EXISTENCE RESULTS FOR A CLASS OF SEMILINEAR DEGENERATE ELLIPTIC EQUATIONS

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Abstract. We prove existence results for the Dirichlet problem associated with an elliptic semilinear second-order equation of divergence form. Degeneracy in the ellipticity condition is allowed.

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1. Introduction

We consider the semilinear boundary value problem

$$\begin{cases} -\sum\limits_{i,j=1}^{m} \frac{\partial}{\partial x_i}\bigl(a_{ij}(x)\frac{\partial u}{\partial x_j}\bigr) = f(u) \text{ in } \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{1.0}$$

where Ω is a bounded open subset of $\mathbb{R}^m$, f is a real valued function defined on $\mathbb{R}$, and the coefficients $a_{i,j}(x)$ satisfy the ellipticity condition

$$\sum_{i,j=1}^{m} a_{ij}(x)p_i p_j \geqslant \alpha \sum_{i=1}^{m} \nu_i(x)p_i^2 \qquad \text{for a.e. } x \in \Omega \text{ and for any } p \in \mathbb{R}^m$$

with $\nu_i(x)$ satisfying sufficiently general hypotheses.

We obtain some results of existence, uniqueness and boundedness for weak solutions of problem (1.0) with minimal hypotheses on f. Similar results, when f has a natural polynomial growth, have been obtained in [3], [5], [7] and in [8] by pseudomonotone operators' theory, while our proof uses fixed-point theorems. The paper is structured as follows. In Sections 2 and 3 we state hypotheses and results. In Section 4 we establish some useful lemmas and, finally, in Section 5 we prove our main theorems.

2. Functional spaces

Let $\mathbb{R}^m$ be the Euclidean m-space with a generic point $x = (x_1, x_2, \dots, x_m)$, Ω a bounded open subset of $\mathbb{R}^m$. The notation meas_x will indicate the m-dimensional Lebesgue measure.

If $u(x)$ is a measurable function defined in Ω, we will denote by $|u|_p$ $(1 \leqslant p \leqslant \infty)$ the usual norm in the space $L^p(\Omega)$.

Hypothesis 2.1. Let $\nu_i(x)$ $(i = 1, 2, \dots, m)$ be a positive and measurable function defined in Ω such that

$$\nu_i(x) \in L^1(\Omega), \quad \nu_i^{-1}(x) \in L^{g_i}(\Omega)$$

where $\sum\limits_{i}^{m} \frac{1}{g_i} < 2$ $(g_i > 1)$ if $m \geqslant 3$ $\quad (m = 2)$.

The symbol $H^1(\nu_i, \Omega)$ stands for the completion of $C^1(\overline{\Omega})$ with respect to the norm

$$\|u\|_1 = \left(\int_\Omega \left(|u|^2 + \sum_{i=1}^{m} \nu_i(x) \left| \frac{\partial u}{\partial x_i} \right|^2 \right) \mathrm{d}x \right)^{\frac{1}{2}};$$

$H_0^1(\nu_i, \Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^1(\nu_i, \Omega)$.

Finally, $H^{-1}(\nu_i^{-1}, \Omega)$ denotes the dual space of $H_0^1(\nu_i, \Omega)$ (see also [5], [6] and [10] for details concerning the weighted Sobolev spaces).

3. Hypotheses, problems and results

Hypothesis 3.1. The coefficients $a_{ij}(x)$ $(i, j = 1, 2, \dots, m)$ are functions defined and measurable in Ω satisfying

$$a_{ij}(x) = a_{ji}(x),$$
$$\frac{a_{ij}(x)}{\sqrt{\nu_i(x)\nu_j(x)}} \in L^\infty(\Omega) \quad (i, j = 1, 2, \dots, m).$$

Hypothesis 3.2. There exists $\alpha > 0$ such that for almost every x in Ω we have

$$\sum_{i,j=1}^{m} a_{ij}(x) p_i p_j \geqslant \alpha \sum_{i=1}^{m} \nu_i(x) p_i^2 \quad \text{for any } p \in \mathbb{R}^m. \tag{3.1}$$

