# PRECOVERS AND GOLDIE'S TORSION THEORY

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(Received December 19, 2002)

*Abstract.* Recently, Rim and Teply [8], using the notion of $\tau$-exact modules, found a necessary condition for the existence of $\tau$-torsionfree covers with respect to a given hereditary torsion theory $\tau$ for the category $R$-mod of all unitary left $R$-modules over an associative ring $R$ with identity. Some relations between $\tau$-torsionfree and $\tau$-exact covers have been investigated in [5]. The purpose of this note is to show that if $\sigma = (\mathscr{T}_\sigma, \mathscr{F}_\sigma)$ is Goldie's torsion theory and $\mathscr{F}_\sigma$ is a precover class, then $\mathscr{F}_\tau$ is a precover class whenever $\tau \geqslant \sigma$. Further, it is shown that $\mathscr{F}_\sigma$ is a cover class if and only if $\sigma$ is of finite type and, in the case of non-singular rings, this is equivalent to the fact that $\mathscr{F}_\tau$ is a cover class for all hereditary torsion theories $\tau \geqslant \sigma$.

*Keywords*: hereditary torsion theory, Goldie's torsion theory, non-singular ring, precover class, cover class

*MSC 2000*: 16S90, 18E40, 16D80

In what follows, $R$ stands for an associative ring with identity and $R$-mod denotes the category of all unitary left $R$-modules. The basic properties of rings and modules can be found in [1]. A class $\mathscr{G}$ of modules is called *abstract*, if it is closed under isomorphic copies, *co-abstract*, if its members are pairwise non-isomorphic and *complete* with respect to a given property, if every module with this property is isomorphic to a member of the class $\mathscr{G}$.

Recall that a *hereditary torsion theory* $\tau = (\mathscr{T}_\tau, \mathscr{F}_\tau)$ for the category $R$-mod consists of two abstract classes $\mathscr{T}_\tau$ and $\mathscr{F}_\tau$, the $\tau$*-torsion class* and the $\tau$*-torsionfree class*, respectively, such that $\mathrm{Hom}(T, F) = 0$ whenever $T \in \mathscr{T}_\tau$ and $F \in \mathscr{F}_\tau$, the class $\mathscr{T}_\tau$ is closed under submodules, factor-modules, extensions and arbitrary direct sums, the class $\mathscr{F}_\tau$ is closed under submodules, extensions and arbitrary direct products

The research has been partially supported by the Grant Agency of the Charles University, grant #GAUK 268/2002/B-MAT/MFF and also by the institutional grant MSM 113 200 007.

and for each module $M$ there exists an exact sequence $0 \to T \to M \to F \to 0$ such that $T \in \mathscr{T}_\tau$ and $F \in \mathscr{F}_\tau$. For two hereditary torsion theories $\tau$ and $\tau'$ the symbol $\tau \leqslant \tau'$ means that $\mathscr{T}_\tau \subseteq \mathscr{T}_{\tau'}$ and consequently $\mathscr{F}_{\tau'} \subseteq \mathscr{F}_\tau$. Associated with each hereditary torsion theory $\tau$ is the *Gabriel filter* $\mathscr{L}_\tau$ of left ideals of $R$ consisting of all left ideals $I \leqslant R$ with $R/I \in \mathscr{T}_\tau$. Recall that $\tau$ is said to be of *finite type*, if $\mathscr{L}_\tau$ contains a cofinal subset $\mathscr{L}'_\tau$ of finitely generated left ideals. A submodule $N$ of the module $M$ is called $\tau$*-closed* (or $\tau$*-pure*), if the factor-module $M/N$ belongs to $\mathscr{F}_\tau$. A module $M$ is said to be $\tau$*-noetherian*, if the set of all $\tau$-closed submodules of $M$ satisfies the maximum condition. A module $Q$ is said to be $\tau$*-injective*, if it is injective with respect to all short exact sequences $0 \to A \to B \to C \to 0$, where $C \in \mathscr{T}_\tau$. Further, a hereditary torsion theory $\tau$ is called *exact*, if $E(Q)/Q$ is $\tau$-torsionfree $\tau$-injective, $E(Q)$ being the injective hull of $Q$, whenever $Q$ is a $\tau$-torsionfree $\tau$-injective module. If, in addition, $\tau$ is of finite type, then it is called *perfect*. For more details on torsion theories we refer to [7] or [6].

