REGULAR, INVERSE, AND COMPLETELY REGULAR CENTRALIZERS OF PERMUTATIONS

Janusz Konieczny, Fredericksburg

(Received January 21, 2002)

Abstract. For an arbitrary permutation σ in the semigroup T_n of full transformations on a set with n elements, the regular elements of the centralizer $C(\sigma)$ of σ in T_n are characterized and criteria are given for $C(\sigma)$ to be a regular semigroup, an inverse semigroup, and a completely regular semigroup.

Keywords: semigroup of full transformations, permutation, centralizer, regular, inverse, completely regular

MSC 2000: 20M20

1. Introduction

Let $X_n = \{1, 2, \ldots, n\}$. The full transformation semigroup T_n is the set of all mappings from X_n to X_n with composition as the semigroup operation. It has the symmetric group S_n of permutations on X_n as its group of units and it is a subsemigroup of the semigroup PT_n of partial transformations on X_n. (A partial transformation on X_n is a mapping from a subset of X_n to X_n.)

Let S be a semigroup and $a \in S$. The *centralizer* of a relative to S is defined as

$$C(a) = \{b \in S\colon\ ab = ba\}.$$

It is clear that $C(a)$ is a subsemigroup of S.

Centralizers in T_n were studied by Higgins [1], Liskovec and Feĭnberg [6], [7], and Weaver [8]. The author studied centralizers in the semigroup PT_n [3], [4], [5].

In [3], the author determined Green's relations and studied regularity of the centralizers of permutations in the semigroup PT_n. The purpose of this paper is to obtain the corresponding regularity results for the centralizers of permutations in the semigroup T_n.

2. Elements of $C(\sigma)$

For $\alpha \in T_n$ we denote the kernel of α (the equivalence relation $\{(x,y) \in X_n \times X_n\colon x\alpha = y\alpha\}$) by $\mathrm{Ker}(\alpha)$ and the image of α by $\mathrm{Im}(\alpha)$. For $Y \subseteq X_n$, $Y\alpha$ will denote the image of Y under α, that is, $Y\alpha = \{x\alpha\colon x \in Y\}$. As customary in transformation semigroup theory, we write transformations on the right (that is, $x\alpha$ instead of $\alpha(x)$). For a cycle $a = (x_0\, x_1 \ldots x_{k-1})$ we denote $\{x_0, x_1, \ldots, x_{k-1}\}$ by $\mathrm{span}(a)$.

For $\sigma \in S_n$, $C(\sigma)$ will denote the centralizer of σ in T_n, that is,

$$C(\sigma) = \{\alpha \in T_n\colon \sigma\alpha = \alpha\sigma\}.$$

Throughout the paper, we shall use the following characterization of the elements of $C(\sigma)$ ($\sigma \in S_n$), which is a special case of [5, Theorem 5].

Theorem 1. *Let $\sigma \in S_n$ and $\alpha \in T_n$. Then $\alpha \in C(\sigma)$ if and only if for every cycle $(x_0\, x_1 \ldots x_{k-1})$ in σ there is a cycle $(y_0\, y_1 \ldots y_{m-1})$ in σ such that m divides k and for some index i,*

$$x_0\alpha = y_i,\ x_1\alpha = y_{i+1},\ x_2\alpha = y_{i+2}, \ldots,$$

where the subscripts on the y's are calculated modulo m.

Let $\sigma \in S_n$ and $\alpha \in C(\sigma)$. It follows from Theorem 1 that for every cycle a in σ there is a cycle b in σ such that $(\mathrm{span}(a))\alpha = \mathrm{span}(b)$. This justifies the following definition.

Let $\sigma \in S_n$ and let A be the set of cycles in σ (including 1-cycles). For $\alpha \in C(\sigma)$, define a full transformation t_α on A by: for every cycle a in σ,

$$at_\alpha = \text{ the cycle } b \text{ in } \sigma \text{ such that } (\mathrm{span}(a))\alpha = \mathrm{span}(b).$$

For example, consider $\sigma = abc = (1\ 2)(3\ 4\ 5)(6\ 7\ 8\ 9) \in S_9$. Then for

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 4 & 5 & 3 & 1 & 2 & 1 & 2 \end{pmatrix} \in C(\sigma), \quad t_\alpha = \begin{pmatrix} a & b & c \\ a & b & a \end{pmatrix}.$$

We shall frequently use the following lemma. For a cycle a in $\sigma \in S_n$ we denote by $\ell(a)$ the length of a.

