A NOTE ON EQUALITY OF FUNCTIONAL ENVELOPES

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Abstract. We characterize generalized extreme points of compact convex sets. In particular, we show that if the polyconvex hull is convex in $\mathbb{R}^{m\times n}$, $\min(m,n)\leqslant 2$, then it is constructed from polyconvex extreme points via sequential lamination. Further, we give theorems ensuring equality of the quasiconvex (polyconvex) and the rank-1 convex envelopes of a lower semicontinuous function without explicit convexity assumptions on the quasiconvex (polyconvex) envelope.

Keywords: extreme points, polyconvexity, quasiconvexity, rank-1 convexity

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1. Introduction

It was proved in [10], [19] that there exist minimal generators of quasiconvex, rank-1 convex and polyconvex sets called quasiconvex, rank-1 convex and polyconvex extreme points, respectively. Here we show that for compact convex sets in $\mathbb{R}^{m\times n}$, $\min(m,n)\leqslant 2$, all these notions of extreme points coincide and are equal to lamination extreme points which are easy to identify. If $\min(m,n)\geqslant 3$ it still holds true that quasiconvex and rank-1 convex extreme points coincide with lamination extreme points of convex compact sets as shown in [11]. This enables us to prove that if $\min(m,n)=2$ and $K\subset\mathbb{R}^{m\times n}$ and its polyconvex hull is convex then this hull is constructed by the sequential lamination from the set of lamination extreme points of K. An analogous assertion holds for the quasiconvex hull in general dimensions. Therefore our results retrieve theorems from [7], [20] that convexity of the polyconvex hull of a compact set is equivalent to convexity of its rank-1 convex hull if $\min(m,n)\leqslant 2$, and an analogous theorem about the quasiconvex hull.

Then we apply our results to give a sufficient condition for the equality of semiconvex envelopes, and we give a necessary condition for a function with a subquadratic growth to have a nonconvex quasiconvexification.

We say that a function $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is quasiconvex if for any $\varphi \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^m)$, $(0,1)^n$-periodic, and any $A \in \mathbb{R}^{m\times n}$

$$f(A) \leqslant \int_{(0,1)^n} f(A + \nabla\varphi(x))\,\mathrm{d}x.$$

Quasiconvexity plays a crucial role in the calculus of variations. Namely, the sequential weak* lower semicontinuity of $I\colon W^{1,\infty}(\Omega;\mathbb{R}^n) \to \mathbb{R}$, $I(u) = \int_\Omega f(\nabla u(x))\,\mathrm{d}x$, $\Omega \subset \mathbb{R}^n$ a bounded domain, is equivalent to the quasiconvexity of $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$, see [1], [13], [14].

Further, we say that $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is rank-1 convex if for any $A, B \in \mathbb{R}^{m\times n}$, $\operatorname{rank}(A - B) \leqslant 1$ and any $0 \leqslant \lambda \leqslant 1$

$$f(\lambda A + (1-\lambda)B) \leqslant \lambda f(A) + (1-\lambda) f(B).$$

Finally, $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is polyconvex if there is a convex function $h\colon \mathbb{R}^{\sigma(m,n)} \to \mathbb{R}$ such that $f(A) = h(T(A))$ for any $A \in \mathbb{R}^{m\times n}$, where $T(A)$ is the vector of all subdeterminants of A and $\sigma(m,n) = \sum\limits_{i=1}^{\min(m,n)} \binom{m}{i}\binom{n}{i}$; cf. [3], [6].

Rank-1 convexity is a necessary condition for quasiconvexity but it is not a sufficient condition if $m \geqslant 3$ and $n \geqslant 2$ as shown in [18]. On the other hand, polyconvexity is a sufficient condition for quasiconvexity. If $\min(m,n) = 1$ then all the three notions are equivalent to the usual convexity.

Let K be a compact set in $\mathbb{R}^{m\times n}$. Besides the convex hull $C(K)$ we define the quasiconvex hull, rank-1 convex hull and the polyconvex hull, $Q(K)$, $R(K)$ and $P(K)$ of K, respectively (see e.g. [15]) by

$$Q(K) := \{A \in \mathbb{R}^{m\times n};\ f(A) \leqslant \sup_K f\ \forall f\colon \mathbb{R}^{m\times n} \to \mathbb{R} \text{ quasiconvex}\},$$

the other two hulls being defined analogously.

