CONNECTED RESOLVING DECOMPOSITIONS IN GRAPHS

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(Received November 19, 2001)

Abstract. For an ordered k-decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph G and an edge e of G, the $\mathcal{D}$-code of e is the k-tuple $c_{\mathcal{D}}(e) = (d(e, G_1), d(e, G_2), \ldots, d(e, G_k))$, where $d(e, G_i)$ is the distance from e to G_i. A decomposition $\mathcal{D}$ is resolving if every two distinct edges of G have distinct $\mathcal{D}$-codes. The minimum k for which G has a resolving k-decomposition is its decomposition dimension $\dim_{\rm d}(G)$. A resolving decomposition $\mathcal{D}$ of G is connected if each G_i is connected for $1 \leqslant i \leqslant k$. The minimum k for which G has a connected resolving k-decomposition is its connected decomposition number $\operatorname{cd}(G)$. Thus $2 \leqslant \dim_{\rm d}(G) \leqslant \operatorname{cd}(G) \leqslant m$ for every connected graph G of size $m \geqslant 2$. All nontrivial connected graphs with connected decomposition number 2 or m are characterized. We provide bounds for the connected decomposition number of a connected graph in terms of its size, diameter, girth, and other parameters. A formula for the connected decomposition number of a nonpath tree is established. It is shown that, for every pair a, b of integers with $3 \leqslant a \leqslant b$, there exists a connected graph G with $\dim_{\rm d}(G) = a$ and $\operatorname{cd}(G) = b$.

Keywords: distance, resolving decomposition, connected resolving decomposition

MSC 2000: 05C12

1. Introduction

A *decomposition* of a graph G is a collection of subgraphs of G, none of which have isolated vertices, whose edge sets provide a partition of $E(G)$. A decomposition into k subgraphs is a *k-decomposition*. A decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ is *ordered* if the ordering $(G_1, G_2, \ldots, G_k)$ has been imposed on $\mathcal{D}$. If each subgraph G_i $(1 \leqslant i \leqslant k)$ is isomorphic to a graph H, then $\mathcal{D}$ is called an *H-decomposition* of G. Decompositions of graphs have been the subject of many studies. J. Bosák [1] has written a book devoted to the subject.

Research supported in part by a Western Michigan University Faculty Research and Creative Activities Fund.

For edges e and f in a connected graph G, the *distance* $d(e,f)$ between e and f is the minimum nonnegative integer k for which there exists a sequence $e = e_0, e_1, \ldots, e_k = f$ of edges of G such that e_i and e_{i+1} are adjacent for $i = 0, 1, \ldots, k-1$. Thus $d(e,f) = 0$ if and only if $e = f$, $d(e,f) = 1$ if and only if e and f are adjacent, and $d(e,f) = 2$ if and only if e and f are nonadjacent edges that are adjacent to a common edge of G. Also, this distance equals the standard distance between vertices e and f in the line graph $L(G)$. For an edge e of G and a subgraph F of G, we define the distance between e and F as

$$d(e,F) = \min_{f \in E(F)} d(e,f).$$

Let $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ be an ordered k-decomposition of a connected graph G. For $e \in E(G)$, the *$\mathcal{D}$-code* (or simply the *code*) of e is the k-vector

$$c_{\mathcal{D}}(e) = (d(e,G_1), d(e,G_2), \ldots, d(e,G_k)).$$

Hence exactly one coordinate of $c_{\mathcal{D}}(e)$ is 0, namely the ith coordinate if $e \in E(G_i)$. The decomposition $\mathcal{D}$ is said to be a *resolving decomposition* for G if every two distinct edges of G have distinct $\mathcal{D}$-codes. The minimum k for which G has a resolving k-decomposition is its *decomposition dimension* $\dim_{\mathrm{d}}(G)$. A resolving decomposition of G with $\dim_{\mathrm{d}}(G)$ elements is a *minimum resolving decomposition* for G. Thus if G is a connected graph of size at least 2, then $\dim_{\mathrm{d}}(G) \geqslant 2$. The following result appeared in [2].

Theorem A. *Let G be a connected graph order $n \geqslant 3$.*
(a) *Then $\dim_{\mathrm{d}}(G) = 2$ if and only if $G = P_n$.*
(b) *If $n \geqslant 5$, then $\dim_{\mathrm{d}}(G) \leqslant n$.*

The concept of resolvability in graphs has appeared in the literature. Slater introduced and studied these ideas with different terminology in [9], [10]. Slater described the usefulness of these ideas when working with U.S. sonar and coast guard Loran (Long range aids to navigation) stations. Harary and Melter [8] discovered these concepts independently. Recently, these concepts were rediscovered by Johnson [6], [7] of the Pharmacia Company while attempting to develop a capability of large datasets of chemical graphs. Resolving decompositions in graphs were introduced and studied in [2] and further studied in [4], [5]. We refer to the book [3] for graph theory notation and terminology not described here.

A resolving decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ of a connected graph G is *connected* if each subgraph G_i $(1 \leqslant i \leqslant k)$ is a connected subgraph in G. The minimum

k for which G has a connected resolving k-decomposition is its *connected decomposition number* $\mathrm{cd}(G)$. A connected resolving decomposition of G with $\mathrm{cd}(G)$ elements is called a *minimum connected resolving decomposition of* G. If G has $m \geqslant 2$ edges, then the m-decomposition $\mathcal{D} = \{G_1, G_2, \ldots, G_m\}$, where each G_i $(1 \leqslant i \leqslant m)$ contains a single edge, is a connected resolving decomposition of G. Thus $\mathrm{cd}(G)$ is defined for every connected graph G of size at least 2. Moreover, every connected resolving k-decomposition is a resolving k-decomposition, and so

$$2 \leqslant \dim_{\mathrm{d}}(G) \leqslant \mathrm{cd}(G) \leqslant m. \tag{1}$$

for every connected graph G of size $m \geqslant 2$.

To illustrate these concepts, consider the graph G of Figure 1. Let $\mathcal{D} = \{G_1, G_2, G_3\}$, where $E(G_1) = \{e_1, e_5, f_1, f_5, f_4\}$, $E(G_2) = \{e_2, e_3, f_2\}$, and $E(G_3) = \{e_4, e_6, f_3, f_6, f_7\}$. The $\mathcal{D}$-codes of the edges of G are:

$$\begin{aligned}
&c_{\mathcal{D}}(e_1) = (0,1,2),\ c_{\mathcal{D}}(e_2) = (1,0,2),\ c_{\mathcal{D}}(e_3) = (2,0,1),\ c_{\mathcal{D}}(e_4) = (2,1,0),\\
&c_{\mathcal{D}}(e_5) = (0,4,1),\ c_{\mathcal{D}}(e_6) = (1,4,0),\ c_{\mathcal{D}}(f_1) = (0,1,1),\ c_{\mathcal{D}}(f_2) = (1,0,1),\\
&c_{\mathcal{D}}(f_3) = (1,1,0),\ c_{\mathcal{D}}(f_4) = (0,2,1),\ c_{\mathcal{D}}(f_5) = (0,3,1),\ c_{\mathcal{D}}(f_6) = (1,3,0),\\
&c_{\mathcal{D}}(f_7) = (1,2,0).
\end{aligned}$$

