THE CROSSING NUMBER OF THE GENERALIZED PETERSEN GRAPH $P[3k, k]$

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Abstract. Guy and Harary (1967) have shown that, for $k \geqslant 3$, the graph $P[2k, k]$ is homeomorphic to the Möbius ladder M_{2k}, so that its crossing number is one; it is well known that $P[2k, 2]$ is planar. Exoo, Harary and Kabell (1981) have shown hat the crossing number of $P[2k + 1, 2]$ is three, for $k \geqslant 2$. Fiorini (1986) and Richter and Salazar (2002) have shown that $P[9, 3]$ has crossing number two and that $P[3k, 3]$ has crossing number k, provided $k \geqslant 4$. We extend this result by showing that $P[3k, k]$ also has crossing number k for all $k \geqslant 4$.

Keywords: graph, drawing, crossing number, generalized Petersen graph, Cartesian product

MSC 2000: 05C10

1. Preliminaries

The theory of crossing numbers owes its birth to a problem posed by P. Turán in 1952. The first graphs to be studied were the complete bipartite graphs. The *complete bipartite graph* $K_{r,s}$ is the graph of order $(r + s)$ whose vertex-set can be split into two disjoint sets A and B such that $|A| = r$ and $|B| = s$, and in which two vertices are joined by an edge if and only if they belong to different sets. From Turán's days, the problem of crossing numbers extended to other graphs, such as the complete graphs, the Cartesian product of graphs, and the generalized Petersen graphs. The *complete graph* K_t on t vertices is the graph in which every pair of distinct vertices are joined by an edge. The *Cartesian product* of two graphs G and H, denoted by $G \times H$, has the vertexset $V(G) \times V(H)$ and the edgeset specified by joining (u, v) to (u', v') by an edge if and only if

(i) $u = u'$ and $(v, v') \in e(H)$; or

(ii) $v = v'$ and $(u, u') \in e(G)$
for all $u, u' \in V(G)$ and $v, v' \in V(H)$.

The *generalized Petersen graph* $P[m, q]$ is defined to be the graph of order $2m$ whose vertexset is $\{u_1, \dots, u_m, x_1, \dots, x_m\}$ and edgeset is $\{u_i x_i, u_i u_{i+1}, x_i x_{i+q}\colon i = 1, \dots, m;$ addition modulo $m\}$.

A *drawing* of a graph G is a mapping of G into a surface, which in our case will be the Euclidean plane. A *good drawing* is one in which no edge crosses itself, any two edges have at most one point in common (which can either be a vertex or a crossing), and no three edges have a common point other than a vertex. A good drawing D exhibiting the least number of crossings is said to be an *optimal drawing*, and the number of crossings in such a drawing is the *crossing number* of G, denoted by $\nu(G)$.

Two graphs G_1 and G_2 are said to be *homeomorphic* if there exists a graph which can be obtained from each by successively contracting edges incident to vertices of valency two. In this case, we write $G_1 = \mathcal{H}(G_2)$ and $G_2 = \mathcal{H}(G_1)$. The following theorem follows.

Theorem 1.1. *If two graphs G_1 and G_2 are homeomorphic, then $\nu(G_1) = \nu(G_2)$.*

One of the most researched results in graph theory dating to before the 1930s is the characterization of planar graphs. One of the characterizations of these graphs uses K_5 and $K_{3,3}$, and is given by Kuratowski's theorem stated below.

Theorem 1.2 Kuratowski's Theorem [5]**.** *A graph G is planar if and only if G contains no subgraph homeomorphic either to K_5 or to $K_{3,3}$.*

In the sequel, it will be convenient to denote the graph $G = \mathcal{H}(K_{3,3})$ with partite sets $\{x, y, z\}$ and $\{a, b, c\}$ by $K\langle x, y, z; a, b, c\rangle$.