Lemma 2. *If $\sigma \in S_n$, a and b are cycles in σ and $\alpha, \beta \in C(\sigma)$ then:*

(1) $t_{\alpha\beta} = t_\alpha t_\beta$.

(2) *If $at_\alpha = b$ then $\ell(b)$ divides $\ell(a)$.*

(3) *$at_\alpha = bt_\alpha$ if and only if $x\alpha = y\alpha$ for some $x \in \mathrm{span}(a)$ and some $y \in \mathrm{span}(b)$.*

P r o o f. Immediate by the definition of t_α and Theorem 1. $\square$

3. Regular $C(\sigma)$

An element a of a semigroup S is called *regular* if $a = axa$ for some $x \in S$. If all elements of S are regular, we say that S is a *regular semigroup* [2, p. 50].

The regular elements of the centralizer of $\sigma \in S_n$ relative to the semigroup PT_n of partial transformations on X_n are described in [3, Lemma 4.1]. This result carries over to the semigroup T_n with slight modifications of the proof.

Let $\sigma \in S_n$ and $\alpha \in C(\sigma)$. For a cycle b in σ we denote by $t_\alpha^{-1}(b)$ the inverse image of b under t_α, that is, the set of all cycles a in σ such that $at_\alpha = b$.

Theorem 3. *Let $\sigma \in S_n$ and $\alpha \in C(\sigma)$. Then α is regular if and only if for every $b \in \operatorname{Im}(t_\alpha)$ there is $a \in t_\alpha^{-1}(b)$ such that $\ell(a) = \ell(b)$.*

Proof. Suppose α is regular, that is, $\alpha = \alpha\beta\alpha$ for some $\beta \in C(\sigma)$. Let $b \in \operatorname{Im}(t_\alpha)$ and select $c \in t_\alpha^{-1}(b)$. Since $t_\alpha = t_\alpha t_\beta t_\alpha$ (by (1) of Lemma 2) and $ct_\alpha = b$, there is a cycle a in σ such that $ct_\alpha = b$, $bt_\beta = a$ and $at_\alpha = b$. Then $a \in t_\alpha^{-1}(b)$ and, by (2) of Lemma 2, $\ell(c) \geqslant \ell(b) \geqslant \ell(a) \geqslant \ell(b)$, implying $\ell(a) = \ell(b)$.

Conversely, suppose that the given condition is satisfied. We define $\beta \in C(\sigma)$ such that $\alpha = \alpha\beta\alpha$. Let $b = (y_0\, y_1 \ldots y_{m-1})$ be a cycle in σ. If $b \notin \operatorname{Im}(t_\alpha)$, define $y_i\beta = y_i$ for $i = 0, 1, \ldots, m-1$. Suppose $b \in \operatorname{Im}(t_\alpha)$. Then, by the given condition, there is a cycle $a = (x_0\, x_1 \ldots x_{m-1})$ in σ such that $at_\alpha = b$. By Theorem 1, there is $i \in \{0, 1, \ldots, m-1\}$ such that

$$x_0\alpha = y_i,\ x_1\alpha = y_{i+1},\ x_2\alpha = y_{i+2}, \ldots,$$

where the subscripts on the y's are calculated modulo m. Define

$$y_i\beta = x_0,\ y_{i+1}\beta = x_1,\ y_{i+2}\beta = x_2, \ldots,$$

where the subscripts on the y's and x's are calculated modulo m. By the construction of β and Theorem 1 we have $\beta \in C(\sigma)$ and $\alpha = \alpha\beta\alpha$, which implies that α is regular. □

The following theorem characterizes the permutations $\sigma \in S_n$ for which the semigroup $C(\sigma)$ is regular. (See [3, Theorem 4.2] for the corresponding result in PT_n.) For positive integers m and k we write $m \mid k$ if m divides k.