For a module $M$, the *singular submodule* $Z(M)$ consists of all elements $a \in M$, the *annihilator left ideal* $(0\colon a) = \{r \in R;\ ra = 0\}$ of which is essential in $R$. *Goldie's torsion theory* for the category $R$-mod is the hereditary torsion theory $\sigma = (\mathscr{T}_\sigma, \mathscr{F}_\sigma)$, where $\mathscr{T}_\sigma = \{M \in R\text{-mod};\ Z(M/Z(M)) = M/Z(M)\}$ and $\mathscr{F}_\sigma = \{M \in R\text{-mod};\ Z(M) = 0\}$. If the ring $R$ is $\sigma$-torsionfree, $Z(R) = 0$, then $R$ is called *non-singular.* Note that in this case the Gabriel filter $\mathscr{L}_\sigma$ consists of essential left ideals only.

If $\mathscr{G}$ is an abstract class of modules, then a homomorphism $\varphi\colon G \to M$ is called a $\mathscr{G}$*-precover* of the module $M$, if $G \in \mathscr{G}$ and every homomorphism $f\colon F \to M$, $F \in \mathscr{G}$, factors through $\varphi$, i.e. there exists a homomorphism $g\colon F \to G$ such that $\varphi g = f$. Moreover, a $\mathscr{G}$-precover $\varphi$ of $M$ is said to be a $\mathscr{G}$*-cover*, if each endomorphism $f$ of $G$ with $\varphi f = \varphi$ is an automorphism of the module $G$. An abstract class $\mathscr{G}$ of modules is called a *precover* (*cover*) *class*, if every module has a $\mathscr{G}$-precover ($\mathscr{G}$-cover). It is well-known that an $\mathscr{F}_\tau$-precover $\varphi\colon G \to M$ is an $\mathscr{F}_\tau$-cover if and only if $\operatorname{Ker}\varphi$ contains no non-zero submodule $\tau$-closed in $G$. For more details concerning the theory of precovers and covers we refer to [10].

It is well-known (see e.g. [7; Proposition 42.9]) that a hereditary torsion theory $\tau$ is of finite type if and only if any directed union of $\tau$-torsionfree $\tau$-injective modules is $\tau$-injective and that this condition is sufficient for the existence of $\tau$-torsionfree covers (see [9] for the $\tau$-torsionfree rings and [2] for the general case). On the other hand, in [8] a necessary condition has been presented saying that the directed union of $\tau$-exact submodules of a given module is $\tau$-injective. By a $\tau$*-exact* module we mean any $\tau$-torsionfree module, every $\tau$-torsionfree homomorphic image of which is $\tau$-injective. The purpose of this note is to prove that for Goldie's torsion theory $\sigma$ the finite type condition is necessary and sufficient for the existence of $\sigma$-torsionfree covers. Moreover, if $\mathscr{F}_\sigma$ is a precover class, then $\mathscr{F}_\tau$ is a precover class whenever

$\tau \geqslant \sigma$ and the same holds for cover classes provided that the ring $R$ is non-singular. More precisely, we are going to prove the following two theorems.

**Theorem 1.** *Let $\sigma = (\mathscr{T}_\sigma, \mathscr{F}_\sigma)$ be Goldie's torsion theory for the category $R$-*mod*. If $\mathscr{F}_\sigma$ is a precover class, then $\mathscr{F}_\tau$ is a precover class for any hereditary torsion theory $\tau \geqslant \sigma$.*

**Theorem 2.** *Let $\sigma = (\mathscr{T}_\sigma, \mathscr{F}_\sigma)$ be Goldie's torsion theory for the category $R$-*mod*. The following conditions are equivalent:*