Thus $\mathcal{D}$ is a resolving decomposition of G. By Theorem A, $\dim_{\mathrm{d}}(G) = |\mathcal{D}| = 3$. However, $\mathcal{D}$ is not connected since G_1 and G_2 are not connected subgraphs in G. On the other hand, let $\mathcal{D}^* = \{G_1^*, G_2^*, G_3^*, G_4^*, G_5^*\}$, where $E(G_1^*) = \{e_1, f_1\}$, $E(G_2^*) = \{e_5, f_4, f_5\}$, $E(G_3^*) = \{e_2, e_3, f_2\}$, $E(G_4^*) = \{e_4, f_3\}$, and $E(G_5^*) = \{e_6, f_6, f_7\}$. Then $\mathcal{D}^*$ is a connected resolving decomposition of G. But $\mathcal{D}^*$ is not minimum since the decomposition $\mathcal{D}' = \{G_1', G_2', G_3', G_4'\}$, where $E(G_1') = \{e_1\}$, $E(G_2') = \{e_3\}$, $E(G_3') = \{e_5\}$, and $E(G_4') = E(G) - \{e_1, e_3, e_5\}$, is a connected resolving decomposition of G with fewer elements. Indeed, it can be verified that $\mathcal{D}'$ is a minimum connected resolving decomposition of G and so $\mathrm{cd}(G) = |\mathcal{D}'| = 4$.

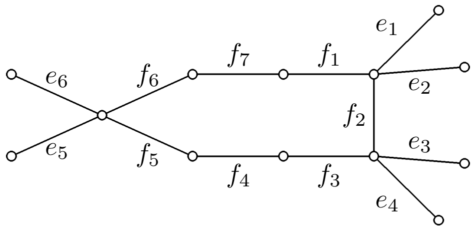

Figure 1. A graph G with $\dim_{\mathrm{d}}(G) = 3$ and $\mathrm{cd}(G) = 4$

The example just presented also illustrates an important point. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ be a resolving decomposition of G. If $e \in E(G_i)$ and $f \in E(G_j)$, where $i \neq j$ and $i, j \in \{1, 2, \ldots, k\}$, then $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$ since $d(e, G_i) = 0$ and $d(e, G_j) \neq 0$. Thus, when determining whether a given decomposition $\mathcal{D}$ of a graph G is a resolving decomposition for G, we need only verify that the edges of G belonging to same element in $\mathcal{D}$ have distinct $\mathcal{D}$-codes. The following two observations are useful.

Observation 1.1. *Let $\mathcal{D}$ be a resolving decomposition of G and $e_1, e_2 \in E(G)$. If $d(e_1, f) = d(e_2, f)$ for all $f \in E(G) - \{e_1, e_2\}$, then e_1 and e_2 belong to distinct elements of $\mathcal{D}$.*

Observation 1.2. *Let G be a connected graph. Then $\dim_{\mathrm{d}}(G) = \mathrm{cd}(G)$ if and only if G contains a minimum resolving decomposition that is connected.*

2. Refinements of decompositions of a graph

Let $\mathcal{D}$ and $\mathcal{D}^*$ be two decompositions of a connected graph G. Then $\mathcal{D}^*$ is called a *refinement* of $\mathcal{D}$ if every element in $\mathcal{D}^*$ is a subgraph of some element of $\mathcal{D}$. A refinement $\mathcal{D}^*$ of $\mathcal{D}$ is *connected* if $\mathcal{D}^*$ is a connected decomposition of G. For the graph G of Figure 1, the decomposition $\mathcal{D}^*$ of G is a connected refinement of $\mathcal{D}$. We have seen that $\mathcal{D}$ is resolving and its refinement $\mathcal{D}^*$ is also resolving. This is not coincident, as we show now.

Theorem 2.1. *Let $\mathcal{D}$ and $\mathcal{D}^*$ be two decompositions of a connected graph G. If $\mathcal{D}$ is a resolving decomposition of G and $\mathcal{D}^*$ is a refinement of $\mathcal{D}$, then $\mathcal{D}^*$ is also a resolving decomposition of G.*

Proof. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ and $\mathcal{D}^* = \{H_1, H_2, \ldots, H_\ell\}$ be two decompositions of G, where $k \leqslant \ell$, such that each H_i $(1 \leqslant i \leqslant \ell)$ is a subgraph of G_j for some j with $1 \leqslant j \leqslant k$. Let e and f be distinct edges of G. We show that $c_{\mathcal{D}^*}(e) \neq c_{\mathcal{D}^*}(f)$. Since $\mathcal{D}$ is a resolving decomposition of G, it follows that $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Thus $d(e, G_j) \neq d(f, G_j)$ for some j with $1 \leqslant j \leqslant k$, say $d(e, G_1) \neq d(f, G_1)$. If G_1 is an element of $\mathcal{D}^*$, then $d(e, G_1) \neq d(f, G_1)$ and so $c_{\mathcal{D}^*}(e) \neq c_{\mathcal{D}^*}(f)$. Thus we may assume that $G_1 = H_{i_1} \cup H_{i_2} \cup \ldots \cup H_{i_s}$, where $1 \leqslant i_1 < i_2 < \ldots < i_s \leqslant \ell$ and $s \geqslant 2$. Observe that at least one of e and f does not belong to G_1, for otherwise, $d(f, G_1) = 0 = d(f, G_1)$. We consider two cases.

Case 1. *Exactly one of e and f is in G_1, say $e \in E(G_1)$ and $f \notin E(G_1)$.* Thus $e \in E(H_{i_p})$ for some p with $1 \leqslant p \leqslant s$ and so $d(e, H_{i_p}) = 0$. Since $f \notin E(G_1)$, it follows that $f \notin E(H_{i_p})$ and so $d(v, H_{i_p}) \neq 0$. Hence $c_{\mathcal{D}^*}(e) \neq c_{\mathcal{D}^*}(f)$.

Case 2. $e, f \notin E(G_1)$. Let $e', f' \in E(G_1)$ such that $d(e, G_1) = d(e, e')$ and $d(f, G_1) = d(f, f')$, where say $d(e, e') < d(f, f')$. If $e', f' \in E(H_{i_p})$ for some p with $1 \leqslant p \leqslant s$, then $d(e, H_{i_p}) = d(e, e') < d(f, f') = d(f, H_{i_p})$, implying that $c_{\mathcal{D}^*}(e) \neq c_{\mathcal{D}^*}(f)$. If $e' \in E(H_{i_p})$ and $f \in E(H_{i_q})$, where $1 \leqslant p \neq q \leqslant s$, then $d(e, H_{i_p}) = d(e, e') < d(f, f') \leqslant d(f, H_{i_p})$, again, implying that $c_{\mathcal{D}^*}(e) \neq c_{\mathcal{D}^*}(f)$.