Theorem 4. *Let $\sigma \in S_n$. Then $C(\sigma)$ is a regular semigroup if and only if there are no distinct cycles a, b and c in σ such that $\ell(c) \mid \ell(b) \mid \ell(a)$ and $\ell(b) < \ell(a)$.*

Proof. Suppose there are distinct cycles

$$a = (x_0\, x_1 \ldots x_{k-1}), \quad b = (y_0\, y_1 \ldots y_{m-1}) \text{ and } c = (z_0\, z_1 \ldots z_{p-1})$$

in σ such that $p \mid m \mid k$ and $m < k$. Define $\alpha \in T_n$ by

$$x_0\alpha = y_0,\ x_1\alpha = y_1,\ x_2\alpha = y_2, \ldots; \quad y_0\alpha = z_0,\ y_1\alpha = z_1,\ y_2\alpha = z_2, \ldots,$$

where the subscripts on the y's are calculated modulo m and the subscripts on the z's are calculated modulo p, and $x\alpha = x$ for any other $x \in X_n$. By the construction of α and Theorem 1, $\alpha \in C(\sigma)$ and $t_\alpha^{-1}(b) = \{a\}$. Since $\ell(a) = k > m = \ell(b)$, it follows by Theorem 3 that α is not regular, and so $C(\sigma)$ is not a regular semigroup.

Conversely, suppose that $C(\sigma)$ is not a regular semigroup. Let $\alpha \in C(\sigma)$ be a nonregular element. By Theorem 3, there is $b \in \mathrm{Im}(t_\alpha)$ such that there is no $a \in t_\alpha^{-1}(b)$ with $\ell(a) = \ell(b)$. Select $a \in t_\alpha^{-1}(b)$. Then, by (2) of Lemma 2 and the fact that $\ell(a) \neq \ell(b)$, we have $\ell(b) \mid \ell(a)$ and $\ell(b) < \ell(a)$. Note that $a \neq b$. Let $c = bt_\alpha$. Then $\ell(c) \mid \ell(b)$, $c \neq a$ (since $\ell(c) \mid \ell(b)$ and $\ell(b) < \ell(a)$) and $c \neq b$ (since $b \notin t_\alpha^{-1}(b)$). Thus a, b and c are distinct cycles in σ such that $\ell(c) \mid \ell(b) \mid \ell(a)$ and $\ell(b) < \ell(a)$. This concludes the proof. □

For example, for $\sigma = (1)(2\ 3)(4\ 5)(6\ 7\ 8)$ and $\varrho = (1\ 2)(3\ 4)(5\ 6\ 7\ 8)$ in S_8, $C(\sigma)$ is a regular semigroup, whereas $C(\varrho)$ is not regular. We note that it follows from [3, Theorem 4.2] that the centralizer of σ relative to PT_n is not a regular semigroup.

4. Inverse $C(\sigma)$ and completely regular $C(\sigma)$

Inverse semigroups and completely regular semigroups are two important classes of regular semigroups.

An element a' in a semigroup S is called an *inverse* of $a \in S$ if $a = aa'a$ and $a' = a'aa'$. If every element of S has exactly one inverse then S is called an *inverse semigroup* [2, p. 145].

If every element of a semigroup S is in some subgroup of S then S is called a *completely regular semigroup* [2, p. 103].

In the class of centralizers of permutations relative to PT_n, inverse semigroups and completely regular semigroups coincide [3, Theorem 4.3]. We find that in the class of centralizers of permutations relative to T_n, the subclass of inverse semigroups is properly included in the subclass of completely regular semigroups.

To prove the next theorem we use the result that a semigroup S is an inverse semigroup if and only if it is regular and its idempotents commute [2, Theorem 5.1.1]. (An element $e \in S$ is called an idempotent if $ee = e$.) Let $\varepsilon \in T_n$ be an idempotent. Then for every $x \in X_n$, $(x\varepsilon)\varepsilon = x(\varepsilon\varepsilon) = x\varepsilon$. It follows that any idempotent in T_n fixes every element of its image.