(i) *$\mathscr{F}_\sigma$ is a cover class;*
(ii) *$\sigma$ is of finite type;*
(iii) *$\sigma$ is perfect.*

*If, moreover, the ring $R$ is non-singular ($Z(R) = 0$), then these conditions are equivalent to the following three conditions:*

(iv) *every non-zero left ideal of $R$ contains a finitely generated essential left ideal;*
(v) *${}_RR$ is $\sigma$-noetherian;*
(vi) *for every hereditary torsion theory $\tau \geqslant \sigma$ the class $\mathscr{F}_\tau$ is a cover class.*

We start with some preliminary lemmas, the symbol $\sigma$ will always denote Goldie's torsion theory.

**Lemma 1.** *Let $\tau \geqslant \sigma$ be a hereditary torsion theory for the category $R$-*mod*. Then*

(i) *a module $Q \in \mathscr{F}_\tau$ is $\tau$-injective if and only if it is injective;*
(ii) *a submodule $K \leqslant Q$ with $Q \in \mathscr{F}_\tau$ injective is $\tau$-closed if and only if it is injective. In this case the factor-module $Q/K$ is also injective.*

P r o o f. (i) If $Q \in \mathscr{F}_\tau$ is $\tau$-injective and $E(Q)$ is the injective hull of $Q$, then $E(Q)/Q \in \mathscr{F}_\tau \subseteq \mathscr{F}_\sigma$ by [7; Corollary 44.3]. In view of the obvious fact $E(Q)/Q \in \mathscr{T}_\sigma$ we have $Q = E(Q)$. The converse is obvious.

(ii) If $K$ is $\tau$-closed in $Q$, then $Q/K \in \mathscr{F}_\tau \subseteq \mathscr{F}_\sigma$. Hence $K$ has no proper essential extension in $Q$ and consequently it is injective. The rest is clear. □

**Lemma 2.** *Let $\tau \geqslant \sigma$ be a hereditary torsion theory for the category $R$-*mod*. If every module has an $\mathscr{F}_\tau$-cover, then every directed union of $\tau$-torsionfree injective modules is $\tau$-torsionfree injective.*

P r o o f. Let $K = \bigcup\limits_{\alpha\in\Lambda} K_\alpha$ be a directed union of $\tau$-torsionfree injective modules, let $M = E(K)$ be the injective hull of $K$ and let $\varphi\colon G \to M/K$ be an $\mathscr{F}_\tau$-cover of the module $M/K$. Denoting by $\pi_\alpha\colon M/K_\alpha \to M/K$ the corresponding natural projections, there are homomorphisms $f_\alpha\colon M/K_\alpha \to G$ such that $\varphi f_\alpha = \pi_\alpha$ for every $\alpha \in \Lambda$. Obviously, $\operatorname{Ker} f_\alpha \subseteq K/K_\alpha$ and we are going to show that the equality holds for each $\alpha \in \Lambda$. If not, then $K_\beta/K_\alpha \nsubseteq \operatorname{Ker} f_\alpha$ for some $\alpha, \beta \in \Lambda$ and so $0 \neq f_\alpha(K_\beta/K_\alpha) \cong K_\beta/L_\beta \in \mathscr{F}_\tau \subseteq \mathscr{F}_\sigma$ yields according to Lemma 1 that $0 \neq f_\alpha(K_\beta/K_\alpha) \subseteq \operatorname{Ker} \varphi$ is injective. This contradicts the fact that $\varphi$ is an $\mathscr{F}_\tau$-cover of the module $M/K$ and consequently $\operatorname{Im} f_\alpha \cong M/K \in \mathscr{F}_\tau$ for each $\alpha \in \Lambda$. Thus $M/K \in \mathscr{F}_\sigma \cap \mathscr{T}_\sigma = 0$, $M = K$ and we are through. □

**Lemma 3.** *Let $\tau = (\mathscr{T}_\tau, \mathscr{F}_\tau)$ be an arbitrary hereditary torsion theory for the category $R$-mod. The following conditions are equivalent:*