Finally, we define the so-called lamination convex hull $L(K)$ of K as $L(K) := \overline{\bigcup\limits_{i\in\mathbb{N}\cup\{0\}} L_i(K)}$, where $L_0(K) := K$ and

$$L_i(K) := \{\lambda A + (1-\lambda)B;\ 0 \leqslant \lambda \leqslant 1, \operatorname{rank}(A-B) = 1, A, B \in L_{i-1}(K)\},\ i \in \mathbb{N}.$$

Clearly, we have $L(K) \subset R(K) \subset Q(K) \subset P(K)$ and contrary to quasiconvex, rank-1 convex and polyconvex hulls the lamination convex hull is relatively easy to construct.

The notion of quasiconvexity is closely related to gradient Young measures. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $K \subset \mathbb{R}^{m\times n}$ compact. It is known ([4], [8]) that for any sequence $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,\infty}(\Omega;\mathbb{R}^m)$ such that $\nabla u_k(x) \in K$ for almost all

$x \in \Omega$, there exists its subsequence (here denoted in the same way) and a family of probability measures $\{\nu_x\}_{x\in\Omega}$, supported on K and such that for any continuous function $v\colon K \to \mathbb{R}$ and any $b \in L^1(\Omega)$

$$\lim_{k\to\infty} \int_\Omega v(\nabla u_k(x)) b(x)\,\mathrm{d}x = \int_\Omega \int_K v(A)\nu_x(\mathrm{d}A) b(x)\,\mathrm{d}x. \tag{1.1}$$

The family of probability measures $\{\nu_x\}_{x\in\Omega}$ for which the above limit passage holds is called a gradient Young measure generated by $\{\nabla u_k\}_{k\in\mathbb{N}}$. If $\{\nu_x\}_{x\in\Omega}$ is independent of x we call such a measure a homogeneous gradient Young measure. It follows from the analysis by Kinderlehrer and Pedregal [8] who found an explicit characterization of gradient Young measures that a probability measure ν supported on a compact set K is a homogeneous gradient Young measure if and only if for any quasiconvex $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$

$$f\biggl(\int_K A\nu(\mathrm{d}A)\biggr) \leqslant \int_K f(A)\nu(\mathrm{d}A). \tag{1.2}$$

If $\{u_k\}$ is bounded only in some $W^{1,p}(\Omega;\mathbb{R}^m)$, $1 < p < +\infty$, a similar characterization is available. In particular, a necessary condition for a measure $\{\nu_x\}$ to be generated by a sequence in $W^{1,p}(\Omega;\mathbb{R}^m)$ is that

$$f\biggl(\int_{\mathbb{R}^{m\times n}} A\nu_x(\mathrm{d}A)\biggr) \leqslant \int_{\mathbb{R}^{m\times n}} f(A)\nu_x(\mathrm{d}A) \tag{1.3}$$

hold for almost all $x \in \Omega$ and for all $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ which are quasiconvex and have at most p-growth at infinity; cf. [9] for details. Moreover, the limit passage in (1.1) is satisfied with $\mathbb{R}^{m\times n}$ instead of K for all $v\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ continuous with at most p-growth at infinity and $b \in L^\infty(\Omega)$.

A homogeneous gradient Young measure which satisfies (1.2) even for all rank-one convex functions will be called a homogeneous *laminate*, cf. [16], and if it satisfies (1.2) for all polyconvex functions it will be called a homogeneous *polyconvex* measure.

It is well known (see e.g. [2], [15]) that the quasiconvex (rank-1 convex, polyconvex) hull of a compact set $K \subset \mathbb{R}^{m\times n}$ coincides with the set of first moments of all homogeneous gradient Young measures (homogeneous laminates, homogeneous polyconvex measures) supported on K.