Therefore, $\mathcal{D}^*$ is a resolving decomposition of G. □

By Theorem 2.1, a connected resolving decomposition of a connected graph can be obtained from a resolving decomposition by means of refinement. However, a connected refinement of a resolving decomposition is not necessary to be minimum. Indeed, using an extensive case-by-case analysis, we can show that the graph G of Figure 1 has two distinct minimum resolving decompositions (up to isomorphic), namely, $\{G_1, G_2, G_3\}$ and $\{H_1, H_2, H_3\}$, where $G_1 = G_2 = P_3 \cup P_4$, $G_3 = P_4$, $H_1 = H_2 = P_2 \cup 2P_3$, and $H_3 = P_4$. For example, $\mathcal{D} = \{G_1, G_2, G_3\}$, where $E(G_1) = \{e_1, e_5, f_1, f_5, f_4\}$, $E(G_2) = \{e_2, e_3, f_2\}$, and $E(G_3) = \{e_4, e_6, f_3, f_6, f_7\}$ and $\widetilde{\mathcal{D}} = \{H_1, H_2, H_3\}$, where $E(H_1) = \{e_1, e_6, f_1, f_4, f_6\}$, $E(H_2) = \{e_2, e_3, f_2\}$, and $E(H_3) = \{e_4, e_5, f_3, f_5, f_7\}$. The decompositions $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ are shown in Figure 2. Since each connected refinement of $\mathcal{D}$ contains at least five elements, each connected refinement of $\widetilde{\mathcal{D}}$ contains at least seven elements, and $\operatorname{cd}(G) = 4$, it follows that no minimum connected resolving decomposition of G is a refinement of any minimum resolving decomposition of G.

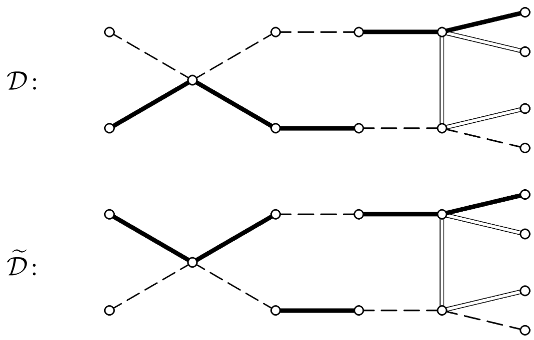

Figure 2. The two distinct minimum resolving decompositions $\mathcal{D}$ and $\widetilde{\mathcal{D}}$ of G

We have seen that if G is a connected graph of size $m \geqslant 2$, then $2 \leqslant \mathrm{cd}(G) \leqslant m$. In this section, we first characterize those connected graphs G of size $m \geqslant 2$ such that $\mathrm{cd}(G) = 2$ or $\mathrm{cd}(G) = m$.

Theorem 3.1. *Let G be a connected graph of order $n \geqslant 3$ and size m. Then*
(a) $\mathrm{cd}(G) = 2$ *if and only if* $G = P_n$, *and*
(b) $\mathrm{cd}(G) = m$ *if and only if* $G = K_3$ *or* $G = K_{1,n-1}$.

Proof. We first verify (a). Let $P_n\colon v_1, v_2, \ldots, v_n$ and let $\mathcal{D} = \{G_1, G_2\}$ be the decomposition of P_n in which $E(G_1) = \{v_1v_2\}$ and G_2 is the path $v_2, v_3, \ldots, v_n$. Thus $\mathcal{D}$ is connected. For $2 \leqslant i \leqslant n-1$, the edge v_iv_{i+1} is the unique edge of G_2 at distance $i-1$ from G_1. Therefore, $\mathcal{D}$ is a connected resolving decomposition of P_n and so $\mathrm{cd}(P_n) = 2$. For the converse, let G be a connected graph of order $n \geqslant 3$ and $\mathrm{cd}(G) = 2$. By (1) $\dim_{\mathrm{d}}(G) = 2$ as well. It then follows by Theorem A that $G = P_n$.

Next we verify (b). It is routine to show that $\mathrm{cd}(K_3) = 2$ and $\mathrm{cd}(K_{1,n-1}) = n-1$ and so the graphs described in (b) have $\mathrm{cd}(G) = m$. For the converse, let G be a connected graph of order $n \geqslant 3$ and size $m \geqslant 2$ such that $\mathrm{cd}(G) = m$. If $m = 2$, then $G = P_3$ and $\mathrm{cd}(P_3) = 2$ by (a). If $m = 3$, then $G \in \{P_4, K_3, K_{1,3}\}$. Since $\mathrm{cd}(P_4) = 2$ and $\mathrm{cd}(K_3) = \mathrm{cd}(K_{1,3}) = 3$, it follows that $G = K_3$ or $G = K_{1,3}$. Now let G be a connected graph of size $m \geqslant 4$ and let $E(G) = \{e_1, e_2, \ldots, e_m\}$. If $G \neq K_{1,n-1}$, then G contains a path P_4 of order 4 with three edges, say e_1, e_2, and e_3, such that $d(e_1, e_2) = 1$, $d(e_1, e_3) = 2$, and $d(e_2, e_3) = 1$. Then $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-1}\}$, where $E(G_1) = \{e_1, e_2\}$ and $E(G_i) = \{e_{i+1}\}$ for $2 \leqslant i \leqslant m-1$, is a connected resolving decomposition of G. Thus $\mathrm{cd}(G) \leqslant |\mathcal{D}| = m-1$. $\square$

It was shown in [2] that $\dim_{\mathrm{d}}(K_3) = 3$ and $\dim_{\mathrm{d}}(K_{1,n-1}) = n-1$. Thus the following corollary is a consequence of (1) and Theorem 3.1.

Corollary 3.2. *Let G be a connected graph of order $n \geqslant 3$ and of size m. Then* $\dim_{\mathrm{d}}(G) = m$ *if and only if* $G = K_3$ *or* $G = K_{1,n-1}$.

Next, we present bounds for $\mathrm{cd}(G)$ of a connected graph G in terms of its size and diameter.

Proposition 3.3. *If G is a connected graph of size $m \geqslant 2$ and diameter d, then*

$$2 \leqslant \mathrm{cd}(G) \leqslant m - d + 2.$$

Proof. We have seen that $\mathrm{cd}(G) \geqslant 2$ for every connected graph G of size $m \geqslant 2$. Thus it remains to verify the upper bound. Let $u, v \in V(G)$ such that $d(u, v) = d$

and let $P\colon\ u = v_1, v_2, \ldots, v_{d+1} = v$ be a $u-v$ path of length d in G. Also, let $E(G) - E(P) = \{e_1, e_2, \ldots, e_{m-d}\}$. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-d+2}\}$, where $E(G_i) = \{e_i\}$ for $1 \leqslant i \leqslant m-d$, $E(G_{m-d+1}) = \{v_1v_2\}$, and $E(G_{m-d+2}) = E(P-v_1)$. Then $\mathcal{D}$ is a connected decomposition of G. Since $d(v_iv_{i+1}, G_{m-d+1}) = i-1$ for $2 \leqslant i \leqslant d$, it follows that $\mathcal{D}$ is a resolving decomposition of G. Therefore, $\mathrm{cd}(G) \leqslant |\mathcal{D}| = m-d+2$. $\square$

By Theorem 3.1, the lower bound in Proposition 3.3 is sharp. If $d = 1$, then $G = K_n$ for some $n \geqslant 3$. Since $\dim_{\mathrm{d}}(K_n) = \mathrm{cd}(K_n)$, it then follows by Theorem A that the upper bound in Proposition 3.3 is not sharp for $d = 1$. If $d = 2$, then $G = K_{1,m}$ is the only graph with $\mathrm{cd}(G) = m-d+2 = m$ by Theorem 3.1. Thus we may assume that $m \geqslant d \geqslant 3$. If $m = d$, then $G = P_{m+1}$ and $\mathrm{cd}(G) = 2 = m-d+2$. If $m \geqslant d+1$, let G be the graph obtained from the path $P_{d+1}\colon\ u_1, u_2, \ldots, u_{d+1}$ by adding the $m-d \geqslant 1$ new vertices $v_1, v_2, \ldots, v_{m-d}$ and joining each of these vertices to u_d. Then the diameter of G is d and size of G is m. Moreover, it can be verified that $\mathrm{cd}(G) = m-d+2$. Thus the upper bound in Proposition 3.3 is sharp for $d \geqslant 2$.