2. Cartesian products

We let S_3 denote the star-graph $K_{1,3}$ and P_n the path-graph with $n+1$ vertices, and consider the graph of the Cartesian product $S_3 \times P_n$, denoting the vertices $(0, i)$, $(1, i)$, $(2, i)$ and $(3, i)$ by h_i, a_i, b_i and c_i, respectively (for $i = 0, \dots, n$), where the vertices h_i represent the hubs of the stars. In the drawing of $S_3 \times P_n$, we delete the path $\Gamma = (h_0, \dots, h_n)$ which passes through the hubs of the stars. We let the subgraph of $((S_3 \times P_n) - \Gamma)$ induced by the vertices h_i, a_i, b_i and c_i be denoted by S^i. Also, the subgraph induced by the vertices h_i, a_i, b_i, c_i, h_{i+1}, a_{i+1}, b_{i+1}, and c_{i+1} is denoted by H^i, so that H^i is made up of S^i and S^{i+1} together with the three edges connecting the two stars. It is easy to prove the result of Lemma 2.1 below, since the upper bound follows from the drawings of Figure 1(a) and (b) for n odd

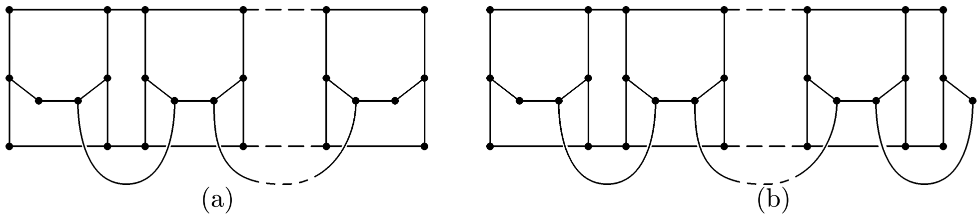

Figure 1

and even, respectively, while the proof of the lower bound follows the same lines as that in Jendrol' and Ščerbová [4].

Lemma 2.1. *Let G denote the graph of the Cartesian product $S_3 \times P_n$ $(n \geqslant 1)$ with the path Γ joining the hubs of the stars deleted, that is, $G := ((S_3 \times P_n) - \Gamma)$. If D is a good drawing of G in which no star S^i $(i = 0, \ldots, n)$ has a crossed edge, then $\nu_D(G) = n - 1$.*

3. The generalized Petersen graph $P[3k, k]$

The generalized Petersen graph $P[3k, k]$ of order $6k$ is made up of an *outer-circuit* $u_i u_{i+1}$, the *spokes* $u_i x_i$, and the 3-*circuits* on the vertices $\{x_i, x_{i+k}, x_{i+2k}\}$, where $i = 1, \ldots, 3k$ and addition is taken modulo $3k$. A drawing of $P[3k, k]$ is shown in Figure 2. We note that by deleting an edge from each of the 3-circuits, we obtain the graph $\mathcal{H}(\mathcal{G}_k)$ where $\mathcal{G}_k$ is shown in Figure 3(a), such that $\mathcal{G}_k \supseteq \mathcal{H}((S_3 \times P_{k-1}) - \Gamma)$ and $\nu(P[3k, k]) \geqslant \nu(\mathcal{G}_k)$. We also note that to obtain $P[3k, k]$ back from $\mathcal{G}_k$ we simply expand the vertex h_i of $\mathcal{G}_k$ into the 3-circuit $x_i x_{i+k} x_{i+2k}$, as illustrated in Figure 3(b). Thus, to get a lower bound for the crossing number of $P[3k, k]$ we can simply consider the crossing number of the graph $\mathcal{G}_k$.