We say that A in K is a quasiconvex extreme point of K if the only homogeneous gradient Young measure supported on K with the first moment A is δ_A (Dirac's mass at A). Following Zhang [19] we denote the set of all quasiconvex extreme points of K by $K_{q,e}$. Analogously, we denote by $K_{r,e}$ the set of rank-1 convex extreme points of K, i.e. the set of points which can only be represented by a Dirac mass among all homogeneous laminates, and $K_{p,e}$ stands for those points in K which can only be represented by a Dirac measure among all homogeneous polyconvex measures in K. As homogeneous laminates form a strict subset of homogeneous gradient Young measures ([18]) we have $K_{p,e} \subset K_{q,e} \subset K_{r,e}$. It has been proved in [10], [19] that polyconvex, quasiconvex and rank-1 convex extreme points are the smallest generators of the polyconvex, quasiconvex and rank-1 convex hulls of a compact set $K \subset \mathbb{R}^{m\times n}$. Specifically, e.g. for the polyconvex hull, $P(M) = P(K)$ for a compact set $M \subset K$ if and only if $K_{p,e} \subset M$.

We say that $A \in K$ is a lamination extreme point if the following holds:

$$A = \lambda B + (1-\lambda)C,\ 0 < \lambda < 1,\ B, C \in K,\ \operatorname{rank}(B - C) \leqslant 1 \Rightarrow B = C = A$$

and we denote their set by $K_{l,e}$.

The above definition is a particular case of D-convex extreme points defined in [12]. The generalized notions introduced above of extreme points are in general different as shown in [10, Example 4] for the case $K_{p,e} \neq K_{q,e} \neq K_{r,e}$.

Quasiconvex hulls play an important role in the theory of martensitic transformations. Namely, the quasiconvex hull of the zero set of a material stored energy density is the set of stress free configurations. The rank-1 convex hull is an inner approximation of this set, while the polyconvex hull is an outer approximation. In this sense, our results below assert that if $Q(K)$ is convex then all stress free states can be obtained from $K_{q,e}$ by finite sequential lamination.

Finally, we denote by K_e the set of convex (usual) extreme points of K, by $\operatorname{Aff} K$ the affine hull of K, $\operatorname{Aff} K := A + \operatorname{span}(K - K)$ for any fixed $A \in K$ and the dimension of K, $\dim K := \dim \operatorname{span}(K - K)$. We say that $A, B \in \mathbb{R}^{m\times n}$ are rank-1 connected if $\operatorname{rank}(A - B) = 1$. $\operatorname{Aff} K$ has rank-1 connections if it contains rank-1 connected matrices.

We will need the following lemma. Its first part about $K_{q,e}$ already appeared in [19, Th. 1.4].

Lemma 1.1. *Let $K \subset \mathbb{R}^{m\times n}$ be compact, convex and such that $\operatorname{Aff} K$ does not have rank-1 connections. Then $K = K_{q,e}$. If $\min(m,n) = 2$ then even $K = K_{p,e}$.*

Proof. We follow the proof of [5, Th. 4.1]. Suppose that $A \in K$. We can assume without loss of generality that $A = 0$, as quasiconvex extreme points are translation invariant, i.e., if $B \in K_{q,e}$ then $B - D \in \{C - D;\ C \in K\}_{q,e}$ for a fixed $D \in \mathbb{R}^{m\times n}$. Therefore we can also assume that $\operatorname{Aff} K$ is a linear subspace without rank-1 matrices. Suppose now that there is a homogeneous gradient Young measure ν supported on K such that its first moment $\bar{\nu} = A = 0$ and $\nu \neq \delta_0$. Let the span of $\operatorname{supp} \nu$ have a dimension $0 < l < mn$. Clearly, $\operatorname{span} \operatorname{supp} \nu \subset \operatorname{Aff} K$ and therefore it contains no rank-1 matrices. Let $\{E_i\}$ be an orthonormal basis for $\mathbb{R}^{m\times n}$ such that the first l elements, $\{E_i\}_{i=1}^{l}$, form a basis of the span of $\operatorname{supp} \nu$. Consider the quadratic form

$$\psi(F) = \sum_{i=l+1}^{mn} (E_i \cdot F)^2 - \varepsilon |F|^2, \tag{1.4}$$

where $\varepsilon > 0$ can be chosen such that ψ is rank-1 convex (see [5, proof of Th. 4.1]) and therefore by [6, Ch. 4, Th. 1.7] also quasiconvex. As ν is a gradient Young measure we have from (1.2) with ψ instead of f that

$$0 = \psi(\overline{\nu}) \leqslant \int_K \psi(F)\nu(\mathrm{d}F) = \int_{\operatorname{supp}\nu} \psi(F)\nu(\mathrm{d}F).$$

But if $0 \neq F \in \operatorname{supp}\nu$ we have $\psi(F) = -\varepsilon|F|^2 < 0$, so this is a contradiction as ν was supposed to be non-atomic.

