $\mathcal{L}$, then the restriction $\chi_A^n|_{\mathcal{L}}$ is a dominant map for a "generic" matrix $A \in \mathcal{M}_n$. We mostly deal with restrictions $\chi^q|_{\mathcal{V}}\colon\ \mathcal{V} \longrightarrow \mathbb{F}^q$, where $\mathcal{V} \subseteq \mathcal{M}_n$ is an irreducible algebraic set (not necessarily a linear subspace) such that $q = \max\limits_{B \in \mathcal{V}} r_B(n)$. (Let us notice that the polynomials s_n^j with $j > q$ identically vanish on each set $\mathcal{E} \subseteq \mathcal{M}_n$ such that $\max\limits_{B \in \mathcal{E}} r_B(n) = q$.) The main goal of the note is to characterize certain classes of sets $\mathcal{V}$ having the property that the restrictions $\chi^q|_{\mathcal{V}}$ are dominant maps. We will focus our attention on linear subspaces of $\mathcal{M}_n$ and on $\mathcal{GL}_n$-invariant irreducible algebraic cones.

We first collect a few lemmas that allow to relate the characteristic map to an elementary theory of the $\mathcal{GL}_n$-invariant irreducible algebraic cones (cf. [7], [8], [9]). We also offer a purely geometrical interpretation of the notion of relative codimension (Lemma 1.5). We next use the lemmas to prove a complete characterization of rank varieties by means of the characteristic map (Theorem 2.1) and to derive from that a characterization of linear subspaces $\mathcal{L} \subseteq \mathcal{M}_n$ with dominant restriction $\chi^q|_{\mathcal{L}}$ via their "$\mathcal{GL}_n$-invariant hulls" (Corollary 2.2). We also discuss conditions to impose upon $\dim \mathcal{L}$ in order to get that the restriction of the characteristic map is dominant.

1. Certain useful lemmas

In what follows, we write $\operatorname{st.rank} \mathcal{E}$ instead of $\max\limits_{A \in \mathcal{E}} r_A(n)$, where $\mathcal{E} \subseteq \mathcal{M}_n$ is a non-empty set.

Lemma 1.1. *Let $\mathcal{L} \subseteq \mathcal{M}_n$ be a linear subspace such that $q := \operatorname{st.rank} \mathcal{L} \geqslant 1$. Define $\widehat{\mathcal{L}} = \overline{\bigcup\limits_{A \in \mathcal{L}} \mathcal{O}(A)}$. Then the following are true:*

(i) $\widehat{\mathcal{L}}$ *is a $\mathcal{GL}_n$-invariant irreducible cone,*
(ii) $\max\{r_A;\ A \in \widehat{\mathcal{L}}\} = \max\{r_A;\ A \in \mathcal{L}\}$,
(iii) $\chi^q|_{\mathcal{L}}\colon\ \mathcal{L} \longrightarrow \mathbb{F}^q$ *is a dominant map if and only if so is* $\chi^q|_{\widehat{\mathcal{L}}}\colon\ \widehat{\mathcal{L}} \longrightarrow \mathbb{F}^q$.

Proof. Assertions (i) and (ii), and the "only if" part of (iii) are obvious. In order to prove the "if" part of (iii), observe that

$$\overline{\chi^q(\mathcal{L})} = \overline{\chi^q(\bigcup_{A\in\mathcal{L}} \mathcal{O}(A))} \supseteq \chi^q(\widehat{\mathcal{L}}),$$

where the inclusion is a consequence of the continuity of $\chi^q\colon\ \mathcal{M}_n \longrightarrow \mathbb{F}^q$. $\square$

For a positive integer $q \leqslant n$ we define $\Delta_n^q \in \mathbb{F}[\mathrm{s}_n^1, \ldots, \mathrm{s}_n^q] \subset \mathbb{F}[\mathcal{M}_n]$ to be the discriminant of the polynomial $T^q + \sum_{j=1}^{q}(-1)^j \mathrm{s}_n^j T^{q-j} \in \mathbb{F}[\mathcal{M}_n][T]$ (cf. [9]). The most important property of Δ_n^q is that a matrix $A \in \mathcal{M}_n$ with $r_A(n) = q$ has a multiple non-zero eigenvalue if and only if $\Delta_n^q(A) = 0$. Below we formulate an obvious but remarkable necessary condition for a restriction of the characteristic map χ^q to be dominant.