(i) *every module has a $\tau$-torsionfree precover;*

(ii) *every injective module has a $\tau$-torsionfree precover;*

(iii) *every injective module has an injective $\tau$-torsionfree precover.*

P r o o f. For an arbitrary injective module $M$ we obviously have the commutative diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "\iota"'] & M \arrow[d, equal] \\ E(G) \arrow[r, "\psi"] & M \end{tikzcd}$$

where $\iota$ is the inclusion map of $G$ into its injective hull $E(G)$ and $\varphi$ is an $\mathscr{F}_\tau$-precover of the module $M$. Then $\psi$ is obviously an $\mathscr{F}_\tau$-precover of $M$ and consequently (ii) implies (iii).

Assuming (iii) let us consider the pullback diagram

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[d, "i"'] & M \arrow[d, "j"] \\ G \arrow[r, "\psi"] & E(M) \end{tikzcd}$$

where $M \in R$-mod is arbitrary and $\psi$ is an $\mathscr{F}_\tau$-precover of $E(M)$ with $G$ injective. Clearly, $i$ is injective, hence $F \in \mathscr{F}_\tau$ and the pullback property yields that $\varphi$ is an $\mathscr{F}_\tau$-precover of the module $M$. The rest is clear. □

**Lemma 4.** *Let $\tau = (\mathscr{T}_\tau, \mathscr{F}_\tau)$ be a hereditary torsion theory for the category $R$-mod. A homomorphism $\varphi\colon G \to M$ with $G \in \mathscr{F}_\tau$ and $M$ injective is an $\mathscr{F}_\tau$-precover of the module $M$ if and only if to each homomorphism $f\colon Q \to M$ with $Q \in \mathscr{F}_\tau$ injective, there exists a homomorphism $g\colon Q \to G$ such that $\varphi g = f$.*

P r o o f. Only the sufficiency requires verification. So, let us consider the commutative diagram

$$\begin{tikzcd} E(F) \arrow[r, equal] \arrow[d, "g"'] & E(F) \arrow[d, "h"] & F \arrow[l, "i"'] \arrow[d, "f"] \\ G \arrow[r, "\varphi"] & M \arrow[r, equal] & M \end{tikzcd}$$

with the given $\varphi$, $M$ injective and $f\colon F \to M$, $F \in \mathscr{F}_\tau$, arbitrary. Then there is $h\colon E(F) \to M$ with $hi = f$, $M$ being injective, and $g\colon E(F) \to G$ with $\varphi g = h$ by the definition of a precover. Thus $\varphi(gi) = hi = f$ and the proof is complete. $\square$

P r o o f (of Theorem 1). Let $\lambda$ be an arbitrary infinite cardinal and let $\mathfrak{M}_\lambda$ be any complete co-abstract set of modules of cardinalities at most $\lambda$. For any $M \in \mathfrak{M}_\lambda$ we fix an $\mathscr{F}_\sigma$-precover $\varphi_M\colon G_M \to M$ and denote by $\kappa$ the first cardinal with $\kappa > |G_M|$ for each $M \in \mathfrak{M}_\lambda$.

Further, let $Q \in \mathscr{F}_\tau$ be an arbitrary injective module with $|Q| \geqslant \kappa$ and let $K \leqslant Q$ be its submodule such that $|Q/K| \leqslant \lambda$. Then, obviously, $Q \in \mathscr{F}_\sigma$ and consequently, by the above part, the factor-module $Q/K$ has an $\mathscr{F}_\sigma$-precover $\varphi\colon G \to Q/K$ with $|G| < \kappa$. Thus, there is a homomorphism $f\colon Q \to G$ such that $\varphi f = \pi$, $\pi$ being the canonical projection $Q \to Q/K$. Now $\operatorname{Ker} f = L$ is contained in $K$ and it is a direct summand of $Q$ by Lemma 1 (ii) owing to the fact that $Q/L \cong \operatorname{Im} f \in \mathscr{F}_\sigma$. Moreover, $|Q/L| = |\operatorname{Im} f| \leqslant |G| < \kappa$.