Let $a\colon H_0^1(\nu_i,\Omega)\times H_0^1(\nu_i,\Omega)\to\mathbb{R}$ be such that

$$a(u,v)=\int_\Omega\sum_{ij=1}^m a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,\mathrm{d}x,$$

and define

$$\tau=\inf_{u\in H_0^1(\nu_i,\Omega)\setminus\{0\}}\frac{a(u,u)}{|u|_2}.$$

In Section 4 we prove the following

Lemma 4.4. *Let us assume that* (2.1), (3.1), (3.2) *hold. Then* $\tau>0$ *and there exists* $u_0\in H_0^1(\nu_i,\Omega)$ *such that* $\tau=a(u_0,u_0)$ *and*

$$a(u,u_0)=\tau\int_\Omega uu_0\,\mathrm{d}x\quad\text{for any } u\in H_0^1(\nu_i,\Omega);$$

moreover, we can choose $u_0\geqslant 0$.

Definition 3.2. Let H be a Hilbert space, f, $g\in C^1(H,\mathbb{R})$, and let

$$E=\{u\in H\colon\ g(u)=0,\quad g'(u)\neq 0\}.$$

A point $u_0\in H$ is a *critical point* of $f|_E$ if $\frac{\mathrm{d}}{\mathrm{d}t}f(h(t))|_{t=0}=0$ for all C^1 paths $h(t)\colon\ ]-\varepsilon,\varepsilon[\to E$ such that $h(0)=u_0$.

Remark 3.3. If there exists $u_0\in E$ such that $f(u_0)=\min\{f(u)\colon\ u\in E\}$, then $(f|_E)'(u_0)=0$.

Theorem 3.4 (see, e.g., [2]). *A point* $u_0\in E$ *is a critical point of* $f|_E$ *if and only if there exists* $\lambda\in\mathbb{R}$ *such that* $f'(u_0)=\lambda g'(u_0)$.

Now, if $f\in C(\mathbb{R})$ satisfies the condition

$$u\in H_0^1(\nu_i,\Omega)\Rightarrow f(u)\in H^{-1}(\nu_i,\Omega),$$

we obtain the following well posed problem

Problem. Find a function $u(x)\in H_0^1(\nu_i,\Omega)$ such that

$$\int_\Omega\sum_{ij=1}^m a_{ij}(x)\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,\mathrm{d}x=(f(u),v)\,(1)\tag{3.1}$$

for any $v(x)\in H_0^1(\nu_i,\Omega)$.

(1) We denote by $(\cdot,\cdot)$ the duality pairing between $H_0^1(\nu_i,\Omega)$ and $H^{-1}(\nu_i,\Omega)$.

A function $u(x)$ satisfying (3.1) is a weak solution of Problem (1.0).

R e m a r k 3.5. When f does not depend on u, $f \in H^{-1}(\nu_i, \Omega)$, the hypotheses (2.1), (3.1), (3.2) are sufficient to ensure existence and uniqueness of a weak solution of problem (1.0), moreover we have

$$\|u\|_{1,0} \leqslant \|f\|_{H^{-1}(\nu_i,\Omega)}.$$

Proof follows from Lemma 4.1 and the Lax-Milgram theorem (see Remark 4.2 for the definition of $\|u\|_{1,0}$).