Lemma 1.2. *Let $\mathcal{Q} \subseteq \mathcal{M}_n$ be an irreducible quasi-algebraic set such that $q :=$ st.rank $\mathcal{Q} \geqslant 1$ and $\chi^q|_{\mathcal{Q}}\colon\ \mathcal{Q} \longrightarrow \mathbb{F}^q$ is a dominant map. Then there is a matrix $A \in \mathcal{Q}$ such that $r_A(n) = q$ and $\Delta_n^q(A) \neq 0$.*

The following Lemmas 1.3, 1.5 and 1.6 seem to be of some independent interest.

Lemma 1.3. *Let $\mathcal{V} \subseteq \mathcal{M}_n$ be a $\mathcal{GL}_n$-invariant irreducible algebraic set such that st.rank $\mathcal{V} =: q \geqslant 1$. Then the restriction $\chi_A^q|_{\mathcal{V}}$ is a dominant map for all $A \in \mathcal{M}_n$ whenever $\chi^q|_{\mathcal{V}}$ is a dominant map.*

Proof. Let $\mu = \max\limits_{B\in\mathcal{V}} r_B$ and let $C \in \mathcal{M}_{n-q}$ be a nilpotent matrix in the Jordan canonical form such that $r_C = \mu - q$. Define

$$Y = \{y = (y_j)_{j=1}^q \in (\mathbb{F}^*)^q;\ C \oplus \mathrm{Diag}\,(y) \in \mathcal{V},\ \text{the elements } y_j \text{ are pairwise distinct}\}.$$

The set $Y \subset \mathbb{F}^q$ is quasi-algebraic. Assume now that $\chi^q|_{\mathcal{V}}$ is a dominant map. Then $Y \neq \emptyset$ thanks to the irreducibility and to the $\mathcal{GL}_n$-invariancy of $\mathcal{V}$ (cf. also Lemma 1.2). Let

$$\Phi\colon\ Y \ni y \mapsto \chi^q(C \oplus \mathrm{Diag}\,(y)) \in \mathbb{F}^q.$$

The range $\Phi(Y)$ is dense in $\mathbb{F}^q$, because for each $B \in \mathcal{V}$ with $r_B = \mu$ and $\Delta_n^q(B) \neq 0$ there is a $y \in Y$ such that $\chi^q(B) = \Phi(y)$. Consequently, the Zariski interior of

$Y \subseteq \mathbb{F}^q$ is non-empty and Y is an irreducible set. Pick a matrix $A \in \mathcal{M}_n$. In virtue of the $\mathcal{GL}_n$-invariancy of $\mathcal{V}$, one can assume without loss of generality that A is upper triangular. Let $a = (a_j)_{j=1}^q \in \mathbb{F}^q$ be the point whose coordinates are equal to the q terminal diagonal entries of A. Define

$$\widetilde{\Phi}\colon\, Y \ni y \mapsto \chi^q(C \oplus \mathrm{Diag}\,(y + a)) \in \mathbb{F}^q.$$

Obviously, $\widetilde{\Phi} = (\widetilde{\varphi}_j|_Y)_{j=1}^q$, where $\widetilde{\varphi}_j \in \mathbb{F}[T_1, \ldots, T_q]$ and $\deg(\widetilde{\varphi}_j) = j$. Focus the attention on the "formal Jacobian" $\det\left(\mathrm{d}_{(T_1,\ldots,T_q)}(\widetilde{\varphi}_j)_{j=1}^q\right) \in \mathbb{F}[T_1, \ldots, T_q]$. It is not difficult to check that $\deg\left(\det\left(\mathrm{d}_{(T_1,\ldots,T_q)}(\widetilde{\varphi}_j)_{j=1}^q\right)\right) = \frac{1}{2}q(q-1)$. Consider finally the map

$$\Psi\colon\, Y \ni y \mapsto \chi^q(A + (C \oplus \mathrm{Diag}\,(y))) \in \mathbb{F}^q.$$