Theorem 5. *Let $\sigma \in S_n$. Then $C(\sigma)$ is an inverse semigroup if and only if there are no distinct cycles a and b in σ such that either $\ell(b) = \ell(a)$ or $1 < \ell(b) < \ell(a)$ and $\ell(b) \mid \ell(a)$.*

Proof. Let a and b be distinct cycles in σ. Suppose $\ell(b) = \ell(a)$ with $a = (x_0\, x_1 \ldots x_{k-1})$ and $b = (y_0\, y_1 \ldots y_{k-1})$. Define $\varepsilon, \xi \in T_n$ by $x_i\varepsilon = y_i$, $y_i\varepsilon = y_i$, $y_i\xi = x_i$, $x_i\xi = x_i$ $(0 \leqslant i \leqslant k-1)$, and $x\varepsilon = x\xi = x$ for any other $x \in X_n$. Note that $x_0(\varepsilon\xi) = x_0$ and $x_0(\xi\varepsilon) = y_0$.

Suppose $a = (x_0\, x_1 \ldots x_{k-1})$ and $b = (y_0\, y_1 \ldots y_{m-1})$ with $1 < m < k$ and $m \mid k$. Define $\varepsilon, \xi \in T_n$ by

$$x_0\varepsilon = y_0,\ x_1\varepsilon = y_1,\ x_2\varepsilon = y_2, \ldots; \quad x_0\xi = y_1,\ x_1\xi = y_2,\ x_2\xi = y_3, \ldots,$$

where the subscripts on the y's are calculated modulo m, and $x\varepsilon = x\xi = x$ for any other $x \in X_n$. Note that $x_0(\varepsilon\xi) = y_0$ and $x_0(\xi\varepsilon) = y_1$.

In both cases, by the construction of ε and ξ and Theorem 1, we have that ε and ξ are idempotents in $C(\sigma)$ such that $\varepsilon\xi \neq \xi\varepsilon$. Thus, since idempotents in an inverse semigroup commute, the existence of distinct cycles a and b in σ with either $\ell(b) = \ell(a)$ or $1 < \ell(b) < \ell(a)$ and $\ell(b) \mid \ell(a)$ implies that $C(\sigma)$ is not an inverse semigroup.

Conversely, suppose that there are no distinct cycles a and b in σ satisfying the given condition (either $\ell(b) = \ell(a)$ or $1 < \ell(b) < \ell(a)$ and $\ell(b) \mid \ell(a)$). Then there are no distinct cycles a, b and c in σ satisfying the condition given in Theorem 4 ($\ell(c) \mid \ell(b) \mid \ell(a)$ and $\ell(b) < \ell(a)$), and so $C(\sigma)$ is a regular semigroup.

Let $\varepsilon, \xi \in C(\sigma)$ be idempotents. Let $a = (x_0\, x_1 \ldots x_{k-1})$ be a cycle in σ. If there is no cycle b in σ such that $b \neq a$ and $\ell(b) \mid \ell(a)$ then, by Theorem 1, $(\operatorname{span}(a))\varepsilon = (\operatorname{span}(a))\xi = \operatorname{span}(a)$, and so $x_0\varepsilon = x_0\xi = x_0$ (since any idempotent in T_n fixes every element of its image).

Suppose there is a cycle b in σ such that $b \neq a$ and $\ell(b) \mid \ell(a)$. Then, since the given condition is satisfied, b must be a 1-cycle, say $b = (y_0)$, and b is the only 1-cycle in σ. Thus, by Theorem 1 and the fact that ε is an idempotent, $y_0\varepsilon = y_0$ and either $x_0\varepsilon = x_0$ or $x_0\varepsilon = y_0$. Similarly, $y_0\xi = y_0$ and either $x_0\xi = x_0$ or $x_0\xi = y_0$. If $x_0\varepsilon = x_0\xi = x_0$ then $x_0(\varepsilon\xi) = x_0 = x_0(\xi\varepsilon)$. In any of the three remaining cases, $x_0(\varepsilon\xi) = x_0(\xi\varepsilon) = y_0$.