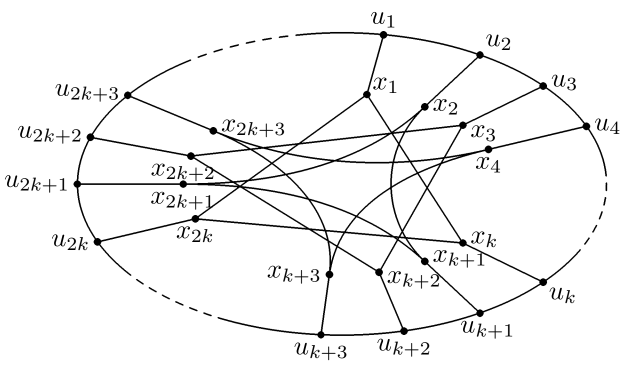

Figure 2

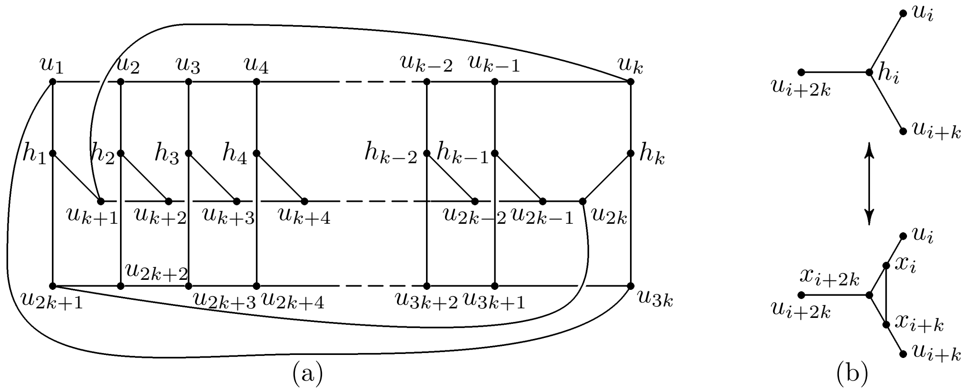

Figure 3

Theorem 3.1. *The crossing number of $\mathcal{G}_k$ is equal to k, for $k \geqslant 4$.*

P r o o f. Figure 3 (a) sets the upper bound equal to k. To show that $\nu(\mathcal{G}_k) \geqslant k$, we assume that t is the least value of k for which $\nu(\mathcal{G}_t) \leqslant t-1$. As we will show in Lemma 3.2, t is greater than 4. We also note that the deletion of the vertex h_i and the edges incident to it (for values of i between 1 and k) from $\mathcal{G}_k$ yields a homeomorph to $\mathcal{G}_{k-1}$. Therefore, since $\mathcal{G}_t$ contains $\mathcal{G}_{t-1}$ as a subgraph, we have $\nu(\mathcal{G}_t) \geqslant \nu(\mathcal{G}_{t-1}) = t-1$, by minimality of t. Thus, we only need to show that $\nu(\mathcal{G}_t) \neq t-1$, by assuming, for contradiction, that it is equal to $t-1$.

We now consider an optimal drawing D of $\mathcal{G}_t$ and assume that a staredge $h_j u_{j+\alpha t}$, for $1 \leqslant j \leqslant t$ and $\alpha = 0, 1, 2$, makes a positive contribution to the crossing number of $\mathcal{G}_t$. In this case, when we delete the hub h_j we get an induced drawing D_1 of a homeomorph of $\mathcal{G}_{t-1}$ such that

$$\begin{aligned} t-1 = \nu(\mathcal{G}_t) &\geqslant \nu_{D_1}(\mathcal{G}_{t-1}) + 1 \\ &\geqslant (t-1)+1, \text{ by the inductive hypothesis,} \\ &= t, \text{ a contradiction.} \end{aligned}$$

Thus, we can assume that all the $(t-1)$ crossings of $\mathcal{G}_t$ are self-intersections of the $3t$-circuit C made up of the edges $u_i u_{i+1}$ for $1 \leqslant i \leqslant 3t$, addition modulo $3t$.

Therefore, there exists an optimal drawing D_2 of $\mathcal{G}_t$ such that in D_2 the edges of the stars do not contribute to the crossing number. We now consider the following two cases.

C a s e 1. If there is an edge e in D_2 which is crossed twice or more, then deleting e together with the two other edges at distance t from e along C, we get a subgraph homeomorphic to $[(S_3 \times P_{t-1}) - \Gamma]$.

Therefore, by Lemma 2.1 above:

$$t-1=\nu(\mathcal{G}_t)\geqslant\nu([(S_3\times P_{t-1})-\Gamma])+2=(t-2)+2=t,\ \text{ a contradiction.}$$

C a s e 2. If there are two edges in D_2 at distance t from each other which are crossed *but do not cross each other*, then repeating the same procedure as in Case 1, we get the same induced drawing, and the result follows similarly.

We can therefore assume hereafter that in D_2 there is no edge which is crossed twice and no two edges at distance t from each other giving a contribution of two to the crossing number.