Evidently, $\Psi = ((\widetilde{\varphi}_j + \psi_j)|_Y)_{j=1}^q$, where $\psi_j \in \mathbb{F}[T_1, \ldots, T_q]$ and $\deg(\psi_j) < \deg(\widetilde{\varphi}_j)$. It is easy to see that the following equality holds for each element y of the Zariski interior of Y: $\det\left(\mathrm{d}_y\Psi\right) = \det\left(\mathrm{d}_y\widetilde{\Phi}\right) + \psi(y)$, where $\psi \in \mathbb{F}[T_1, \ldots, T_q]$ is such that $\deg(\psi) < \frac{1}{2}q(q-1)$. Consequently, $\det(\mathrm{d}_y\Psi) \neq 0$ for a "generic" element $y \in Y$, which means that Ψ is a dominant map. It turns out that so is $\chi_A^q|_{\mathcal{V}}$. The proof is complete. $\square$

Let $A \in \mathcal{M}_n$ be such that $q := r_A(n) \geqslant 2$ and $\mathrm{tr}(A) \neq 0$. For $j = 2, \ldots, q$ we define

$$\xi_j^q(A) = \begin{cases} \dfrac{[\mathrm{tr}(A)]^j}{\mathrm{s}_n^j(A)} & \text{if } \mathrm{s}_n^j(A) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we put $\Xi^q(A) = (\xi_j^q(A))_{j=2}^q \in \mathbb{F}^{q-1}$. The subsequent lemma is a minor modification of [9, Lemma 1] and can be proved in the same way.

Lemma 1.4. *Let $A, B \in \mathcal{M}_n$ be such that $q := r_A(n) = r_B(n) \geqslant 2$, $\mathrm{tr}(A)\mathrm{tr}(B) \neq 0$, and $\Delta_n^q(A)\Delta_n^q(B) \neq 0$. Then the following conditions are equivalent:*

(a) $\mathbb{F}^*\mathcal{O}(A) = \mathbb{F}^*\mathcal{O}(B)$,

(b) $r_A = r_B$ *and* $\Xi^q(A) = \Xi^q(B)$.

Let us remark that the characteristic map $\chi^q\colon\, \mathcal{M}_n \longrightarrow \mathbb{F}^q$ (with $q \geqslant 2$) is intimately related to the map

$$\Xi^q\colon\, \{A \in \mathcal{M}_n;\ r_A(n) = q,\, \mathrm{tr}(A) \neq 0\} \longrightarrow \mathbb{F}^{q-1}.$$

Lemma 1.5. *Let $\mathcal{V} \subseteq \mathcal{M}_n$ be an irreducible algebraic cone with $q :=$ st.rank $\mathcal{V} \geqslant 2$. Then the following conditions are equivalent:*

(a) *$\chi^q|_{\mathcal{V}}$ is a dominant map,*

(b) tr *does not identically vanish on* $\mathcal{V}$ *and* $\Xi^q|_{\mathcal{U}}\colon \mathcal{U} \longrightarrow \mathbb{F}^{q-1}$ *is a dominant map for a non-empty open subset* $\mathcal{U} \subseteq \{A \in \mathcal{V};\ r_A(n) = q,\ \operatorname{tr}(A) \neq 0\}$.

Proof. Assume that (a) is satisfied. Then none of the polynomials $\mathrm{s}_n^1 = \operatorname{tr}, \ldots, \mathrm{s}_n^q$ identically vanishes on $\mathcal{V}$. Define $\mathcal{U} = \{A \in \mathcal{V};\ \mathrm{s}_n^1(A) \neq 0, \ldots, \mathrm{s}_n^q(A) \neq 0\}$. The set $\mathcal{U}$ is open in $\mathcal{V}$ and non-empty (in virtue of the irreducibility of $\mathcal{V}$). Furthermore, $\mathcal{U} \subseteq \{A \in \mathcal{V};\ r_A(n) = q,\ \operatorname{tr}(A) \neq 0\}$. The restriction $\chi^q|_{\mathcal{U}}\colon \mathcal{U} \longrightarrow \mathbb{F}^q$ is dominant. So is the map

$$\Theta\colon\ (\mathbb{F}^*)^q \ni (x_1, \ldots, x_q) \mapsto \left(\frac{x_1^2}{x_2}, \ldots, \frac{x_1^q}{x_q}\right) \in \mathbb{F}^{q-1}.$$

Since $\Xi^q|_{\mathcal{U}} = \Theta \circ (\chi^q|_{\mathcal{U}})$, condition (b) follows.