Since a was an arbitrary cycle in σ and x_0 was an arbitrary element in $\operatorname{span}(a)$, it follows that $\varepsilon\xi = \xi\varepsilon$. Thus $C(\sigma)$ is a regular semigroup in which idempotents commute, and so it is an inverse semigroup. $\square$

E. g., for the permutations $\sigma = (1)(2\ 3\ 4)(5\ 6\ 7\ 8)$, $\varrho = (1)(2\ 3)(4\ 5)(6\ 7\ 8)$ and $\delta = (1\ 2)(3\ 4\ 5\ 6\ 7\ 8)$ in S_8, $C(\sigma)$ is an inverse semigroup, whereas $C(\varrho)$ and $C(\delta)$

are regular but not inverse semigroups. We note that it follows from [3, Theorem 4.3] that the centralizer of σ relative to PT_n is not an inverse semigroup.

To determine the permutations $\sigma \in S_n$ for which $C(\sigma)$ is a completely regular semigroup, we need a characterization of Green's $\mathcal{H}$-relation in $C(\sigma)$.

If S is a semigroup and $a, b \in S$, we say that $a\,\mathcal{H}\,b$ if $aS^1 = bS^1$ and $S^1a = S^1b$, where S^1 is S with an identity adjoined. In other words, $a, b \in S$ are $\mathcal{H}$-related if and only if they generate the same right ideal and the same left ideal. The relation $\mathcal{H}$ is one of the five equivalences on S known as *Green's relations* [2, p. 45]. If an $\mathcal{H}$-class H contains an idempotent then H is a maximal subgroup of S [2, Corollary 2.2.6]. Note that it follows that completely regular semigroups are semigroups in which every $\mathcal{H}$-class is a group.

The following theorem was proved in [3, Corollary 3.5] for partial transformations. However, the result is also true for full transformations (with slight modifications of the proof).

Theorem 6. *Let $\sigma \in S_n$ and let $\alpha, \beta \in C(\sigma)$. Then $\alpha\,\mathcal{H}\,\beta$ if and only if* $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\beta)$, $\mathrm{Im}(t_\alpha) = \mathrm{Im}(t_\beta)$, *and for every $c \in \mathrm{Im}(t_\alpha)$ we have*

(a) *if $a \in t_\alpha^{-1}(c)$ then there is $b \in t_\beta^{-1}(c)$ such that $\ell(b)$ divides $\ell(a)$; and*

(b) *if $a \in t_\beta^{-1}(c)$ then there is $b \in t_\alpha^{-1}(c)$ such that $\ell(b)$ divides $\ell(a)$.*

Theorem 7. *Let $\sigma \in S_n$. Then $C(\sigma)$ is a completely regular semigroup if and only if there are no distinct cycles a, b and c in σ such that $\ell(c) \mid \ell(b) \mid \ell(a)$.*

Proof. We shall use the result that a semigroup S is completely regular if and only if for every $a \in S$, a and a^2 are $\mathcal{H}$-related [2, Theorem 2.2.5 and Proposition 4.1.1].

Suppose there are distinct cycles $a = (x_0\, x_1 \ldots x_{k-1})$, $b = (y_0\, y_1 \ldots y_{m-1})$ and $c = (z_0\, z_1 \ldots z_{p-1})$ in σ such that $p \mid m \mid k$. Define $\alpha \in T_n$ by

$$x_0\alpha = y_0,\ x_1\alpha = y_1,\ x_2\alpha = y_2, \ldots; \quad y_0\alpha = z_0,\ y_1\alpha = z_1,\ y_2\alpha = z_2, \ldots,$$

where the subscripts on the y's are calculated modulo m and the subscripts on the z's are calculated modulo p, and $x\alpha = x$ for any other $x \in X_n$. By the construction of α and Theorem 1, $\alpha \in C(\sigma)$. Moreover, by Theorem 6, α and α^2 are not $\mathcal{H}$-related (since $b \in \mathrm{Im}(t_\alpha)$ and $b \notin \mathrm{Im}(t_{\alpha^2})$). Thus $C(\sigma)$ is not a completely regular semigroup.