The *girth* of a graph is the length of its shortest cycle. Next, we provide bounds for the connected decomposition number of a connected graph in terms of its size and girth.

Theorem 3.4. *If G is a connected graph of size $m \geqslant 3$ and girth $\ell \geqslant 3$, then*

$$3 \leqslant \mathrm{cd}(G) \leqslant m - \ell + 3.$$

Moreover, $\mathrm{cd}(G) = m - \ell + 3$ if and only if G is a cycle of order at least 3.

Proof. Since $\ell \geqslant 3$, it follows that G is not a path and so $\mathrm{cd}(G) \geqslant 3$ by Theorem 3.1. It remains to verify the upper bound. If $\ell = 3$, then $\mathrm{cd}(G) \leqslant m$ by (1) and so the upper bound holds. Thus we may assume that $\ell \geqslant 4$. Let $C_\ell\colon\ v_1, v_2, \ldots, v_\ell, v_1$ be a cycle of length ℓ in G, let $d = \lfloor \ell/2 \rfloor$, and let $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-\ell+3}\}$ be a decomposition of G, where $E(G_1) = \{v_1v_2\}$, $E(G_2) = \{v_2v_3, v_3v_4, \ldots, v_dv_{d+1}\}$, $E(G_3) = \{v_{d+1}v_{d+2}, v_{d+2}v_{d+3}, \ldots, v_{\ell-1}v_\ell, v_\ell v_1\}$, and each of G_i $(4 \leqslant i \leqslant m-\ell+3)$ contains exactly one edge in $E(G) - E(C_\ell)$. Thus $\mathcal{D}$ is connected. Furthermore, $c_{\mathcal{D}}(v_1v_2) = (0, 1, 1, \ldots)$, $c_{\mathcal{D}}(v_iv_{i+1}) = (i-1, 0,\ \min\{i, d-i+1\}, \ldots)$ for $2 \leqslant i \leqslant d$, $c_{\mathcal{D}}(v_{d+1}v_{d+2}) = (d, 1, 0, \ldots)$, $c_{\mathcal{D}}(v_iv_{i+1}) = (\ell-i+1,\ \min\{i-d, \ell-i+2\}, 0, \ldots)$ for $d+2 \leqslant i \leqslant \ell-1$, and $c_{\mathcal{D}}(v_\ell v_1) = (1, 2, 0, \ldots)$, it follows that the $\mathcal{D}$-codes of vertices of G are distinct. Thus $\mathcal{D}$ is a connected resolving decomposition of G and so $\mathrm{cd}(G) \leqslant |\mathcal{D}| = m - \ell + 3$.

If G is a cycle C_n of order $n \geqslant 3$, then $\ell = m = n$ and so $\mathrm{cd}(G) = 3$. For the converse, let $G \neq C_n$ be a connected graph of order $n \geqslant 3$, size $m \geqslant 3$, and girth $\ell \geqslant 3$ and let $C_\ell\colon v_1, v_2, \ldots, v_\ell, v_1$ be a smallest cycle in G, where $\ell < n$. Since G is connected and $G \neq C_n$, it follows that $m \geqslant 4$ and there exists a vertex $v \in V(G) - V(C_\ell)$ such that v is adjacent to a vertex of C_ℓ, say $vv_1 \in E(G)$. We consider three cases.

Case 1. $\ell = 3$. Then G contains an induced subgraph H_1 of Figure 3(a), where dashed lines indicate that the given edges may or may not be present. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-\ell+2}\}$, where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3\}$, $E(G_3) = \{v_1v_3\}$, and each of G_i $(4 \leqslant i \leqslant m - \ell + 2)$ contains exactly one edge in $E(G) - (E(C_\ell) \cup \{vv_1\})$. Since $d(vv_1, G_2) = 1$ and $d(v_1v_2, G_2) = 2$, it follows that $\mathcal{D}$ is a connected resolving decomposition of G and so $\operatorname{cd}(G) \leqslant |\mathcal{D}| = m - \ell + 2$.

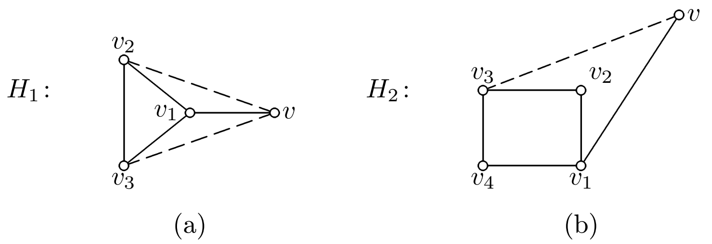

Figure 3. The subgraphs H_1 and H_2

Case 2. $\ell = 4$. Then G contains an induced subgraph H_2 of Figure 3(b), where the dashed line indicate that the given edge may or may not be present. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-\ell+2}\}$, where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3\}$, $E(G_3) = \{v_1v_4,\ v_3v_4\}$, and each of G_i $(4 \leqslant i \leqslant m - \ell + 2)$ contains exactly one edge in $E(G) - (E(C_\ell) \cup \{vv_1\})$. Since $d(vv_1, G_2) = 2$, $d(v_1v_2, G_2) = 1$, $d(v_1v_4, G_2) = 2$, and $d(v_3v_4, G_2) = 1$, it follows that $\mathcal{D}$ is a connected resolving decomposition of G and so $\operatorname{cd}(G) \leqslant |\mathcal{D}| = m - \ell + 2$.

Case 3. $\ell \geqslant 5$. Since C_ℓ is a smallest cycle in G, it follows that v is adjacent exactly one vertex of C_ℓ. Let $d = \lfloor \ell/2 \rfloor$ and let $\mathcal{D} = \{G_1, G_2, \ldots, G_{m-\ell+2}\}$ be a decomposition of G, where $E(G_1) = \{vv_1, v_1v_2\}$, $E(G_2) = \{v_2v_3, v_3v_4, \ldots, v_dv_{d+1}\}$, $E(G_3) = \{v_{d+1}v_{d+2}, v_{d+2}v_{d+3}, \ldots, v_{\ell-1}v_\ell, v_\ell v_1\}$, and each of G_i $(4 \leqslant i \leqslant m - \ell + 2)$ contains exactly one edge in $E(G) - (E(C_\ell) \cup \{vv_1\})$. Thus $\mathcal{D}$ is connected. Since $c_{\mathcal{D}}(vv_1) = (0, 2, 2, \ldots)$, $c_{\mathcal{D}}(v_1v_2) = (0, 1, 1, \ldots)$, $c_{\mathcal{D}}(v_iv_{i+1}) = (i - 1, 0, \min\{i, d - i + 1\}, \ldots)$ for $2 \leqslant i \leqslant d$, $c_{\mathcal{D}}(v_{d+1}v_{d+2}) = (d, 1, 0, \ldots)$, $c_{\mathcal{D}}(v_iv_{i+1}) = (\ell - i + 1, \min\{i - d, \ell - i + 2\}, 0, \ldots)$ for $d + 2 \leqslant i \leqslant \ell - 1$, and $c_{\mathcal{D}}(v_\ell v_1) = (1, 2, 0, \ldots)$, it follows that $\mathcal{D}$ is a connected resolving decomposition of G. Thus $\operatorname{cd}(G) \leqslant |\mathcal{D}| = m - \ell + 2$. $\square$

Next, we present an upper bound for $\mathrm{cd}(G)$ of a connected graph G in terms of its order. For a connected graph G, let

$$f(G) = \min\{k(G - E(T)) \colon \ T \text{ is a spanning tree of } G\},$$

where $k(G - E(T))$ is the number of components of $G - E(T)$.