There remains to show that if in D_2

(i) there are no two edges at a distance t which are not intersected, or

(ii) there are two edges at a distance t from each other which are pairwise intersecting,

then in both cases we get a contradiction.

Let us first assume that no two edges at a distance t from each other can be found such that they are both intersected. We divide the $3t$-circuit C into three t-sectors such that the number of crossed edges in each sector is p, q and r, respectively. This implies that $p+q+r=2(t-1)$. Since in Sector 1 there are p crossed edges which cannot be matched to crossed edges in Sectors 2 and 3, hence in each of Sectors 2 and 3 there are p edges which are not intersected. Similarly for q and r. Thus, the number of uncrossed edges is at least $2(p+q+r)$. However,

total number of edges $\geqslant$ (number of crossed edges) + (number of uncrossed edges)

$\Rightarrow\ 3t\geqslant(p+q+r)+2(p+q+r)$

$\Rightarrow\ t\geqslant(p+q+r)=2t-2$

$\Rightarrow\ t\leqslant 2,$ a contradiction.

We now assume that there are two edges in D_2 at a distance t from each other that intersect each other. That is, if u_iu_{i+1} $(1\leqslant i\leqslant t$, addition taken modulo $t)$ is intersected by an edge e, then $e\in\{u_{i+t}u_{i+t+1},u_{i+2t}u_{i+2t+1}\}$. Without loss of generality, we assume that $e=u_{i+t}u_{i+1+t}$ and consequently, that the edge $u_{i+2t}u_{i+1+2t}$ is not intersected.

We consider the subgraph H induced by $S(h_i)\cup S(h_{i+1})\setminus\{h_iu_{i+t},h_{i+1}u_{i+1+t}\}$ (shown in Figure 4(a)). This is a 6-circuit none of whose edges is intersected, with the sole exception of u_iu_{i+1} which is intersected once by $u_{i+t}u_{i+1+t}$. Thus, H is planarly embedded and, without loss of generality, we let $u_{i+t}\in\operatorname{Int}(H)$ and $u_{i+1+t}\in\operatorname{Ext}(H)$ (since if these vertices are both in $\operatorname{Int}(H)$ or both in $\operatorname{Ext}(H)$, then the edge u_iu_{i+1} is crossed an even number of times). Therefore, we have the subgraph shown in the drawing of Figure 4(b). Now, h_{i+2} either lies in $\operatorname{Int}(H)$ or in $\operatorname{Ext}(H)$, and none of the edges of $S(h_{i+2})$ can be crossed. Also, the edges of the subgraph in

Figure 4(b) cannot be crossed (apart from the crossing shown), giving us the required contradiction. Therefore, $\nu(\mathcal{G}_t) \neq t-1$, implying that it is at least t. $\square$

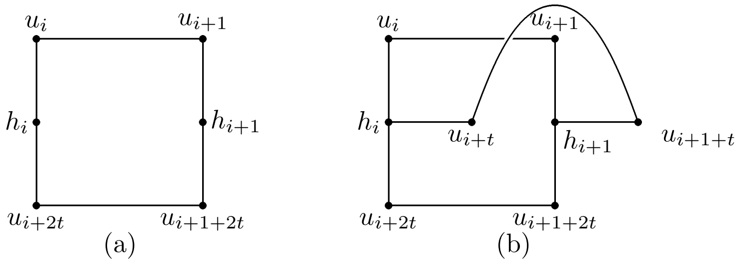

Figure 4

It remains to show that we can start induction with $k=4$.

Lemma 3.2. *The crossing number of $\mathcal{G}_4$ is four.*

Proof. Let α be the least number of edges in $\mathcal{G}_4$ whose deletion yields a planar subgraph. We show that this graph $\mathcal{G}_4$ has crossing number at least four, by assuming that $\alpha = 3$ and get a contradiction. To do this, we consider all the possible combinations in which the three crossed edges may appear. For definiteness, we assume that the vertices are labelled as in Figure 5.

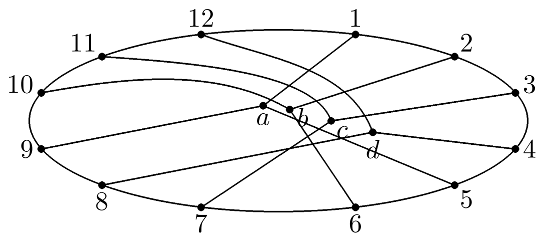

Figure 5

Case 1. All the three crossed edges are incident to the same vertex.