In Section 5 we prove

Theorem 5.1 (Existence, uniqueness and boundedness)**.** *Let us assume that* (2.1), (3.1), (3.2) *hold and let f be Lipschitz continuous with a Lipschitz constant $L < \tau$.*

Then there exists a unique weak solution $u(x)$ of problem (1.0)*; moreover, $u(x) \in L^\infty(\Omega)$ and*

$$\|u\|_\infty \leqslant \gamma(L, g, m, \operatorname{meas}_x \Omega). \tag{5.0}$$

Theorem 5.2. *Let us assume that* (2.1), (3.1), (3.2) *hold and let f be a bounded continuous function. Then Problem* (1.0) *has a weak solution $u(x)$. Moreover, $u(x) \in L^\infty(\Omega)$ and* (5.0) *holds.*

4. Preliminary lemmas

Lemma 4.1. *If the hypothesis* (2.1) *is satisfied then there exists a constant $C = C(m, g_i, |\nu_i^{-1}|_{g_i})$ such that*

$$|u|_{2^\star} \leqslant C\Big(\int_\Omega \sum_{i=1}^m \nu_i(x)\Big|\frac{\partial u}{\partial x_i}\Big|^2 \,\mathrm{d}x\Big)^{\frac12} \qquad \textit{for all } u \in H_0^1(\nu_i, \Omega), \tag{4.1}$$

where $2^\star = 2m\big(m - 2 + \sum\limits_{i=1}^m \frac{1}{g_i}\big)^{-1}$.

Moreover, the imbedding of $H_0^1(\nu_i, \Omega)$ into $L^2(\Omega)$ is compact.

P r o o f. Let us fix $m_i = \frac{2g_i}{g_i+1}$. Then

$$\left|\frac{\partial u}{\partial x_i}\right|_{m_i} \leqslant |\nu_i^{-1}|_{g_i}^{\frac{1}{2}} \left|\nu_i^{\frac{1}{2}} \frac{\partial u}{\partial x_i}\right|_2. \tag{4.2}$$

Since $\sum\limits_{i=1}^{m} \frac{1}{m_i} = \sum\limits_{i=1}^{m} \frac{g_i+1}{2g_i} = \frac{1}{2}\Bigl(m + \sum\limits_{i=1}^{m} \frac{1}{g_i}\Bigr) > 1$, Sobolev's imbedding theorem yields (see, for instance, [12])

$$|u|_q \leqslant C(m, m_i, q) \prod_{i=1}^{m} \left|\frac{\partial u}{\partial x_i}\right|_{m_i}^{\frac{1}{m}} \tag{4.3}$$

where $q = m\bigl(-1 + \sum\limits_{i=1}^{m} \frac{1}{m_i}\bigr)^{-1}$.

From (4.2) and (4.3) we obtain

$$|u|_{2^\star} \leqslant C \prod_{i=1}^{m} \Bigl(|\nu_i^{-1}|_{g_i}^{\frac{1}{2m}} \left|\nu_i^{\frac{1}{2}} \frac{\partial u}{\partial x_i}\right|_2^{\frac{1}{m}}\Bigr).$$

Now, let $\{u_n\}$ be a sequence of functions of $H_0^1(\nu_i, \Omega)$ with equibounded norms and let $\{\Pi_k\}$ be a sequence of open intervals in Ω such that

1. $\Pi_k \subset \Pi_{k+1}$ for any $k \in \mathbb{N}$,
2. $\lim\limits_{k\to+\infty} \Pi_k = \Omega$,
3. for any closed, bounded subset C of Ω there exists $\bar{k}\colon\ C \subset \Pi_k,\ k \geqslant \bar{k}$.

Let us denote by $W^{1,1}(\Pi_1)$ the usual Sobolev space on the set Π_1.

It follows that the norms of $\{u_n\}$ in $W^{1,1}(\Pi_1)$ are equibounded; in fact, applying the Hölder inequality we obtain the following estimate:

$$\begin{aligned}
\|u_n\|_{W^{1,1}(\Pi_1)} &= \int_{\Pi_1} |u_n| \,\mathrm{d}x + \int_{\Pi_1} \sum_{i=1}^{m} \left|\frac{\partial u_n}{\partial x_i}\right| \mathrm{d}x \\
&\leqslant \Bigl(\int_{\Pi_1} |u_n|^2 \,\mathrm{d}x\Bigr)(\operatorname{meas} \Pi_1)^{\frac{1}{2}} + \sum_{i=1}^{m} \Bigl(\int_{\Pi_1} \frac{1}{\nu_i(x)} \,\mathrm{d}x\Bigr)^{\frac{1}{2}} \|u_n\|_1 \\
&\leqslant \text{const } \|u_n\|_1.
\end{aligned}$$

Due to the compact imbedding of $W^{1,1}(\Pi_1)$ into $L^1(\Pi_1)$ (see e.g. [1]) there is a subsequence $\{u_{1,n}\}$ from $\{u_n\}$ that converges a.e. in Π_1.