Theorem 3.5. *If G is a connected graph of order $n \geqslant 5$, then*

$$\mathrm{cd}(G) \leqslant n + f(G) - 1.$$

P r o o f. If G is a tree of order n, then $f(G) = 0$. Since the size of G is $n - 1$, it follows by (1) that $\mathrm{cd}(G) \leqslant n - 1$ and so the result is true for a tree. Thus we may assume that G is a connected graph that is not a tree. Suppose that $f(G) = k$. Let T be a spanning tree of G such that $k(G - E(T)) = k$, where $E(T) = \{e_1, e_2, \ldots, e_{n-1}\}$ and $H_1, H_2, \ldots, H_k$ are k components of $G - E(T)$. Let

$$\mathcal{D} = \{G_1, G_2, \ldots, G_{n-1}, H_1, H_2, \ldots, H_k\},$$

where $E(G_i) = \{e_i\}$ for $1 \leqslant i \leqslant n - 1$. Then $\mathcal{D}$ is a connected decomposition of G with $n + k - 1$ elements.

We now show that $\mathcal{D}$ is a resolving decomposition of G. Let e and f be two edges of G. If e and f belongs to distinct elements of $\mathcal{D}$, then $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Thus we may assume that e and f belong to the same element H_i in $\mathcal{D}$, where $1 \leqslant i \leqslant k$. We show that $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Let $e = uv$ and let P be the unique $u - v$ path in T, and let u' and v' be the vertices on P adjacent to u and v, respectively. If f is adjacent to at most one of uu' and vv', then either $d(e, uu') \neq d(f, uu')$ or $d(e, vv') \neq d(f, vv')$, and so $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Hence we may assume that f is adjacent to both uu' and vv'. If $u' = v'$, then f is incident with the vertex u'. Since $n \geqslant 5$ and T is a spanning tree, there is a vertex $x \in V(G) - \{u, v, u'\}$ such that x is adjacent in T with exactly one of u, v and u'. If $u'x \in E(T)$, then $d(f, u'x) = 1 \neq 2 = d(e, u'x)$; otherwise, $d(e, ux) = 1 \neq 2 = d(f, ux)$ or $d(e, vx) = 1 \neq 2 = d(f, vx)$, according to whether ux or vx is an edge of T. So $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. If $u' \neq v'$, then we may assume that f is incident with u'. Let g be an edge of T distinct from uu' that is incident with u'. Then $d(e, g) = 2 \neq 1 = d(f, g)$. Therefore, $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$. Therefore, $\mathcal{D}$ is a connected resolving decomposition of G and so $\mathrm{cd}(G) \leqslant |\mathcal{D}| = n + k - 1 = n + f(G) - 1$. $\square$

Note that if $G = K_{1,n-1}$, where $n \geqslant 5$, then $f(G) = 0$ and $\mathrm{cd}(G) = n - 1$. Thus the upper bound in Theorem 3.5 is attainable for stars. On the other hand, the inequality in Theorem 3.5 can be strict. For example, the graph G of Figure 4 has order $n = 8$

and $f(G) = 2$. Since $\mathcal{D} = \{G_1, G_2, G_3\}$, where $E(G_1) = \{e_1, e_2, e_3, e_5, e_7, e_8, e_9\}$, $E(G_2) = \{e_4\}$, and $E(G_3) = \{e_6\}$, is a connected resolving decomposition of G, it then follows by Theorem 3.1 that $\mathrm{cd}(G) = 3$. Therefore, $\mathrm{cd}(G) < n + f(G) - 1$ for the graph of Figure 4.

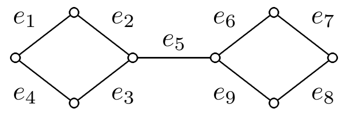

Figure 4. A graph G with $\mathrm{cd}(G) < n + f(G) - 1$

4. Connected decomposition numbers of trees

Although the decomposition dimensions of trees that are not paths have been studied in [2], [4], there is no general formula for the decomposition dimension of a tree that is not a path. However, we are able to establish a formula for the connected decomposition number of a tree that is not a path. First, we need some additional definitions.

A vertex of degree at least 3 in a connected graph G is called a *major vertex* of G. An end-vertex u of G is said to be a *terminal vertex of a major vertex* v of G if $d(u, v) < d(u, w)$ for every other major vertex w of G. The *terminal degree* $\mathrm{ter}(v)$ of a major vertex v is the number of terminal vertices of v. A major vertex v of G is an *exterior major vertex* of G if it has positive terminal degree. Let $\sigma(G)$ denote the sum of the terminal degrees of the major vertices of G and let $\mathrm{ex}(G)$ denote the number of exterior major vertices of G. If G is a tree that is not path, then $\sigma(G)$ is the number of end-vertices of G. For example, the tree T of Figure 5 has four major vertices, namely, v_1, v_2, v_3, v_4. The terminal vertices of v_1 are u_1 and u_2, the terminal vertices of v_3 are u_3, u_4, and u_5, and the terminal vertices of v_4 are u_6 and u_7. The major vertex v_2 has no terminal vertex and so v_2 is not an exterior major vertex of T. Therefore, $\sigma(T) = 7$ and $\mathrm{ex}(T) = 3$.

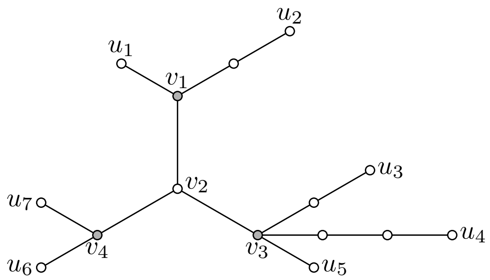

Figure 5. A tree with its exterior major vertices

In this section, we present a formula for the connected decomposition number of a tree T that is not a path in term of $\sigma(T)$ and $\mathrm{ex}(T)$. In order to do this, we first present a useful lemma. For an ordered set $W = \{e_1, e_2, \ldots, e_k\}$ of edges in a connected graph G and an edge e of G, the k-vector

$$c_W(e) = (d(e, e_1), d(e, e_2), \ldots, d(e, e_k))$$

is referred to as the *code of e with respect to W*. For a cut-vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ in G is called a *branch of G at v*. For a bridge e in a connected graph G and a component F of $G - e$, the subgraph F together the bridge e is called a *branch of G at e*. For two edges $e = u_1u_2$ and $f = v_1v_2$ in G, an *$e - f$ path* in G is a path with its initial edge e and terminal edge f.