If $\min(m,n) = 2$ and ν is a non-atomic homogeneous polyconvex Young measure then the rank-1 convex ψ in (1.4) is always polyconvex (see [6, Ch. 4, Th. 1.7]) and we reach a contradiction by the same argument. $\square$

Remark 1.2. If $\min(m,n) = 1$ and $K \subset \mathbb{R}^{m\times n}$ has more than one point then, obviously, $\operatorname{Aff} K$ has always rank-1 connections.

2. Generators of convex compact sets

Every compact set $K \subset \mathbb{R}^{m\times n}$ satisfies $P(K) = P(K_{p,e})$, cf. [10], and, in general, $L(K_{p,e}) \subset P(K)$, strictly. If K is convex we have the following result which relies on Lemma 1.1.

Proposition 2.1. *Let $K \subset \mathbb{R}^{m\times n}$ be a compact convex set,* $\min(m,n) \leqslant 2$. *Then* $K = L_{\dim K}(K_{p,e})$.

Proof. We proceed by induction. If $\dim K = 1$ and $\operatorname{Aff} K$ has rank-1 connections then the assertion clearly holds due to the Krein-Milman theorem because $K_e \subset K_{p,e}$; cf. [2, Cor. I.2.4]. If $\operatorname{Aff} K$ has no rank-1 connections then $K = K_{p,e}$ by Lemma 1.1 and $K = L_1(K)$. Suppose that the assertion holds for sets of dimensions less than d and that $\dim K = d$. Take $P \in \partial K$ and let $H \subset \operatorname{Aff} K$ be the supporting hyperplane to K through P. Then $K \cap H$ is compact, convex and its dimension is less than d. By the induction hypothesis $K \cap H = L_{\dim K\cap H}((K\cap H)_{p,e})$. But if $A \in (K\cap H)_{p,e} \subset K \cap H$ then $A \in K_{p,e}$ because K is in one of the halfspaces determined by H. Thus $P \in K \cap H \subset L_{\dim K\cap H}(K_{p,e}) \subset L_{d-1}(K_{p,e})$. If $\operatorname{Aff} K$ has rank-1 connections then $L_d(K) = L_1(\partial K)$. Therefore $K \subset L_1(L_{d-1}(K_{p,e}))$. Hence, $K \subset L_d(K_{p,e})$. On the other hand, as $K_{p,e} \subset K$ we have $L_d(K_{p,e}) \subset K$, i.e., $K = L_d(K_{p,e})$.

If $\operatorname{Aff} K$ has no rank-1 connections then $K = K_{p,e}$ by Lemma 1.1 and $K = L_{\dim K}(K_{p,e})$. $\square$

Corollary 2.2. *Let $K \subset \mathbb{R}^{m\times n}$ be compact,* $\dim K = d$, $\min(m,n) \leqslant 2$, *and such that $P(K)$ is convex. Then $L_d(K_{p,e}) = P(K)$.*

Proof. We have $P(K) = L_d(P(K)_{p,e})$ because $P(K)$ is convex. On the other hand, $P(K)_{p,e} \subset K_{p,e}$. Indeed, if $A \in K$ does not belong to $K_{p,e}$ then there is a homogeneous polyconvex measure ν with the first moment A which is not the Dirac measure and which is supported on K. As $K \subset P(K)$ we see that ν is also supported on $P(K)$ and therefore $A \notin P(K)_{p,e}$. The proof is complete. □

The following result was first obtained by Zhang [20] for $m = n = 2$ and was extended in [7] to the case $\min(m,n) \leqslant 2$. It easily follows from Corollary 2.2.

Corollary 2.3 [7]**.** *Let $K \subset \mathbb{R}^{m\times n}$ be compact,* $\dim K = d$, $\min(m,n) \leqslant 2$ *and such that $P(K)$ is convex. Then $L_d(K) = P(K)$.*

Proof. We have $L_d(K_{p,e}) \subset L_d(K) \subset P(K)$. But due to Corollary 2.2, $L_d(K_{p,e}) = P(K)$ and therefore $L_d(K) = P(K)$. □

We can give a simple geometric description of polyconvex extreme points of convex sets.