Let condition (b) be satisfied. Then none of the polynomials $\mathrm{s}_n^1, \ldots, \mathrm{s}_n^q$ identically vanishes on $\mathcal{V}$. Consider the set $\widetilde{\mathcal{U}} := \mathbb{F}^*\mathcal{U}_0$, where $\mathcal{U}_0 = \{A \in \mathcal{U};\ \mathrm{s}_n^1(A) \neq 0, \ldots, \mathrm{s}_n^q(A) \neq 0\}$. Obviously, $\widetilde{\mathcal{U}} \subseteq \{A \in \mathcal{V};\ r_A(n) = q,\ \operatorname{tr}(A) \neq 0\}$. (Recall that $\mathcal{V}$ is a cone!) Furthermore, $\widetilde{\mathcal{U}} \neq \emptyset$, $\widetilde{\mathcal{U}} \cup \{O_n\}$ is a cone, $\widetilde{\mathcal{U}}$ is an open subset of $\mathcal{V}$, and $\Xi^q|_{\widetilde{\mathcal{U}}}\colon \widetilde{\mathcal{U}} \longrightarrow \mathbb{F}^{q-1}$ is a dominant map. Consequently, the set

$$U_\lambda := \{(\lambda^2 x_2^{-1}, \ldots, \lambda^q x_q^{-1});\ (x_2, \ldots, x_q) \in \Xi^q(\widetilde{\mathcal{U}})\}$$

is dense in $\mathbb{F}^{q-1}$ for all $\lambda \in \mathbb{F}^*$. Now, pick a $\lambda \in \mathbb{F}^*$ and observe that $\{\lambda\} \times U_\lambda \subseteq \chi^q(\widetilde{\mathcal{U}})$. Indeed, if $A \in \widetilde{\mathcal{U}}$, then $(\lambda, \lambda^2[\xi_2^q(A)]^{-1}, \ldots, \lambda^q[\xi_q^q(A)]^{-1}) = \chi^q(B)$ for $B = \dfrac{\lambda}{\operatorname{tr}(A)} A \in \widetilde{\mathcal{U}}$ (because $\widetilde{\mathcal{U}} \cup \{O_n\}$ is a cone). It turns out that

$$\bigcup_{\lambda \in \mathbb{F}^*} (\{\lambda\} \times U_\lambda) \subseteq \chi^q(\widetilde{\mathcal{U}}).$$

Since the set on the left hand side of the above inclusion is dense in $\mathbb{F}^q$, condition (a) follows. $\square$

The subsequent lemma provides a purely geometrical interpretation of the notion of relative codimension. (The idea of the interpretation arises from a part of the proof of [9, Theorem 3].)

Lemma 1.6. *Let* $\mathcal{V} \subseteq \mathcal{M}_n$ *be a* $\mathcal{GL}_n$*-invariant irreducible algebraic cone such that* $q := \operatorname{st.rank} \mathcal{V} \geqslant 2$ *and let neither of the polynomials* tr *and* Δ_n^q *vanish identically on* $\mathcal{V}$. *Then there is a non-empty open subset* $\mathcal{U} \subseteq \{A \in \mathcal{V};\ r_A(n) = q,\ \operatorname{tr}(A) \neq 0\}$ *such that the restriction* $\Xi^q|_{\mathcal{U}}$ *is a regular map and*

$$\operatorname{r.codim} \mathcal{V} = \operatorname{codim}_{\mathbb{F}^{q-1}} \overline{\Xi^q(\mathcal{U})}.$$

P r o o f. Define $Z = \{j \in \{1, \ldots, q\};\ \mathrm{s}_n^j$ does not identically vanish on $\mathcal{V}\}$. Then $\{1, q\} \subseteq Z$. Define also $\mu = \max_{A \in \mathcal{V}} r_A$. The set

$$\mathcal{U} := \{A \in \mathcal{V};\ r_A = \mu,\ \Delta_n^q(A) \neq 0,\ \mathrm{s}_n^j(A) \neq 0 \text{ for all } j \in Z\}$$

is non-empty and open in $\mathcal{V}$. Moreover, it is $\mathcal{GL}_n$-invariant and $\mathcal{U} \cup \{O_n\}$ is a cone. Consider the restriction $\Xi^q|_{\mathcal{U}}\colon \mathcal{U} \longrightarrow \mathbb{F}^{q-1}$. It is a regular map. By the formulae for the dimension of a rank variety and for the dimension of fibres of a dominant map, the following equalities hold:

$$\begin{aligned}
\mathrm{r.codim}\,\mathcal{V} &= n^2 - \sum_{j=1}^{\infty} [\mu(j) - \mu(j-1)]^2 - \dim \mathcal{V} \\
&= n^2 - \sum_{j=1}^{\infty} [\mu(j) - \mu(j-1)]^2 - \dim \mathcal{U} \\
&= n^2 - \sum_{j=1}^{\infty} [\mu(j) - \mu(j-1)]^2 - \dim \overline{\Xi^q(\mathcal{U})} - \dim\,(\Xi^q)^{-1}(\Xi^q(A_0)),
\end{aligned}$$

where A_0 is an element of $\mathcal{U}$. By Lemma 1.4, $(\Xi^q)^{-1}(\Xi^q(A_0)) = \mathbb{F}^*\,\mathcal{O}(A_0)$. (Recall that $\mathcal{U} \cup \{O_n\}$ is a $\mathcal{GL}_n$-invariant cone!) It is routine to check that

$$\dim \mathbb{F}^*\,\mathcal{O}(A_0) = n^2 - (q-1) - \sum_{j=1}^{\infty} [\mu(j) - \mu(j-1)]^2.$$

Finally,

$$\begin{aligned}
\mathrm{r.codim}\,\mathcal{V} = n^2 &- \sum_{j=1}^{\infty} [\mu(j) - \mu(j-1)]^2 - \dim \overline{\Xi^q(\mathcal{U})} \\
&- n^2 + (q-1) + \sum_{j=1}^{\infty} [\mu(j) - \mu(j-1)]^2 = \mathrm{codim}_{\mathbb{F}^{q-1}} \overline{\Xi^q(\mathcal{U})}.
\end{aligned}$$

The proof is complete. □

It seems to be worth of noticing that the above formula for the relative codimension is not true in case Δ_n^q identically vanishes on $\mathcal{V}$. Consider an example.

E x a m p l e 1.7. Let $B \in \mathcal{M}_m$ be a nilpotent matrix. Define $\mathcal{V} = \overline{\mathbb{F}\mathcal{O}(B \oplus I_2)}$, where $I_2 \in \mathcal{M}_2$ is the unit matrix. The set $\mathcal{V} \subset \mathcal{M}_{m+2}$ is a $\mathcal{GL}_{m+2}$-invariant irreducible algebraic cone. It is evident that $\mathrm{st.rank}\,\mathcal{V} = 2$ and that Δ_{m+2}^2 identically vanishes on $\mathcal{V}$ while tr does not. By an easy computation, one can verify that $\mathrm{r.codim}\,\mathcal{V} = 3$. At the same time, $\Xi^2|_{\mathcal{U}}$ is a regular map and $\Xi^2(\mathcal{U}) = \{4\}$ for $\mathcal{U} := \{A \in \mathcal{V};\ r_A(n) = 2,\ \mathrm{tr}(A) \neq 0\} = \mathbb{F}^*\,\overline{\mathcal{O}(B \oplus I_2)}$.

The characteristic map enables us to give a handy complete characterization of rank varieties.

Theorem 2.1. *Let $\mathcal{V} \subseteq \mathcal{M}_n$ be a $\mathcal{GL}_n$-invariant irreducible algebraic cone with $q := \operatorname{st.rank} \mathcal{V} \geqslant 1$. Then the following conditions are equivalent:*
(a) $\chi_A^q|_{\mathcal{V}}$ *is a dominant map for all* $A \in \mathcal{M}_n$,
(b) $\chi^q|_{\mathcal{V}}$ *is a dominant map,*
(c) $\chi^q|_{\mathcal{V}}$ *is an "onto" map,*
(d) $\mathcal{V}$ *is a rank variety.*

Proof. Implication (a) $\Rightarrow$ (b) is obvious. Implication (d) $\Rightarrow$ (c) can be readily verified. Implication (c) $\Rightarrow$ (a) is a direct consequence of Lemma 1.3. Assume finally that $\chi^q|_{\mathcal{V}}$ is a dominant map. Then neither of the polynomials tr and Δ_n^q vanishes identically on $\mathcal{V}$ (cf. Lemma 1.2). If $q = 1$, then condition (d) is satisfied thanks to [7, Theorem 1.2]. So, assume in addition that $q \geqslant 2$. By Lemmas 1.6 and 1.5, the equalities

$$\operatorname{r.codim} \mathcal{V} = \operatorname{codim}_{\mathbb{F}^{q-1}} \overline{\Xi^q(\mathcal{U})} = 0$$

hold, where $\mathcal{U} \subseteq \mathcal{V}$ is the open subset from Lemma 1.6. It turns out that $\mathcal{V}$ is a rank variety. The proof is complete. □