The same procedure can be done on each Π_j for $j = 2, 3, \ldots$. Hence we get a system of sequences $\{u_{j,n}\}$, $n, j = 1, 2, \ldots$ (where $\{u_{j,n}\}$ is a subsequence of $\{u_{j-1,n}\}$) such that $\{u_{j,n}\}$ is convergent a.e. in Π_j for $j = 1, 2, \ldots$.

By the diagonals method we obtain that $\{u_{n,n}\}$ converges a.e. in Ω and, by virtue (4.1), in $L^2(\Omega)$.

R e m a r k 4.2. If the hypothesis (2.1) holds, then $\left(\int_\Omega \sum\limits_{i=1}^{m} \nu_i(x)|\frac{\partial u}{\partial x_i}|^2 \,\mathrm{d}x\right)^{1/2}$ constitutes an equivalent norm in $H_0^1(\nu_i,\Omega)$. We will denote this norm by $\|u\|_{1,0}$.

Lemma 4.3. *Let $u(x) \in H_0^1(\nu_i,\Omega)$ and $k \geqslant 0$, then the function $\min(u,k)$ belongs to $H_0^1(\nu_i,\Omega)$.*

P r o o f. Define $v = \min(u,k)$ for $u \in H_0^1(\nu_i,\Omega)$ and let $\{\varphi_n\}$ be a sequence of functions of $C_0^\infty(\Omega)$ such that

$$\lim_{n\to+\infty} \|\varphi_n - u\|_1 = 0.$$

Let $\psi_n = \min(\varphi_n, k)$ for any $n \in \mathbb{N}$.

By regularization, we can prove that ψ_n belongs to $H_0^1(\nu_i,\Omega)$; moreover, because the norms of $\{\psi_n\}$ are equibounded in $H_0^1(\nu_i,\Omega)$, there exists a subsequence that weakly converges in $H_0^1(\nu_i,\Omega)$. On the other hand,

$$|v(x) - \psi_n(x)| \leqslant |u(x) - \varphi_n(x)| \quad \text{a.e. in } \Omega,$$

so $\{\psi_n\}$ converges to v in $L^2(\Omega)$. □

The conclusion now follows easily.

P r o o f o f L e m m a 4.4. We observe that

$$\tau = \inf\left\{a(u,u)\colon\ u \in H_0^1(\nu_i,\Omega),\quad \int_\Omega u^2\,\mathrm{d}x = 1\right\}, \tag{4.4}$$

and we define $f, g\colon\ H_0^1(\nu_i,\Omega) \to \mathbb{R}$ as

$$f(u) = a(u,u), \qquad g(u) = \int_\Omega u^2\,\mathrm{d}x - 1.$$

Let

$$E = \{u \in H_0^1(\nu_i,\Omega)\colon\ g(u) = 0\}.$$

Then

$$\tau = \inf_{u\in E} f(u).$$

Let $\{u_n\}$ be a sequence such that $a(u_n,u_n) \to \tau$; from (3.2) and Remark 4.2 we have that $\{u_n\}$ is bounded in $H_0^1(\nu_i,\Omega)$, so there exist $\{u_{n_k}\}$, $u_0 \in H_0^1(\nu_i,\Omega)$ such that $u_{n_k} \rightharpoonup u_0$ weakly in $H_0^1(\nu_i,\Omega)$. By the compact imbedding of $H_0^1(\nu_i,\Omega)$ into $L^2(\Omega)$ (Lemma 4.1), $u_{n_k} \to u_0$ strongly in $L^2(\Omega)$, which gives $\int_\Omega u_0^2\,\mathrm{d}x = 1$. Therefore $u_0 \in E$.