Proposition 2.4. *Let $K \subset \mathbb{R}^{m\times n}$,* $\min(m,n) \leqslant 2$ *be a compact convex set. Then $K_{p,e} = K_{q,e} = K_{r,e} = K_{l,e}$. Therefore, $A \in K$ is a polyconvex extreme point of K if and only if there is no segment in K with distinct rank-1 connected endpoints and with A as an inner point.*

Proof. As in general $K_{p,e} \subset K_{p,e} \subset K_{r,e} \subset K_{l,e}$ we must only show that $K_{l,e} \subset K_{p,e}$. We proceed by induction. If $\dim K = 1$ and $\operatorname{Aff} K$ has rank-1 connections then the assertion clearly holds. If $\operatorname{Aff} K$ has no rank-1 connections then $K = K_{p,e}$ by Lemma 1.1 and the assertion holds again. Suppose that the assertion holds for sets of dimensions less than d and that $\dim K = d$. If $\operatorname{Aff} K$ has no rank-1 connections then the argument is the same as for the dimension one, so suppose that $\operatorname{Aff} K$ has rank-1 connections. Take $A \in K_{l,e}$. Then A must be in ∂K and let $H \subset \operatorname{Aff} K$ be the supporting hyperplane to K through A. Then $K \cap H$ is convex, compact and its dimension is less than d. Moreover, by the induction hypothesis and the lamination extremality $A \in (K \cap H)_{l,e} \subset (K \cap H)_{p,e}$. If ν is a polyconvex measure supported on K with the first moment A it must be supported on $K \cap H$ as K entirely stays in one of the halfspaces defined by H and we have that $\nu = \delta_A$, i.e., $A \in K_{p,e}$. This yields $K_{l,e} \subset K_{p,e}$. □

Note that the implication: $\operatorname{Aff} K$ has no rank-1 connections implies $K = K_{p,e}$ has played a key role in the proofs of Propositions 2.1, 2.4. It has been pointed out in [7] that if $\min(m,n) > 2$ then there are homogeneous polyconvex measures supported on subspaces without rank-1 matrices which are not Dirac masses. It follows that Propositions 2.1, 2.4 and Corollaries 2.2 and 2.3 do not extend to this case. Nevertheless, even if $\min(m,n) > 2$ it still holds true that the only homogeneous gradient Young measure supported on subspaces without rank-1 connections is a Dirac mass; cf. [5, Th. 4.1] and Lemma 1.1. Consequently, analogous assertions to Propositions 2.1, 2.4 and Corollaries 2.2, 2.3 hold if one replaces $K_{p,e}$ by $K_{q,e}$ and $P(K)$ by $Q(K)$.

Proposition 2.5 [11]. *Let $K \subset \mathbb{R}^{m\times n}$ be a compact convex set, $\dim K = d$. Then $K_{q,e} = K_{r,e} = K_{l,e}$. Therefore, $A \in K$ is a quasiconvex extreme point of K if and only if there is no segment in K with distinct rank-1 connected endpoints and with A as an inner point. Moreover, $K = L_d(K_{q,e})$.*

The following result was first proved in [20]. Its simpler proof was then given in [7]. It easily follows from Proposition 2.5.

Corollary 2.6 [20]. *Let $K \subset \mathbb{R}^{m\times n}$ be compact. If $Q(K)$ is convex then $L_d(K) = R(K) = Q(K)$, where $d = \dim K$.*

3. Equality of the envelopes

Having a function $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ one can define its quasiconvex envelope $Qf := \sup\{g \leqslant f;\ g \text{ quasiconvex}\}$ and analogously we can define the rank-1 convex envelope Rf, the convex envelope Cf and the polyconvex envelope Pf of f. We have $f \geqslant Rf \geqslant Qf \geqslant Pf \geqslant Cf$; cf. [6]. The following proposition gives a sufficient condition under which $Pf = Qf = Rf$.

First we recall that for a function $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ we define the α-sublevel set $\mathrm{Lev}_\alpha f = \{A \in \mathbb{R}^{m\times n};\ f(A) \leqslant \alpha\}$. It is easy to see that compact sublevel sets of quasiconvex (rank-1 convex, polyconvex) functions are quasiconvex (rank-1 convex, polyconvex) sets, i.e., they coincide with their quasiconvex (rank-1 convex, polyconvex) hulls.