As an immediate consequence of the above theorem, Lemma 1.1, and formula (•) for the dimension of a rank variety, we get a complete characterization of linear subspaces $\mathcal{L} \subseteq \mathcal{M}_n$ having the property that the restriction $\chi^q|_{\mathcal{L}}$ is a dominant map.

Corollary 2.2. *For a linear subspace $\mathcal{L} \subseteq \mathcal{M}_n$ with $q := \operatorname{st.rank} \mathcal{L} \geqslant 1$ the following conditions are equivalent:*
(a) $\chi^q|_{\mathcal{L}}$ *is a dominant map,*
(b) $\widehat{\mathcal{L}}$ *is a rank variety,*
(c) $\dim \widehat{\mathcal{L}} = n^2 - \sum_{j=1}^{\infty} [\mu(j) - \mu(j-1)]^2$,
where $\widehat{\mathcal{L}} \subseteq \mathcal{M}_n$ is defined as in Lemma 1.1 and $\mu = \max_{A \in \mathcal{L}} r_A$.

Conditions (b) and (c) of Corollary 2.2 are clear but, unfortunately, rather not practicable because of troubles with the "$\mathcal{GL}_n$-invariant hull" $\widehat{\mathcal{L}}$. One would like to get a characterization of linear subspaces with the dominant restriction of the map χ^q in terms of their dimension (cf. [3]). It is evident that if a linear subspace $\mathcal{L} \subseteq \mathcal{M}_n$ such that $q := \operatorname{st.rank} \mathcal{L} \geqslant 1$ allows the dominant restriction $\chi^q|_{\mathcal{L}}$, then
(i) $\dim \mathcal{L} \geqslant q$,

(ii) tr and Δ_n^q do not identically vanish on $\mathcal{L}$.

Below we offer a theorem that provides a comparison between the "generic case" of the characterization of linear subspaces with the dominant restriction of the characteristic map and the case we have dealt with in Corollary 2.2. In what follows, we denote by $\mathbf{G}_k(\mathcal{M}_n)$ the Grassmann variety of the linear subspaces of dimension k contained in $\mathcal{M}_n$. We equip $\mathbf{G}_k(\mathcal{M}_n)$ and its subsets with the Zariski topology (arising from that on $\mathbb{F}^{kn^2}$ via the identification $\mathbb{F}^{kn^2} = \bigoplus_{j=1}^{k} \mathcal{M}_n$). For an integer s such that $0 \leqslant s \leqslant n$ we define $\mathbf{G}_k^s(\mathcal{M}_n) = \{\mathcal{L} \in \mathbf{G}_k(\mathcal{M}_n);\ \operatorname{st.rank} \mathcal{L} = s\}$. Notice that $\mathbf{G}_k^s(\mathcal{M}_n)$ is a locally closed (in other words, quasi-algebraic) subset of $\mathbf{G}_k(\mathcal{M}_n)$. For all information needed about Grassmann varieties we refer to [5]. Furthermore, let $\mathcal{T}_m$, $\mathcal{T}_m^0 \subseteq \mathcal{M}_m$ be, respectively, the set of all upper triangular matrices and the set of all nilpotent upper triangular matrices.

Theorem 2.3. (I) *Let k, $s \in \mathbb{N}$ be such that $1 \leqslant s \leqslant n$ and $s \leqslant k \leqslant \frac{1}{2}(n-s) \times (n-s-1) + \frac{1}{2}s(s+1)$. Then the set $\mathbf{E} := \{\mathcal{L} \in \mathbf{G}_k^s(\mathcal{M}_n) \colon \chi^s|_{\mathcal{L}}$ is a dominant map$\}$ is non-empty and open in $\mathbf{G}_k^s(\mathcal{M}_n)$.*