1.1. The three edges are incident to a star-hub.

In this case, deleting the star-hub yields a graph $G_1 = \mathcal{H}(\mathcal{G}_3) = \mathcal{H}(P[9,3])$ which is not planar.

1.2. The three edges are incident to a circuit-vertex.

Without loss of generality, we can assume that it is the vertex 12. Deleting this vertex, we get a subgraph containing the graph G_2 shown in Figure 6, which contains as a subgraph $K\langle a,b,c;2,6,9\rangle$, so that it is not planar.

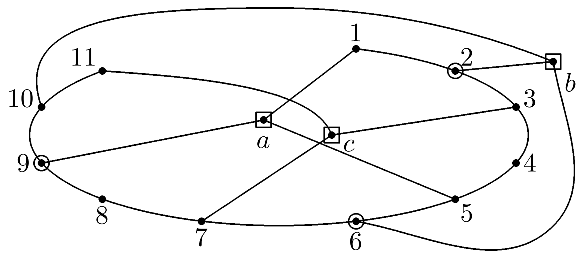

Figure 6

C a s e 2. Two of the crossed edges are incident to the same vertex x, while the third edge e is not.

Since deleting two edges incident to a vertex x from a cubic graph yields a pendant edge xy, whose presence or otherwise does not affect planarity, we will equivalently consider deleting x and an edge e.

2.1. The vertex x is a star-hub.

2.1.1. The edge e is a circuit-edge.

By deleting the star-hub x, the edge e and every fourth circuit-edge from e, we get a graph $\mathcal{H}((S_3 \times P_2) - \Gamma)$. In this graph no edge is crossed, for, otherwise, $\alpha > 3$. Therefore, in particular, no star edge is crossed, and Lemma 2.1 implies that $\nu((S_3 \times P_2) - \Gamma) = 1$, a contradiction.

2.1.2. The edge e is a star-edge.

Without loss of generality, we choose the star-hub to be d. We note that e can belong to a successive star (either $S(c)$ or $S(a)$), or else e belongs to the non-successive star $S(b)$. In case (i), without loss of generality, we let e be the edge $(c, 11)$, while in case (ii), without loss of generality, we let e be the edge $(b, 10)$. It is readily checked that the resulting graph G_3 contains $K\langle a, b, 7; 3, 6, 9\rangle$ or $K\langle a, c, 6; 1, 5, 9\rangle$, respectively, so that in either case, G_3 is non-planar.

2.2. The vertex x is a circuit-vertex.

Without loss of generality, we choose x to be the vertex 12, which we delete to get the graph G_4 of Figure 7.

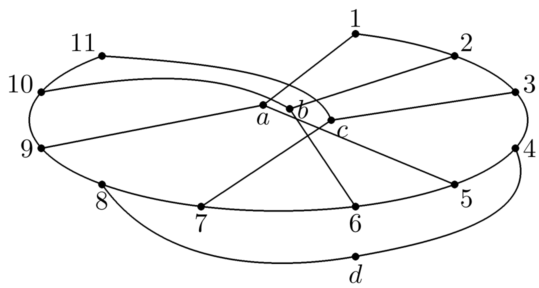

Figure 7

2.2.1. The edge e is a star-edge.

If e is an edge of $S(d)$, then by deleting what remains of $S(d)$ we get a non-planar subgraph containing G_2 shown in Figure 6, giving a contradiction (as shown in Case 1.2). If e is not an edge of $S(d)$, then e is an edge of either $S(a), S(b)$ or $S(c)$. This implies that we have two untouched stars and, thus, at least one of them must have an edge adjacent to one of the deleted circuit-edges. Without loss of generality, we let this star be $S(a)$, and consider the subgraph containing the graph G_5 shown in Figure 8. Introducing either $S(b)$ or $S(c)$ yields another crossed edge.