Conversely, suppose there are no distinct cycles a, b and c in σ such that $\ell(c) \mid \ell(b) \mid \ell(a)$. Let $\alpha \in C(\sigma)$. We shall use Theorem 6 to prove that $\alpha\,\mathcal{H}\,\alpha^2$.

Let a and b be cycles in σ such that $b = at_\alpha$. We claim that $bt_\alpha = a$ or $bt_\alpha = b$. Indeed, if $b = a$ then $bt_\alpha = at_\alpha = b$. Suppose $b \neq a$ and let $c = bt_\alpha$. Then

$\ell(c) \mid \ell(b) \mid \ell(a)$ and so, by the given condition, a, b and c cannot be distinct. Thus, since $b \neq a$, we must have $c = a$ or $c = b$. Hence $bt_\alpha = a$ or $bt_\alpha = b$.

We show that $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\alpha^2)$. It is clear that $\mathrm{Ker}(\alpha) \subseteq \mathrm{Ker}(\alpha^2)$. For the reverse inclusion, suppose that $x, y \in X_n$ are such that $x\alpha^2 = y\alpha^2$. Let a and a' be the cycles in σ such that $x \in \mathrm{span}(a)$ and $y \in \mathrm{span}(a')$, and let $b = at_\alpha$ and $b' = a't_\alpha$. Note that $x\alpha \in \mathrm{span}(b)$ and $y\alpha \in \mathrm{span}(b')$. By the claim, $bt_\alpha = a$ or $bt_\alpha = b$. We consider two cases.

Case 1. $bt_\alpha = a$.

Then $at_\alpha = b$ and $bt_\alpha = a$. Thus, by (2) of Lemma 2, $\ell(a) = \ell(b)$, and so, by Theorem 1, α restricted to $\mathrm{span}(a)$ is one-to-one and α restricted to $\mathrm{span}(b)$ is one-to-one. Since $x\alpha^2 = y\alpha^2$, we have $a(t_\alpha t_\alpha) = at_{\alpha^2} = a't_{\alpha^2} = a'(t_\alpha t_\alpha)$ (by (1) and (3) of Lemma 2). Thus $b't_\alpha = a'(t_\alpha t_\alpha) = a(t_\alpha t_\alpha) = bt_\alpha = a$. Since $b't_\alpha = a'$ or $b't_\alpha = b'$ (by the claim), we have $a = a'$ or $a = b'$.

Suppose $a = a'$. Then $b' = a't_\alpha = at_\alpha = b$, and so $x\alpha, y\alpha \in \mathrm{span}(b)$. Hence, since $(x\alpha)\alpha = (y\alpha)\alpha$ and α restricted to $\mathrm{span}(b)$ is one-to-one, we have $x\alpha = y\alpha$.

Suppose $a = b'$. Then, since $b = at_\alpha$ and $b' = a't_\alpha$, we have $\ell(b) \mid \ell(a) \mid \ell(a')$, and so the cycles a', a and b cannot be distinct. That is, $a' = a$ or $a' = b$ or $a = b$. If $a' = a$ then $b = at_\alpha = a't_\alpha = b' = a$, and so $x\alpha, y\alpha \in \mathrm{span}(a)$. If $a' = b$ then $a = bt_\alpha = a(t_\alpha t_\alpha) = a'(t_\alpha t_\alpha) = b(t_\alpha t_\alpha) = at_\alpha = b$, and so $x\alpha, y\alpha \in \mathrm{span}(a)$. If $a = b$ then clearly $x\alpha, y\alpha \in \mathrm{span}(a)$. Thus, since $(x\alpha)\alpha = (y\alpha)\alpha$ and α restricted to $\mathrm{span}(a)$ is one-to-one, we have $x\alpha = y\alpha$.

Case 2. $bt_\alpha = b$.