(II) *Let d, $q \in \mathbb{N}$ satisfy the inequalities $2 \leqslant q \leqslant n$ and $q \leqslant d \leqslant 1 + \frac{1}{2}q(q-1) + \frac{1}{2}(n-q)(n-q-1) + q(n-q)$. Then there is a linear subspace $\mathcal{L}_d \subset \mathcal{M}_n$ such that $\dim \mathcal{L}_d = d$, $\operatorname{st.rank} \mathcal{L}_d = q$, the polynomials* tr *and Δ_n^q do not identically vanish on $\mathcal{L}_d$, and $\chi^q|_{\mathcal{L}_d}$ is not a dominant map.*

P r o o f. In the proof of (I) we follow the idea of the proof of [3, Theorem 2.3]. Define $\mathcal{L}' = \{O_{n-s}\} \oplus \mathcal{D}_s$ and $\mathcal{L}'' = \mathcal{T}_{n-s}^0 \oplus \mathcal{T}_s$, where $\mathcal{D}_s \subseteq \mathcal{M}_s$ is the set of all diagonal matrices. Then both $\mathcal{L}'$ and $\mathcal{L}''$ are linear subspaces of $\mathcal{M}_n$. Observe that $\dim \mathcal{L}' = s$ and $\dim \mathcal{L}'' = \frac{1}{2}(n-s)(n-s-1) + \frac{1}{2}s(s+1)$. If $\mathcal{K} \subseteq \mathcal{M}_n$ is any linear subspace of dimension k such that $\mathcal{L}' \subseteq \mathcal{K} \subseteq \mathcal{L}''$, then obviously $\mathcal{K} \in \mathbf{E}$. Remark now that the set $\mathbf{U} := \{\mathcal{L} \in \mathbf{G}_k(\mathcal{M}_n)$; there is $A \in \mathcal{L}$ such that the differential $\mathrm{d}_A(\chi^s|_{\mathcal{L}}) \colon \mathcal{L} \longrightarrow \mathbb{F}^s$ is "onto"$\}$ is open in $\mathbf{G}_k(\mathcal{M}_n)$. (To see that, define

$$\mathcal{B}_n^k = \Big\{(B_1, \dots, B_k) \in \bigoplus_{j=1}^{k} \mathcal{M}_n;\ B_1, \dots, B_k \text{ are linearly independent over } \mathbb{F}\Big\},$$

denote by $\operatorname{Span}(B_1, \dots, B_k)$ the linear span over $\mathbb{F}$ of the set $\{B_1, \dots, B_k\} \subset \mathcal{M}_n$, and consider the map

$$\begin{aligned}\{(A, B_1, \dots, B_k) \in \mathcal{M}_n \times \mathcal{B}_n^k;\ A \in \operatorname{Span}(B_1, \dots, B_k)\}&\\ \ni (A, B_1, \dots, B_k) \mapsto [\mathrm{d}_A \chi^s]_{(B_1, \dots, B_k)} \in \mathcal{M}_{s \times k},&\end{aligned}$$

where $[\mathrm{d}_A\chi^s]_{(B_1,\dots,B_k)}$ stands for the matrix of the differential at A of the restriction of χ^s to $\mathrm{Span}\,(B_1,\dots,B_k)$ with respect to the basis $(B_1,\dots,B_k)$ and the canonical basis in $\mathbb{F}^s$.) By the well-known differential characterization of dominant maps,

$$\mathbf{U} = \{\mathcal{L} \in \mathbf{G}_k(\mathcal{M}_n)\colon\ \chi^s|_{\mathcal{L}} \text{ is a dominant map}\}.$$

Consequently, $\mathbf{E}$ is an open subset of $\mathbf{G}_k^s(\mathcal{M}_n)$. The assertion of (I) follows.

We turn to the proof of (II). Pick pairwise distinct elements $\lambda_1,\dots,\lambda_q \in \mathbb{F}^*$ such that $\sum_{j=1}^{q}\lambda_j \neq 0$. Define $w = 1 + \frac{1}{2}q(q-1) + \frac{1}{2}(n-q)(n-q-1) + q(n-q)$ and

$$\mathcal{L}_w = \left\{ \begin{bmatrix} B & C \\ O_{q\times(n-q)} & \widetilde{B} + \kappa\mathrm{Diag}\,(\lambda_1,\dots,\lambda_q) \end{bmatrix} :\ B \in \mathcal{T}^0_{n-q},\ \widetilde{B} \in \mathcal{T}^0_q,\ C \in \mathcal{M}_{(n-q)\times q},\ \kappa \in \mathbb{F} \right\}.$$