Lemma 4.1. *Let T be a tree that is not a path, having order $n \geqslant 4$ and p exterior major vertices $v_1, v_2, \ldots, v_p$. For $1 \leqslant i \leqslant p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of v_i, let P_{ij} be the $v_i - u_{ij}$ path $(1 \leqslant j \leqslant k_i)$, and let x_{ij} be a vertex in P_{ij} that is adjacent to v_i. Let*

$$W = \{v_i x_{ij} \colon 1 \leqslant i \leqslant p \ \text{ and } \ 2 \leqslant j \leqslant k_i\}.$$

Then $c_W(e) \neq c_W(f)$ for each pair e, f of distinct edges of T that are not edges of P_{ij} for $1 \leqslant i \leqslant p$ and $2 \leqslant j \leqslant k_i$.

Proof. Let e and f be two edges of T that are not edges of P_{ij} for $1 \leqslant i \leqslant p$ and $2 \leqslant j \leqslant k_i$. We consider two cases.

Case 1. *e lies on some path P_{i1} for some i with $1 \leqslant i \leqslant p$.* There are two subcases.

Subcase 1.1. *There is an edge $w \in W$ such that f lies on the $e - w$ path or e lies on the $f - w$ path.* Then either $d(f, w) < d(e, w)$ or $d(e, w) < d(f, w)$. In either case, $c_W(e) \neq c_W(f)$.

Subcase 1.2. *Every path between f and an edge of W does not contain e and every path between e and an edge of W does not contain f.* Necessarily, then f lies on some path $P_{\ell 1}$ in T for some $1 \leqslant \ell \leqslant p$. Observe that $i \neq \ell$, for otherwise, f lies on $e - w$ path, where $w = v_i x_{i2} \in W$. Since v_i and v_ℓ are exterior major vertices, it follows that $\deg v_i \geqslant 3$ and $\deg v_\ell \geqslant 3$. Thus there exist a branch B_1 at v_i that does not contain x_{i1} and a branch B_2 at v_ℓ that does not contain $x_{\ell 1}$. Necessarily, each of B_1 and B_2 must contain an edge of W. Let w_1 and w_2 be two edges in W such that w_i belongs to B_i for $i = 1, 2$. If $d(e, w_2) \neq d(f, w_2)$, then $c_W(e) \neq c_W(f)$. Thus we may assume that $d(e, w_2) = d(f, w_2)$. However, then $d(e, w_1) < d(f, w_1)$, again implying that $c_W(e) \neq c_W(f)$.

C a s e 2. *e lies on no path P_{i1} for all i with $1 \leqslant i \leqslant p$.* Then there are at least two branches at e, say B_1^* and B_2^*, each of which contains some exterior major vertex of terminal degree at least 2. Thus each branch B_i^* $(i = 1, 2)$ contains an edge in W. Let $w_i^* \in W$ such that w_i^* belongs to B_i^* for $i = 1, 2$. First, assume that $f \in E(B_1^*)$. Then the $f - w_2^*$ path of T contains e. So $d(e, w_2^*) < d(f, w_2^*)$, implying that $c_W(e) \neq c_W(f)$. Next, assume that $f \notin E(B_1^*)$. Then the $f - w_1^*$ path of T contains e. Thus $d(e, w_1^*) < d(f, w_1^*)$ and so $c_W(e) \neq c_W(f)$. □

We are now prepared to establish a formula for the connected decomposition number of a tree that is not a path.

Theorem 4.2. *If T is a tree that is not a path, then*

$$\mathrm{cd}(T) = \sigma(T) - \mathrm{ex}(T) + 1.$$

P r o o f. Suppose that T contains p exterior major vertices $v_1, v_2, \ldots, v_p$. For each i with $1 \leqslant i \leqslant p$, let $u_{i1}, u_{i2}, \ldots, u_{ik_i}$ be the terminal vertices of v_i. For each pair i, j of integers with $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant k_i$, let P_{ij} be the $v_i - u_{ij}$ path in T and let x_{ij} be a vertex in P_{ij} that is adjacent to v_i.

First, we claim that if $\mathcal{D}$ is a connected resolving decomposition of T, then, for each fixed exterior major vertex v_i $(1 \leqslant i \leqslant p)$, there is at least one edge, say e_{ij}, from each path P_{ij} $(1 \leqslant j \leqslant k_i)$ such that the k_i edges e_{ij} $(1 \leqslant j \leqslant k_i)$ of T belong to distinct elements in $\mathcal{D}$. To verify this claim, assume, to the contrary, that this is not the case. Since each element in $\mathcal{D}$ is connected, we assume, without loss of generality, that P_{i1} and P_{i2} are contained in the same element of $\mathcal{D}$. However, then, $d(v_i x_{i1}, e) = d(v_i x_{i2}, e)$ for all $e \in E(G - (P_{i1} \cup P_{i2}))$, and so $c_{\mathcal{D}}(v_i x_{i1}) = c_{\mathcal{D}}(v_i x_{i2})$, which is a contradiction. Therefore, for each fixed i with $1 \leqslant i \leqslant p$, the k_i edges $e_{ij} \in E(P_{ij})$ $(1 \leqslant j \leqslant k_i)$ belong to distinct elements in $\mathcal{D}$, as claimed.

First, we show that $\mathrm{cd}(T) \geqslant \sigma(T) - \mathrm{ex}(T) + 1$. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_\ell\}$ be a minimum connected resolving decomposition of T. Let $V = \{v_1, v_2, \ldots, v_p\}$ be the set of the exterior major vertices of T. First, assume that $p = 1$. Since the k_1 edges $e_{1j} \in E(P_{1j})$ $(1 \leqslant j \leqslant k_1)$ belong to distinct elements in $\mathcal{D}$, it follows that $\mathrm{cd}(G) \geqslant k_1 = \sigma(T) - \mathrm{ex}(T) + 1$. Thus we may assume that $p \geqslant 2$. We proceed by the following steps:

S t e p 1. Since $p \geqslant 2$, there exists an exterior major vertex v_i with $1 \leqslant i \leqslant p$ such that $\deg v_i = k_i + 1$. Start with such an exterior major vertex, say v_1 with $\deg v_1 = k_1 + 1$. Since the k_1 edges $e_{1j} \in E(P_{1j})$ $(1 \leqslant j \leqslant k_1)$ belong to distinct elements in $\mathcal{D}$, we may assume, without loss of generality, that $e_{1j} \in E(G_j)$ for $1 \leqslant j \leqslant k_1$. Thus

$$\mathrm{cd}(G) = |\mathcal{D}| \geqslant k_1 = (k_1 - 1) + 1.$$

S t e p 2. Consider an exterior major vertex $v \in V - \{v_1\}$ such that the $v_1 - v$ path in T contains no other exterior major vertices in $V - \{v_1, v\}$. We may assume that $v = v_2$. Then the k_2 edges $e_{2j} \in E(P_{2j})$ $(1 \leqslant j \leqslant k_2)$ belong to distinct elements in $\mathcal{D}$. We claim that at most one of the edges e_{2j} $(1 \leqslant j \leqslant k_2)$ belongs to the elements $G_1, G_2, \ldots, G_{k_1}$ of $\mathcal{D}$. Assume, to the contrary, that two edges in $\{e_{2j}\colon 1 \leqslant j \leqslant k_2\}$ belong to $G_1, G_2, \ldots, G_{k_1}$, say e_{21} and e_{22} belong to $G_1, G_2, \ldots, G_{k_1}$. Since e_{21} and e_{22} belong to distinct elements in $\mathcal{D}$, it follows that e_{21} and e_{22} belong to two distinct elements of $G_1, G_2, \ldots, G_{k_1}$, say $e_{21} \in E(G_1)$ and $e_{22} \in E(G_2)$. However, then, either G_1 or G_2 must be disconnected, which is a contradiction. Hence, as claimed, at most one of the edges e_{2j} $(1 \leqslant j \leqslant k_2)$ belongs to the elements $G_1, G_2, \ldots, G_{k_1}$ in $\mathcal{D}$. Then assume, without loss of generality, that $e_{2j} \in E(G_{j+k_1})$ for $1 \leqslant j \leqslant k_2 - 1$. Thus $G_1, G_2, \ldots, G_{k_1}, G_{k_1+1}, \ldots, G_{k_1+k_2-1}$ must be distinct elements of $\mathcal{D}$, implying that