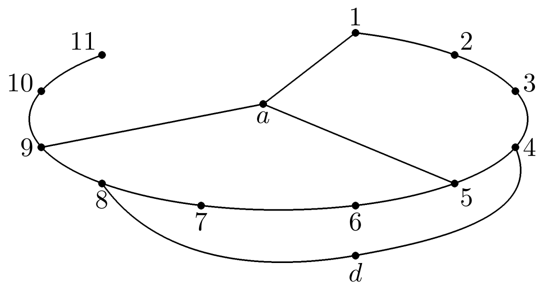

Figure 8

2.2.2. The edge e is a circuit-edge.

In this case we have three untouched stars $S(a)$, $S(b)$ and $S(c)$. Now, to avoid a contradiction, $e \notin \{(3,4),(4,5),(7,8),(8,9)\} := \Theta$ since otherwise we get a planar drawing of a graph $\mathcal{H}$. Consider the graph G_6 induced by $S(a)$, $S(c)$, the set of edges Θ, and the chord $(4,8)$ (shown in Figure 9). This graph has three faces F_1, F_2 and F_3 in one of which b must lie. For each of the three possibilities, we always get at least two circuit-edges joining the vertices of $S(b)$ to the corresponding vertices of the other stars which cross some other edges, thus giving a contradiction.

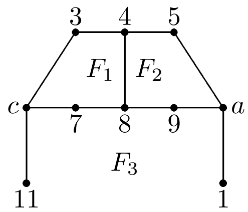

Figure 9

Case 3. No pair of crossed edges share a common vertex.

3.1. All the three crossed edges are star-edges.

In this case, the circuit and one of the stars, say $S(d)$, are untouched. By deleting the crossed edges, each crossed star becomes a 'chord' of the circuit, homeomorphic to K_2. First, we consider the chord obtained from the star $S(a)$ and, without loss of generality, let this chord be $(1,9)$. Then we consider the graph G_7 shown in Figure 10. There are nine cases left to consider, depending on the chords $(2,6)$, $(6,10)$, $(2,10)$ for $S(b)$ and $(3,7)$, $(7,11)$, $(3,11)$ for $S(c)$. None of the chords $(2,10)$, $(6,10)$, $(3,11)$ and $(7,11)$ can be inserted in G_7 without giving another intersection. This leaves the chords $(3,7)$ and $(2,6)$ which, again, cannot both be inserted in G_7 without giving a crossing.

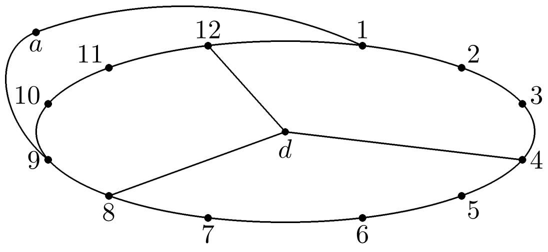

Figure 10

3.2. Two of the crossed edges are star-edges, the third is a circuit-edge e.

In this case we have two stars which are intact, which can be either successive or non-successive. We delete the crossed star-edges and consider the following subcases.

3.2.1. Let the two intact stars be successive, which we choose to be $S(a)$ and $S(b)$.

3.2.1.1. We consider first the case when e is adjacent to at most one of $S(a)$ and $S(b)$, that is, $e \notin \{(1,2),(5,6),(9,10)\}$. Without loss of generality, we let $e=(2,3)$ and consider the graph G_8 shown in Figure 11 obtained by further deleting e. We note that the stars $S(c)$ and $S(d)$ are simply chords and we try to insert them in G_8. There are nine different ways how this can be done, depending on the chords $(3,7)$, $(7,11)$, $(3,11)$ for $S(c)$ and $(4,8)$, $(8,12)$, $(4,12)$ for $S(d)$. Each of the nine possible pairs of these chords yields a crossing.