Proposition 3.1. *Let $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be lower semicontinuous, $\min(m,n) \leqslant 2$. Let for some $\alpha \in \mathbb{R}$ the following hold:*

$$\mathrm{Lev}_\alpha Pf \text{ is convex and compact,} \tag{3.1}$$

$$\forall A \in [\mathrm{Lev}_\alpha Pf]_{l,e}\colon\ f(A) = Pf(A). \tag{3.2}$$

Then $\mathrm{Lev}_\alpha Rf = \mathrm{Lev}_\alpha Pf$. In particular, if (3.1)–(3.2) hold for any $\alpha \in \mathbb{R}$ then $Rf = Qf = Pf$.

P r o o f. We have from (3.2) that $[\mathrm{Lev}_\alpha Pf]_{l,e} \subset \mathrm{Lev}_\alpha f$. Further we have

$$\mathrm{Lev}_\alpha Pf \subset P([\mathrm{Lev}_\alpha Pf]_{p,e}) \subset P([\mathrm{Lev}_\alpha Pf]_{l,e}) \subset P(\mathrm{Lev}_\alpha f).$$

As in general $\mathrm{Lev}_\alpha Pf \supset P(\mathrm{Lev}_\alpha f)$ we get

$$\mathrm{Lev}_\alpha Pf = P(\mathrm{Lev}_\alpha f). \tag{3.3}$$

As f is lower semicontinuous we have that $\mathrm{Lev}_\alpha f$ is closed and as it is a subset of the compact set $\mathrm{Lev}_\alpha Pf$ it is also compact. Applying Corollary 2.3 we have $R(\mathrm{Lev}_\alpha f) = \mathrm{Lev}_\alpha Pf$. On the other hand, as $Rf \geqslant Pf$ we have $R(\mathrm{Lev}_\alpha f) \subset \mathrm{Lev}_\alpha Rf \subset \mathrm{Lev}_\alpha Pf$. Therefore, $R(\mathrm{Lev}_\alpha f) = \mathrm{Lev}_\alpha Rf = \mathrm{Lev}_\alpha Pf$. Clearly, if (3.1)–(3.2) hold for any $\alpha \in \mathbb{R}$ then $Rf = Pf$. □

Analogously we obtain the following results which holds independently of m and n.

Proposition 3.2. *Let $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ be lower semicontinuous. Let for some $\alpha \in \mathbb{R}$ the following hold:*

$$\mathrm{Lev}_\alpha Qf \text{ is convex and compact,} \tag{3.4}$$

$$\forall A \in [\mathrm{Lev}_\alpha Qf]_{l,e}\colon\ f(A) = Qf(A). \tag{3.5}$$

Then $\mathrm{Lev}_\alpha Rf = \mathrm{Lev}_\alpha Qf$. In particular, if (3.4)–(3.5) hold for any $\alpha \in \mathbb{R}$ then $Rf = Qf$.

R e m a r k 3.3. As the set of polyconvex extreme points is the smallest generator of the polyconvex hull, (3.2) can be replaced by (3.3) in Proposition 3.1. Indeed, we have shown in the proof of Proposition 3.1 that (3.2) implies (3.3). Conversely, if (3.1) and (3.3) hold, we see that $[\mathrm{Lev}_\alpha Pf]_{p,e} \subset \mathrm{Lev}_\alpha f$. Due to (3.1) and Proposition 2.4 we have that $[\mathrm{Lev}_\alpha Pf]_{l,e} \subset \mathrm{Lev}_\alpha f$, i.e.,

$$\forall\ A \in [\mathrm{Lev}_\alpha Pf]_{l,e}\colon\ f(A) \leqslant \alpha.$$

As $[\mathrm{Lev}_\alpha Pf]_{l,e} \subset \partial(\mathrm{Lev}_\alpha Pf)$ we get that if $A \in [\mathrm{Lev}_\alpha Pf]_{l,e}$ then $\alpha = Pf(A) \leqslant f(A) \leqslant \alpha$. In other words, $f(A) = Pf(A)$ and (3.2) holds.