As in Case 1, we have $a(t_\alpha t_\alpha) = a'(t_\alpha t_\alpha)$. Thus $b't_\alpha = a'(t_\alpha t_\alpha) = a(t_\alpha t_\alpha) = bt_\alpha = b$. Recall that $b't_\alpha = a'$ or $b't_\alpha = b'$. In the latter case, $b' = b't_\alpha = b$. If $b't_\alpha = a'$ then $b' = a't_\alpha = (b't_\alpha)t_\alpha = bt_\alpha = b$. Thus in any case $b' = b$ and so $x\alpha, y\alpha \in \mathrm{span}(b)$. Since $bt_\alpha = b$, it follows by Theorem 1 that α restricted to $\mathrm{span}(b)$ is one-to-one. Hence, since $(x\alpha)\alpha = (y\alpha)\alpha$, we have $x\alpha = y\alpha$.

Thus in every case $x\alpha = y\alpha$, implying $\mathrm{Ker}(\alpha^2) \subseteq \mathrm{Ker}(\alpha)$. Hence $\mathrm{Ker}(\alpha) = \mathrm{Ker}(\alpha^2)$.

Next we show that $\mathrm{Im}(t_\alpha) = \mathrm{Im}(t_{\alpha^2})$. By (1) of Lemma 2, $\mathrm{Im}(t_{\alpha^2}) = \mathrm{Im}(t_\alpha t_\alpha) \subseteq \mathrm{Im}(t_\alpha)$. Let $b \in \mathrm{Im}(t_\alpha)$, that is, $b = at_\alpha$ for some cycle a in σ. By the claim, $bt_\alpha = a$ or $bt_\alpha = b$. In the former case, $bt_{\alpha^2} = b(t_\alpha t_\alpha) = at_\alpha = b$. In the latter case, $at_{\alpha^2} = a(t_\alpha t_\alpha) = bt_\alpha = b$. Thus $b \in \mathrm{Im}(t_{\alpha^2})$. It follows that $\mathrm{Im}(t_\alpha) = \mathrm{Im}(t_{\alpha^2})$.

Finally we show that (a) and (b) of Theorem 6 are satisfied for every $c \in \mathrm{Im}(t_\alpha)$. Let $c \in \mathrm{Im}(t_\alpha)$. To prove (a), let $a \in t_\alpha^{-1}(c)$, that is, $at_\alpha = c$. By the claim, $ct_\alpha = a$ or $ct_\alpha = c$. Suppose $ct_\alpha = a$. Then $ct_{\alpha^2} = c(t_\alpha t_\alpha) = at_\alpha = c$. Thus $c \in t_{\alpha^2}^{-1}(c)$ and $\ell(c) \mid \ell(a)$ (since $at_\alpha = c$). If $ct_\alpha = c$ then $at_{\alpha^2} = a(t_\alpha t_\alpha) = ct_\alpha = c$, and so $a \in t_{\alpha^2}^{-1}(c)$. It follows that (a) of Theorem 6 is satisfied.

To prove (b), let $a \in t_{\alpha^2}^{-1}(c)$, that is, $at_{\alpha^2} = c$. Thus, by (1) of Lemma 2, $a(t_\alpha t_\alpha) = c$, and so there is a cycle b in σ such that $at_\alpha = b$ and $bt_\alpha = c$. Then $\ell(b) \mid \ell(a)$ and $b \in t_\alpha^{-1}(c)$. It follows that (b) of Theorem 6 is satisfied.

Thus, by Theorem 6, α and α^2 are $\mathcal{H}$-related, and so $C(\sigma)$ is a completely regular semigroup. $\square$

Note that the condition in Theorem 5 (no distinct cycles a and b in σ such that either $\ell(b) = \ell(a)$ or $1 < \ell(b) < \ell(a)$ and $\ell(b) \mid \ell(a)$) is stronger than the condition in Theorem 7 (no distinct cycles a, b and c in σ such that $\ell(c) \mid \ell(b) \mid \ell(a)$). Thus in the class of centralizers of permutations relative to T_n, the subclass of inverse semigroups is included in the subclass of completely regular semigroups.

The inclusion is proper. For example, for $\sigma = (1\ 2)(3\ 4\ 5)(6\ 7\ 8) \in S_8$, $C(\sigma)$ is a completely regular semigroup but not an inverse semigroup.

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Author's address: *Janusz Konieczny*, Department of Mathematics, Mary Washington College, Fredericksburg, VA 22401, U.S.A., e-mail: `jkoniecz@mwc.edu`.