Finally, by virtue of

$$\tau \leqslant a(u_0, u_0) \leqslant \liminf_{k\to+\infty} a(u_{n_k}, u_{n_k}) = \tau$$

we obtain

$$\tau = a(u_0, u_0)$$

and f attains its minimum at $u_0 \in E$. By Remark 3.3 we have

$$(f|_E)'(u_0) = 0.$$

Accordingly, Theorem 3.4 yields

$$(f)'(u_0) = \lambda(g)'(u_0) \text{ for some } \lambda \in \mathbb{R}$$

or

$$a(u, u_0) = \lambda \int_\Omega u u_0 \,\mathrm{d}x \quad \text{ for any } u \in H_0^1(\nu_i, \Omega).$$

Choosing $u = u_0$ we have

$$\tau = a(u_0, u_0) = \lambda \int_\Omega u_0^2 \,\mathrm{d}x \Rightarrow \tau = \lambda.$$

Obviously $u_0 \in H_0^1(\nu_i, \Omega)$ is such that

$$a(u, u_0) = \tau \int_\Omega u u_0 \,\mathrm{d}x \quad \text{ for any } u \in H_0^1(\nu_i, \Omega).$$

Next, Lemma 4.3 implies that if u satisfies (4.4) then $|u|$ also satisfies (4.4), therefore we can choose u_0 to be non-negative.

5. Proof of main results

Define $G\colon H^{-1}(\nu_i^{-1}, \Omega) \to H_0^1(\nu_i, \Omega)$ as

$$G(g) = w \quad \text{where } w \text{ is a weak solution of } \begin{cases} -\sum\limits_{i,j=1}^{m} \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial w}{\partial x_j}\big) = g \text{ in } \Omega \\ w = 0 \quad \text{on } \partial\Omega \end{cases}$$

Remark 3.5 ensures that G is a linear continuous map. For $u \in H_0^1(\nu_i, \Omega)$ define $F(u) = G(f(u))$. Then a fixed point u of F is a solution of problem (1.0).

P r o o f o f T h e o r e m 5.1. We claim that

$$u \in L^2(\Omega) \Rightarrow f(u) \in L^2(\Omega).$$

Indeed,

$$|f(u)| \leqslant |f(u) - f(0)| + |f(0)| \leqslant L|u| + |f(0)|,$$

thus

$$\int_\Omega |f(u)|^2 \,\mathrm{d}x \leqslant 2L^2 \int_\Omega |u|^2 \,\mathrm{d}x + 2|f(0)|^2 \operatorname{meas}_x \Omega.$$

We proceed to show that F is a contractive mapping. We see at once that

$$|f(u) - f(v)|_2 \leqslant L|u - v|_2 \quad \text{for any } u, v \in H_0^1(\nu_i, \Omega).$$

By (3.1) and Remark 4.2 we deduce that

$$\alpha\|u\|_{1,0}^2 \leqslant a(u,u) = (f(u), u) \leqslant c|f(u)|_2\|u\|_{1,0}$$

or

$$\|u\|_{1,0} \leqslant \frac{c}{\alpha}|f(u)|_2.$$

Consequently, G is continuous from $L^2(\Omega) \to L^2(\Omega)$. Therefore

$$\begin{aligned} |F(u) - F(v)|_2 = |G(f(u) - f(v))|_2 &\leqslant \|G\|_\star |f(u) - f(v)|_2 \\ &\leqslant L\|G\|_\star |u - v|_2. \end{aligned} \tag{5.1}$$

Since $\tau|u|_2^2 \leqslant a(u,u) = \int_\Omega f(u)u \,\mathrm{d}x \leqslant |f(u)|_2|u|_2$ or

$$\frac{|G(f(u))|_2}{|f(u)|_2} \leqslant \frac{1}{\tau},$$

it results that

$$\|G\|_\star \leqslant \frac{1}{\tau}.$$

We conclude from (5.1) that

$$|F(u) - F(v)|_2 \leqslant \frac{L}{\tau}|u - v|_2$$

and finally that, since $L < \tau$, F has a fixed point in $H_0^1(\nu_i, \Omega)$.