Similarly, (3.4) & (3.5) is equivalent to (3.4) & $\mathrm{Lev}_\alpha Qf = Q(\mathrm{Lev}_\alpha f)$.

In general it is rather difficult to verify whether $\mathrm{Lev}_\alpha Qf = Q(\mathrm{Lev}_\alpha f)$ but (3.5) turns this problem into a fairly simple algebraic condition.

Note also that we do not require convexity of polyconvex (quasiconvex) envelopes, only convexity of the level sets.

Proposition 3.2 implies the following corollary which can be found (in a more general version) in [21]. We recall that for $f\colon \mathbb{R}^{m\times n} \to \mathbb{R}$ we denote $f^{-1}(0) = \{A \in$

$\mathbb{R}^{m\times n};\ f(A)=0\}$ and that the epigraph of f is the set $\{(A,b)\in\mathbb{R}^{m\times n}\times\mathbb{R};\ b\geqslant f(A)\}$.

Corollary 3.4 [21]**.** *Let* $g\colon\mathbb{R}^{m\times n}\to\mathbb{R}$, $\gamma(|\cdot|^p-1)\leqslant g\leqslant\Gamma(|\cdot|^p+1)$, $p>1$, $\Gamma>\gamma>0$, *and let* $Qg=Cg$. *Then* $Rg=Qg$.

P r o o f. We use a supporting hyperplane construction which appeared in [21]. Let H be the supporting hyperplane to the epigraph of $Qg=Cg$. Then H is the graph of an affine function $\ell\colon\mathbb{R}^{m\times n}\to\mathbb{R}$. Define $f:=g-\ell$. Note that $f\geqslant 0$. We verify the assumptions of Proposition 3.2 for f and $\alpha=0$. Due to affinity of ℓ we have $Qf=Qg-\ell$. We get $\mathrm{Lev}_0Qf=\{A;\ Qf(A)=0\}$. As g has a superlinear growth, so does Cg and Lev_0Qf is compact. Moreover, it is a zero set of a nonnegative convex function $Cg-\ell$. Altogether we have that (3.4) holds.

As $f\geqslant 0$ we have $Qf\geqslant 0$ and by [17, Ch. 4, Lemma 4.2] for any $A\in\mathbb{R}^{m\times n}$ there is a homogeneous gradient Young measure ν^A supported on $\mathbb{R}^{m\times n}$ such that $Qf(A)=\int_{\mathbb{R}^{m\times n}}f(B)\nu^A(\mathrm{d}B)$. In particular, if $Qf(A)=0$ then ν^A must be supported on the zero set of f. We get

$$Q(\mathrm{Lev}_0f)=Q(f^{-1}(0))=\{A;\ Qf(A)=0\}=\mathrm{Lev}_0Qf. \tag{3.6}$$

Provided (3.4) and (3.6) are satisfied for $\alpha=0$, Remark 3.3 implies that the assumptions of Proposition 3.2 are fulfilled. Hence $\mathrm{Lev}_0Rf=\mathrm{Lev}_0Qf$, i.e., $Rg\cap H$ is convex. As H has been an arbitrary hyperplane supporting the epigraph of Cg we get that $Rg=Cg$. □

Corollary 3.5. *Let* $f\colon\mathbb{R}^{m\times n}\to\mathbb{R}$ *be continuous and such that its quasiconvex envelope is nonconvex, let* $\min(m,n)=2$ *and* $\gamma(|\cdot|-1)\leqslant f\leqslant\Gamma(|\cdot|^p+1)$, $p<2$, $\Gamma>\gamma>0$. *Then there is* $\alpha\geqslant\min\limits_{\mathbb{R}^{m\times n}}Cf$ *such that* $P(\mathrm{Lev}_\alpha f)\neq\mathrm{Lev}_\alpha Cf$.

P r o o f. As f has a subquadratic growth its polyconvex envelope is convex and if $P(\mathrm{Lev}_\alpha f)=\mathrm{Lev}_\alpha Pf=\mathrm{Lev}_\alpha Cf$ were true for any $\alpha\geqslant\min\limits_{\mathbb{R}^{m\times n}}Cf$ we would have by Proposition 3.1 that $Rf=Qf=Pf=Cf$ but this cannot be true as Qf is nonconvex. □

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