It is evident that $\dim \mathcal{L}_w = w$, $\mathrm{st.rank}\,\mathcal{L}_w = q$, and neither of the polynomials tr and Δ_n^q vanishes identically on $\mathcal{L}_w$. Since $\Xi^q(\{A \in \mathcal{L}_w;\ \mathrm{tr}(A) \neq 0\})$ is obviously a singleton, $\chi^q|_{\mathcal{L}_w}$ is not a dominant map (cf. Lemma 1.5). In order to get $\mathcal{L}_d$ satisfying the conditions of the statement with $d < w$, it is enough to take an appropriate subspace of $\mathcal{L}_w$. The proof is complete. □

To conclude the note, we give a more practicable sufficient condition for a linear subspace to allow the dominant restriction of the map χ^q. The result is based on a certain theorem concerning the *linear capacity.*

Proposition 2.4. *Let $\mathcal{L} \subseteq \mathcal{M}_n$ be a linear subspace such that* $\mathrm{st.rank}\,\mathcal{L} = 2$. *Define $\mu = \max_{A\in\mathcal{L}} r_A$. Then $\chi^2|_{\mathcal{L}}$ is a dominant map whenever* tr *does not identically vanish on $\mathcal{L}$ and*

$$\dim \mathcal{L} > \frac{1}{2}\Big(n^2 - \sum_{j=1}^{\infty}[\mu(j) - \mu(j-1)]^2\Big).$$

P r o o f. In virtue of Corollary 2.2, it is enough to prove that $\widehat{\mathcal{L}} = \mathcal{X}_\mu$. Since $\mathrm{st.rank}\,\widehat{\mathcal{L}} = 2$, there are only two disjoint possibilities: either $\widehat{\mathcal{L}} = \mathcal{X}_\mu$ or there is $B \in \mathcal{M}_n$ such that $\widehat{\mathcal{L}} = \overline{\mathbb{F}\mathcal{O}(B)}$ (cf. [8, Theorem 2.2]). If $\widehat{\mathcal{L}} = \overline{\mathbb{F}\mathcal{O}(B)}$, then $\mathrm{tr}(B) \neq 0$ (because tr does not vanish on $\mathcal{L}$) and $r_B = \mu$. Thus, by [8, Theorem 3.1],

$$\dim \mathcal{L} \leqslant \frac{1}{2}\Big(n^2 - \sum_{j=1}^{\infty}[\mu(j) - \mu(j-1)]^2\Big),$$

a contradiction. The proof is complete. □

References

[1] *Eisenbud, D., Saltman, D.*: Rank varieties of matrices. Commutative Algebra (Berkeley, CA). Math. Sci. Res. Inst. Publ. 15. Springer, New York, 1989, pp. 173–212.

[2] *Gantmacher, F. R.*: Théorie des matrices. Dunod, Paris, 1966.

[3] *Helton, W., Rosenthal, J., Wang, X.*: Matrix extensions and eigenvalue completions, the generic case. Trans. Amer. Math. Soc. *349* (1997), 3401–3408.

[4] *Lang, S.*: Algebra. W. A. Benjamin, Inc., New York, 1970.

[5] *Łojasiewicz, St.*: Introduction to Complex Analytic Geometry. Birkhäuser, Basel, 1991.

[6] *Shafarevich, I. R.*: Basic Algebraic Geometry. Springer, Berlin, 1977.

[7] *Skrzyński, M.*: Remarks on applications of rank functions to algebraic sets of matrices. Demonstratio Math. *32* (1999), 263–271.

[8] *Skrzyński, M.*: On $\mathcal{GL}_n$-invariant cones of matrices with small stable ranks. Demonstratio Math. *33* (2000), 243–254.

[9] *Skrzyński, M.*: On $\mathcal{GL}_n$-invariant algebraic cones of matrices with relative codimension equal to 1. Commentationes Math. *40* (2000), 167–174.

[10] *Weyman, J.*: The equations of conjugacy classes of nilpotent matrices. Invent. Math. *98* (1989), 229–245.

Author's address: *Marcin Skrzyński*, Institute of Mathematics, Cracow University of Technology, ul. Warszawska 24, PL 31-155 Kraków, e-mail: `pfskrzyn@cyf-kr.edu.pl`.