$$\mathrm{cd}(G) = |\mathcal{D}| \geqslant k_1 + k_2 - 1 = (k_1 - 1) + (k_2 - 1) + 1.$$

If $p = 2$, then $k_1 + k_2 - 1 = \sigma(T) - \mathrm{ex}(T) + 1$ and the proof is complete. Otherwise, we continue to the next step.

S t e p 3. Consider an exterior major vertex $v \in V - \{v_1, v_2\}$ such that the $v_1 - v$ path in T contains no other exterior major vertices in $V - \{v_1, v_2\}$. We may assume that $v = v_3$. Then the k_3 edges $e_{3j} \in E(P_{3j})$ $(1 \leqslant j \leqslant k_3)$ belong to distinct elements in $\mathcal{D}$. Again, we claim that at most one of the edges $e_{3j} \in E(P_{3j})$ $(1 \leqslant j \leqslant k_3)$ belongs to some element G_i of $\mathcal{D}$, where $1 \leqslant i \leqslant k_1 + k_2 - 1$. Assume, to the contrary, that two edges in $\{e_{3j}\colon 1 \leqslant j \leqslant k_2\}$ belong to G_s and G_t, respectively, where $1 \leqslant s < t \leqslant k_1 + k_2 - 1$, say $e_{31} \in E(G_s)$ and $e_{32} \in E(G_t)$. If $1 \leqslant s < t \leqslant k_1$ or $k_1 + 1 \leqslant s < t \leqslant k_1 + k_2 - 1$, then at least one of G_s and G_t must be disconnected, which is impossible. On the other hand, if $1 \leqslant s \leqslant k_1$ and $k_1 + 1 \leqslant t \leqslant k_1 + k_2 - 1$, then, since G_s and G_t are connected, there must be a cycle in T, which is again impossible. Thus, we may assume, without loss of generality, that $e_{3j} \in E(G_{k_1+k_2-1+j})$ for $1 \leqslant j \leqslant k_3 - 1$. Hence all subgraphs G_i $(1 \leqslant i \leqslant k_1 + k_2 + k_3 - 2)$ are distinct elements of $\mathcal{D}$ and so

$$\mathrm{cd}(G) = |\mathcal{D}| \geqslant k_1 + k_2 + k_3 - 2 = (k_1 - 1) + (k_2 - 1) + (k_3 - 1) + 1.$$

We continue this procedure to the remaining exterior major vertices in $V - \{v_1, v_2, v_3\}$ and repeat the argument similar to the one in the previous step until we exhaust all vertices in V. Then we obtain

$$\mathrm{cd}(G) = |\mathcal{D}| \geqslant \left(\sum_{i=1}^{p}(k_i - 1)\right) + 1 = \sigma(G) - \mathrm{ex}(G) + 1.$$

Next we show that $\mathrm{cd}(T) \leqslant \sigma(T) - \mathrm{ex}(T) + 1$. Let $k = \sigma(T) - \mathrm{ex}(T) + 1$. Let $f_{ij} = v_i x_{ij}$ for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant k_i$. Let $U = \{v_1, u_{11}, u_{21}, \ldots, u_{p1}\}$ and let T_0 be the subtree of T of smallest size such that T_0 contains U. Let

$$\mathcal{D} = \{T_0, P_{12}, P_{13}, \ldots, P_{1k_1}, P_{22}, P_{23}, \ldots, P_{2k_2}, \ldots, P_{p2}, P_{p3}, \ldots, P_{pk_p}\}.$$

Certainly, $\mathcal{D}$ is a connected k-decomposition of T. We show that $\mathcal{D}$ is a resolving decomposition of T. It suffices to show that the edges of T belonging to same element of $\mathcal{D}$ have distinct $\mathcal{D}$-codes. Let $e, f \in E(T)$. We consider two cases.

Case 1. $e, f \in E(T_0)$. Then $d(e, P_{ij}) = d(e, f_{ij})$ and $d(f, P_{ij}) = d(f, f_{ij})$ for all pairs i, j with $1 \leqslant i \leqslant p$ and $2 \leqslant j \leqslant k_i$. Let

$$W = \{f_{ij} \colon 1 \leqslant i \leqslant p \text{ and } 2 \leqslant j \leqslant k_i\}.$$

By Lemma 4.1, $c_W(e) \neq c_W(f)$. Observe that the first coordinate in each of $c_{\mathcal{D}}(e)$ and $c_{\mathcal{D}}(f)$ is 0, the remaining $k-1$ coordinates of $c_{\mathcal{D}}(e)$ are exactly those of $c_W(e)$, and the remaining $k-1$ coordinates of $c_{\mathcal{D}}(f)$ are exactly those of $c_W(f)$. Since $c_W(e) \neq c_W(f)$, it follows that $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$.

Case 2. $e, f \in E(P_{ij})$, where $1 \leqslant i \leqslant p$ and $2 \leqslant j \leqslant k_i$. Then $d(e, T_0) = d(e, f_{i1})$ and $d(f, T_0) = d(f, f_{i1})$. Since e and f are two distinct edges in the path P_{ij}, it follows that $d(e, f_{i1}) \neq d(f, f_{i1})$ and so $d(e, T_0) \neq d(f, T_0)$. Therefore, $c_{\mathcal{D}}(e) \neq c_{\mathcal{D}}(f)$.

Therefore, $\mathcal{D}$ is a connected resolving k-decomposition of T and so $\mathrm{cd}(T) \leqslant k = \sigma(T) - \mathrm{ex}(T) + 1$, as desired. □

5. Graphs with prescribed decomposition dimension and connected decomposition number

We have seen that if G is a connected graph of size at least 2 with $\dim_{\mathrm{d}}(G) = a$ and $\mathrm{cd}(G) = b$, then $2 \leqslant a \leqslant b$. Furthermore, paths of order at least 3 are the only connected graphs G of size at least 2 with $\dim_{\mathrm{d}}(G) = \mathrm{cd}(G) = 2$. Thus there is no connected graph G with $\dim_{\mathrm{d}}(G) = 2$ and $\mathrm{cd}(G) > 2$. On the other hand, every pair a, b of integers with $3 \leqslant a \leqslant b$ is realizable as the decomposition dimension and connected decomposition number, respectively, of some graph. In order to show this, we first present a useful lemma.