Now, let us fix $k \geqslant 0$, then from (3.1) for $v = u - \min(u,k)$ we get

$$\alpha\|v\|_{1,0}^2 \leqslant L \int_\Omega |u||v| \,\mathrm{d}x + \int_\Omega |f(0)||v|. \tag{5.2}$$

Lemma 4.1 and the definition of v imply

$$\begin{aligned}
\int_\Omega |u||v|\,\mathrm{d}x &\leqslant \int_{\Omega(u>k)} v^2\,\mathrm{d}x + k\int_{\Omega(u>k)} v\,\mathrm{d}x\\
&\leqslant |v|_{2^\star}^2[\mathrm{meas}_x\,\Omega(u>k)]^{1-\frac{2}{2^\star}} + k\int_{\Omega(u>k)} v\,\mathrm{d}x\\
&\leqslant c^2\|v\|_{1,0}^2[\mathrm{meas}_x\,\Omega(u>k)]^{1-\frac{2}{2^\star}} + k\int_{\Omega(u>k)} v\,\mathrm{d}x.
\end{aligned}$$

Therefore from (5.2) we obtain

$$\|v\|_{1,0}^2(\alpha - Lc^2[\mathrm{meas}_x\,\Omega(u>k)]^{1-\frac{2}{2^\star}}) \leqslant (Lk+|f(0)|)\int_{\Omega(u>k)} v\,\mathrm{d}x. \tag{5.3}$$

Recalling that

$$\lim_{k\to+\infty} \mathrm{meas}_x\,\Omega(u>k) = 0$$

we can certainly choose $\tilde{k}\geqslant 0$ such that for any $k\geqslant\tilde{k}$ we have

$$Lc^2[\mathrm{meas}_x\,\Omega(u>k)]^{1-\frac{2}{2^\star}} \leqslant \frac{\alpha}{2}.$$

We apply this inequality to (5.3) obtaining

$$\|v\|_{1,0} \leqslant \frac{2c}{\alpha}[\mathrm{meas}_x\,\Omega(u>k)]^{1-\frac{1}{2^\star}}\,(|f(0)|+Lk)\ \text{ for any } k\geqslant\tilde{k}. \tag{5.4}$$

Let h, k be real numbers, $h>k\geqslant\tilde{k}$. Then one has

$$|v|_{2^\star} = \left[\int_{\Omega(u>k)} |u-k|^{2^\star}\,\mathrm{d}x\right]^{\frac{1}{2^\star}} \geqslant (h-k)[\mathrm{meas}_x\,\Omega(u>h)]^{\frac{1}{2^\star}};$$

furthermore, (5.4) and Lemma 4.1 yield

$$[\mathrm{meas}_x\,\Omega(u>h)]^{\frac{1}{2^\star}} \leqslant \frac{2c^2}{\alpha(h-k)}(|f(0)|+Lk)[\mathrm{meas}_x\,\Omega(u>k)]^{1-\frac{1}{2^\star}}. \tag{5.5}$$

Next, if $k>0$, we get

$$\begin{aligned}
\mathrm{meas}_x\,\Omega(u>k) \leqslant \frac{1}{k^{2^\star}}\int_{\Omega(u>k)} u^{2^\star}\,\mathrm{d}x,\ &\frac{2c^2}{\alpha k}(|f(0)|+2Lk)2^{\frac{2^\star-1}{2^\star-2}}[\mathrm{meas}_x\,\Omega(u>k)]^{1-\frac{2}{2^\star}}\\
&\leqslant \frac{2c^2}{\alpha k^{2^\star-1}}(|f(0)|+2Lk)2^{\frac{2^\star-1}{2^\star-2}}\left(\int_{\Omega(u>k)} u^{2^\star}\,\mathrm{d}x\right)^{1-\frac{2}{2^\star}}.
\end{aligned}$$