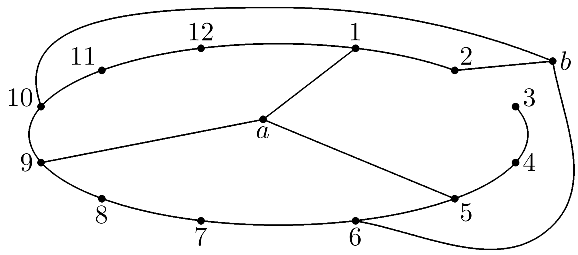

Figure 11

3.2.1.2. We consider now the case when e is adjacent to both $S(a)$ and $S(b)$ and, without loss of generality, assume $e=(1,2)$. By deleting e, we obtain the graph G_9 shown in Figure 12. Again, there are nine cases which need to be considered, depending on the chords $(3,7)$, $(7,11)$, $(3,11)$ for $S(c)$ and $(4,8)$, $(8,12)$, $(4,12)$ for $S(d)$. The chords $(3,7)$, $(7,11)$, $(4,8)$ and $(8,12)$ cannot be inserted in G_9 without producing another crossing. It remains to insert the pair of chords $(3,11)$ and $(4,12)$ in G_9. These cannot both be inserted, since then we either get another crossed edge, or else the edge $(1,2)$ is crossed twice and, thus, $\nu(G_4) \geqslant 4$.

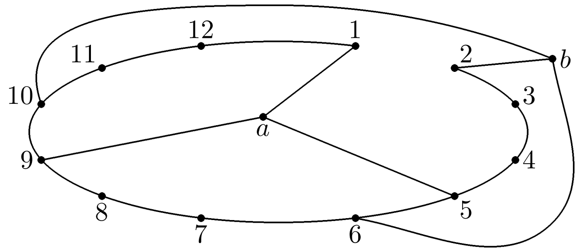

Figure 12

3.2.2. Let the two intact stars be non-successive, and let them be $S(a)$ and $S(c)$.

The edge e must be adjacent to exactly one of the stars and so, without loss of generality, we can choose e to be the edge $(1,2)$. By deleting the edge e we get a subgraph containing the graph G_{10} shown in Figure 13. But then none of the three possibile chords obtained from the star $S(b)$, namely the chords $(2,6)$, $(6,10)$, and $(2,10)$, can be inserted in G_{10} without producing another crossing.

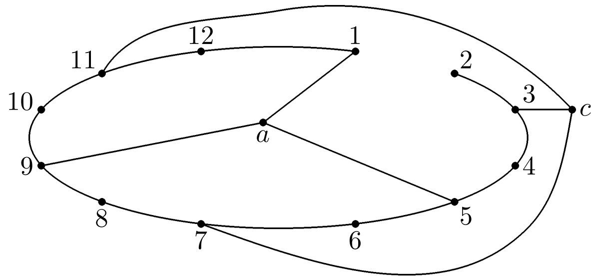

Figure 13

3.3. At least two of the crossed edges are the circuit-edges e and f (hence, at most one of the crossed edges is a star-edge).

Firstly, we note that the minimum distance between e and f must be at least one, otherwise the two edges are incident to the same vertex, which reduces to Case 2. In this case we have at least three stars that are intact, which we denote by $S(a)$, $S(b)$ and $S(c)$. Considering only these three stars, we get a subgraph $\mathcal{H}(G_3)$. Without loss of generality, we can fix e to be the edge $(1,2)$. Since f cannot be adjacent to e, we have $f \notin \{(11,1),(2,3)\}$. Also, $f \notin \{(3,5),(7,9)\}$ since otherwise, by deleting the edges $(1,2),(3,5)$ and $(7,9)$ we get a planar drawing of $\mathcal{H}((S_3 \times P_2) - \Gamma)$, a contradiction. Therefore, f must be one of $\{(5,6),(6,7),(9,10),(10,11)\}$. Now, we consider the graph G_{11} induced by $S(a)$ and $S(c)$ and the edges $(11,1)$, $(3,5)$ and $(7,9)$, which consists of three hexagonal faces (refer to Figure 14). The edge $(2,3)$ must be in one of the faces F_1 or F_2 of G_9; we choose F_1. This implies that $S(b)$ is also in F_1 and thus, the vertex 6 is in F_1. When we try to introduce the circuit-edges $(6,7)$ and $(9,10)$, we note that both of them must intersect some other edge, thus giving us a contradiction. □

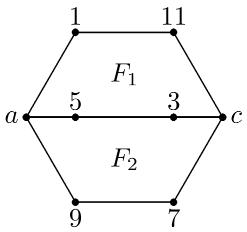

Figure 14

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