Lemma 5.1. *Let G be a connected graph that is not a star. If G contains a vertex that is adjacent to $k \geqslant 1$ end-vertices, then* $\dim_{\mathrm{d}}(G) \geqslant k+1$ *and* $\mathrm{cd}(G) \geqslant k+1$.

Proof. By Observation 1.1, $\dim_{\mathrm{d}}(G) \geqslant k$. Next we show that $\dim_{\mathrm{d}}(G) \neq k$. Assume, to the contrary, that $\dim_{\mathrm{d}}(G) = k$. Let $\mathcal{D} = \{G_1, G_2, \ldots, G_k\}$ be a resolving decomposition of G. Let v be a vertex of G that is adjacent to k end-vertices $v_1, v_2, \ldots, v_k$. Let $e_i = vv_i$, where $1 \leqslant i \leqslant k$. By Observation 1.1, the k edges e_i ($1 \leqslant i \leqslant k$) belong to distinct elements of $\mathcal{D}$. Without loss of generality, assume that $e_i \in E(G_i)$ for $1 \leqslant i \leqslant k$. Since G is not a star, there exists a vertex w distinct from v_i ($1 \leqslant i \leqslant k$) such that w is adjacent to v and w is not an end-vertex of G. We may assume the edge $e = vw$ belongs to G_1. However, then, $c_{\mathcal{D}}(e) = c_{\mathcal{D}}(e_1) = (0, 1, 1, \ldots, 1)$, which is a contradiction. Thus $\dim_{\mathrm{d}}(G) \geqslant k + 1$. The fact that $\mathrm{cd}(G) \geqslant k + 1$ follows by (1). □

Theorem 5.2. *For every pair a, b of integers with $3 \leqslant a \leqslant b$, there exists a connected graph G such that $\dim_{\mathrm{d}}(G) = a$ and $\mathrm{cd}(G) = b$.*

Proof. For $a = b \geqslant 3$, let $G = K_{1,a}$. Since $\dim_{\mathrm{d}}(K_{1,a}) = \mathrm{cd}(K_{1,a}) = a$, the result holds for $a = b$. Thus we may assume that $a < b$. We consider two cases, according to whether $a = 3$ or $a \geqslant 4$.

Case 1. $a = 3$. For each i with $1 \leqslant i \leqslant b-1$, let T_i be the tree obtained from the path $P_i\colon v_{i1}, v_{i2}, \ldots, v_{ii}$ of order i by adding two new vertices u_i and u_i^* and joining u_i and u_i^* to v_{ii}. Then the graph G is obtained from the graphs T_i ($1 \leqslant i \leqslant b-1$) by adding edges $v_{i1}v_{i+1,1}$ for $1 \leqslant i \leqslant b-2$. The graph G is shown in Figure 6 for $b = 5$. Since G is a tree with $\sigma(G) = 2(b-1)$ and $\mathrm{ex}(G) = b - 1$, it follows by Theorem 4.2 that $\mathrm{cd}(G) = b$. It remains to show that $\dim_{\mathrm{d}}(G) = 3$. Let $\mathcal{D} = \{G_1, G_2, G_3\}$, where $E(G_1) = \{u_1v_{11}\}$, $E(G_2) = \{u_iv_{ii}\colon 2 \leqslant i \leqslant d-1\}$, and $E(G_3) = E(G) - (E(G_1) \cup E(G_2))$. We show that $\mathcal{D}$ is a resolving decomposition of G. Observe that $c_{\mathcal{D}}(u_iv_{ii}) = (2i-1, 0, 1)$ for $2 \leqslant i \leqslant b-1$, $c_{\mathcal{D}}(u_1^*v_{11}) = (1, 3, 0)$, $c_{\mathcal{D}}(v_{11}v_{21}) = (1, 2, 0)$, $c_{\mathcal{D}}(v_{i1}v_{i+1,1}) = (i, i, 0)$ for $2 \leqslant i \leqslant b-2$, $c_{\mathcal{D}}(v_{ij}v_{i,j+1}) = (i+j-1, i-j, 0)$ for $j \leqslant i$ and $2 \leqslant i \leqslant b-1$ and $1 \leqslant j \leqslant b-2$, and $c_{\mathcal{D}}(u_i^*v_{ii}) = (2i-1, 1, 0)$ for $2 \leqslant i \leqslant b-1$. Since all $\mathcal{D}$-codes of vertices G are distinct, $\mathcal{D}$ is a resolving decomposition of G and so $\dim_{\mathrm{d}}(G) \leqslant |\mathcal{D}| = 3$. By Theorem A, $\dim_{\mathrm{d}}(G) = 3$.

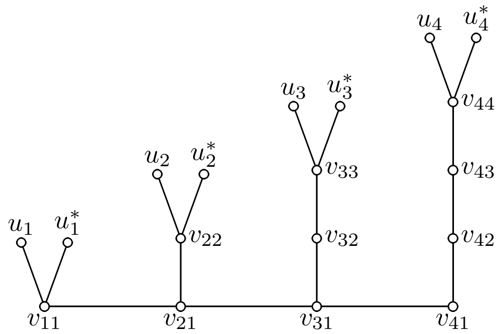

Figure 6. A graph G in Case 1 for $b = 5$

C a s e 2. $a \geqslant 4$. Let G be the graph obtained from the path $P_{b-a+4}\colon u_1, u_2, \ldots, u_{b-a+4}$ of order $b-a+4$ by (1) adding $a-2$ new vertices $v_1, v_2, \ldots, v_{a-2}$ and joining each vertex v_i $(1 \leqslant i \leqslant a-2)$ to u_2 (2) adding a new vertex v_{a-1} and joining v_{a-1} to u_{b-a+3}, and (2) adding $2(b-a)$ new vertices $w_3, w_3^*, w_4, w_4^*, \ldots, w_{b-a+2}, w_{b-a+2}^*$ and joining w_j and w_j^* to u_j for $3 \leqslant j \leqslant b-a+2$. Since G is a tree with $\sigma(G) = (a-1)+2(b-a+1) = 2b-a+1$ and $\operatorname{ex}(G) = b-a+2$, it follows by Theorem 4.2 that $\operatorname{cd}(G) = b$. Next we show that $\dim_{\mathrm{d}}(G) = a$. Since u_2 is adjacent to $a-1$ end-vertices and T is not a star, it then follows by Lemma 5.1 that $\dim_{\mathrm{d}}(G) \geqslant a$. On the other hand, let $\mathcal{D} = \{G_1, G_2, \ldots, G_a\}$, where $E(G_1) = E(P_{b-a+4}) \cup \{u_i w_i\colon 3 \leqslant i \leqslant b-a+2\}$, $E(G_2) = \{u_2 v_1\} \cup \{u_i w_i^*\colon 3 \leqslant i \leqslant b-a+2\}$, $E(G_3) = \{u_{b-a+3} v_{a-1}\}$, and $E(G_i) = \{u_2 v_{i-2}\}$ for $4 \leqslant i \leqslant a$. It can be verified that $\mathcal{D}$ is a resolving decomposition of G, and so $\dim_{\mathrm{d}}(G) \leqslant |\mathcal{D}| = a$. Therefore, $\dim_{\mathrm{d}}(G) = a$, as desired. $\square$

A c k n o w l e d g m e n t s. We are grateful to Professor Gary Chartrand for suggesting the concept of connected resolving decomposition to us and kindly providing useful information on this topic. Also, we thank Professor Peter Slater for the useful conversation.

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