Now, the first term of the above inequality goes to zero as $k \to +\infty$, so we can fix k_1 ($\geqslant \tilde{k}$) such that

$$\frac{2c^2}{\alpha}\,(|f(0)| + 2Lk_1)\,[\operatorname{meas}_x \Omega(u > k_1)]^{1-\frac{2}{2^\star}}\, 2^{\frac{2^\star-1}{2^\star-2}} \leqslant k_1. \tag{5.6}$$

Moreover, one has

$$\frac{2c^2}{\alpha(h-k)}\,(|f(0)| + Lk) \leqslant \frac{2c^2}{(h-k)}\,(|f(0)| + 2Lk_1) \ \text{ if } 0 \leqslant k \leqslant k_1. \tag{5.7}$$

Combining (5.5) and (5.7) we obtain

$$[\operatorname{meas}_x \Omega(u > h)]^{\frac{1}{2^\star}} \leqslant \frac{2c^2}{\alpha(h-k)}\,(|f(0)| + 2Lk_1)\,[\operatorname{meas}_x \Omega(u > k)]^{1-\frac{1}{2^\star}}$$

for any $h, k \in \mathbb{R}$ such that $k_1 \leqslant k < h \leqslant 2k_1$.

Assuming in $[k_1, +\infty[$ that

$$\varphi(k) = \begin{cases} [\operatorname{meas}_x \Omega(u > k)]^{\frac{1}{2^\star}} \text{ if } k_1 \leqslant k \leqslant 2k_1 \\ 0 \ \text{ if } \ k > 2k_1 \end{cases}$$

we get

$$\varphi(h) \leqslant \frac{2c^2}{\alpha(h-k)}\,(|f(0)| + 2Lk_1)\,[\varphi(k)]^{2^\star-1}$$

for any $h, k \in \mathbb{R}$ such that $k_1 \leqslant k < h \leqslant 2k_1$, and from Stampacchia's lemma (see [11], p. 212) we deduce

$$\varphi(k_1 + d) = 0,$$

where d is the first term of (5.6).

We can obtain the same conclusion for $-u$, so the proof of the theorem is complete.

Proof of Theorem 5.2. Set F as in Theorem 5.1. Since the imbedding of $H_0^1(\nu_i, \Omega)$ into $L^2(\Omega)$ is compact, we have that F is also compact from $L^2(\Omega)$ into $L^2(\Omega)$; therefore, by Schaefer's fixed point theorem, it will be sufficient to prove that the set of all solutions of the equation

$$u = \mu F(u) \quad \text{ for } 0 < \mu < 1 \tag{5.8}$$

is unbounded.

Indeed, if u satisfies (5.8), then u is solution of

$$\begin{cases} -\sum\limits_{i,j=1}^{m} \dfrac{\partial}{\partial x_i}\Big(a_{ij}(x)\dfrac{\partial u}{\partial x_j}\Big) = \mu f(u) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

therefore

$$\tau |u|_2^2 \leqslant a(u,u) = \mu \int_\Omega f(u)u \,\mathrm{d}x \leqslant M\,(\operatorname{meas}_x \Omega)^{\frac{1}{2}}\,|u|_2$$

or

$$|u|_2 \leqslant \frac{M\,(\operatorname{meas}_x \Omega)^{\frac{1}{2}}}{\tau}.$$

Now, if we fix in (3.1) $v = u - \min(u,k)$, $k \geqslant 0$ we get

$$\alpha \|u\|_{1,0}^2 \leqslant M \int_\Omega v \,\mathrm{d}x \leqslant M |v|_{2^\star}\,[\operatorname{meas}_x \Omega(u > k)]^{\frac{2^\star - 1}{2^\star}}.$$

This inequality, as in the previous theorem, implies

$$\|u\|_\infty < +\infty.$$

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