Now let $M \in R$-mod be an arbitrary injective module, $\lambda = \max(|M|, \aleph_0)$, and let $\kappa$ be the cardinal corresponding to $\lambda$ by the beginning of this proof. Further, let $\mathfrak{N}_\kappa$ be any complete co-abstract set of $\tau$-torsionfree injective modules of cardinalities less than $\kappa$. We put $G = \bigoplus_{Q \in \mathfrak{N}_\kappa} Q^{(\operatorname{Hom}(Q,M))}$ and $\varphi\colon G \to M$ will denote the corresponding natural evaluation map. To verify that $\varphi$ is a $\tau$-torsionfree precover of the module $M$ we shall use Lemma 4. So, let $Q \in \mathscr{F}_\tau$ be an arbitrary injective module and let $f\colon Q \to M$ be an arbitrary homomorphism. For $|Q| < \kappa$ there exists an isomorphic copy of $Q$ lying in $\mathfrak{N}_\kappa$ and the existence of the homomorphism $g\colon Q \to G$ with $\varphi g = f$ can be easily verified. In the opposite case, for $|Q| \geqslant \kappa$, denoting $K = \operatorname{Ker} f$ we have $|Q/K| = |\operatorname{Im} f| \leqslant |M| \leqslant \lambda$. Thus, by the above part, there is a direct summand $L$ of $Q$ contained in $K$ and such that $|Q/L| < \kappa$. Moreover, $f$ naturally induces the homomorphism $\overline{f}\colon Q/L \to M$ such that $\overline{f}\pi = f$, $\pi\colon Q \to Q/L$ being the canonical projection. Thus there is $\bar{g}\colon Q/L \to G$ with $\varphi\bar{g} = \overline{f}$ by the previous case, so $\varphi(\bar{g}\pi) = \overline{f}\pi = f$ and to complete the proof it suffices now to apply Lemma 3. $\square$

P r o o f (of Theorem 2). (i) implies (ii). It suffices to use Lemma 2 and [7; Proposition 42.9].

(ii) implies (i). This has been proved in [9] in the case of a faithful torsion theory and in [2] in the general case.

(ii) is equivalent to (iii). This is obvious, $\sigma$ being exact by Lemma 1 (see also [7; Corollary 44.3]).

Assume now that the ring $R$ is non-singular.

(ii) implies (iv). Since $R$ is non-singular, the Gabriel filter $\mathscr{L}_\sigma$ consists of essential left ideals only, and consequently every essential left ideal contains an essential finitely generated left ideal by the hypothesis. So, let $0 \neq I \leqslant R$ be an arbitrary non-essential left ideal of $R$ and let $J \leqslant R$ be any left ideal maximal with respect to $I \cap J = 0$. Then $I \oplus J$ is essential in $R$ and consequently there is a finitely generated left ideal $K = \sum\limits_{i=1}^{n} Ra_i \subseteq I \oplus J$ essential in $R$. Now $a_i = b_i + c_i$, $b_i \in I$, $c_i \in J$, $i = 1, \ldots, n$, and it remains to verify that the left ideal $\sum\limits_{i=1}^{n} Rb_i$ is essential in $I$. However, for an arbitrary element $0 \neq u \in I$ we have $0 \neq ru = \sum\limits_{i=1}^{n} r_i a_i = \sum\limits_{i=1}^{n} r_i b_i + \sum\limits_{i=1}^{n} r_i c_i$ for suitable elements $r, r_1, \ldots, r_n \in R$, and consequently, $0 \neq ru = \sum\limits_{i=1}^{n} r_i b_i$, as we wished to show.

(iv) is equivalent to (v). See [7; Proposition 20.1].

(iv) implies (vi). Let $I \in \mathscr{L}_\tau$ be arbitrary and let $K \leqslant I$ be a finitely generated left ideal essential in $I$. Then $I/K \in \mathscr{T}_\sigma \subseteq \mathscr{T}_\tau$, hence $K \in \mathscr{L}_\tau$ and the torsion theory $\tau$ is of finite type. Now it suffices to use [2].

(vi) implies (i). This is trivial. $\square$

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