ON A CERTAIN CONSTRUCTION OF LATTICE EXPANSIONS

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Abstract. We obtain a simple construction for particular subclasses of several varieties of lattice expansions. The construction allows a unified approach to the characterization of the subdirectly irreducible algebras

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1. Introduction

In [5], C. Chen and G. Grätzer introduced a method of construction of Stone lattices from Boolean algebras and distributive lattices. This was a construction by means of the so-called "triples" which were successfully used in constructions of distributive p-algebras [11], pseudocomplemented semilattices [10] and $\mathbf{K}_2$-algebras (a subvariety of MS-algebras) [3], [12], [8].

In this article we first obtain a simple construction for particular subclasses of several varieties, including Ockham algebras with De Morgan skeleton, MS-algebras, demi-p-lattices, p-algebras, Stone lattices and semi-De Morgan algebras (Section 3). The construction allows a unified approach to the characterization of the subdirectly irreducible algebras (Section 4). In Section 5 we generalize this construction. Several examples illustrate the results.

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For notation and assumed results about lattice theory we refer the reader to [1]. Let L be a lattice expansion. ∇^L is the universal congruence on L and Δ^L is the trivial congruence on L. $\mathrm{Con}(L)$ denotes the congruence lattice of L. By $\theta(x,y)$ we denote the principal congruence relation generated by the pair $(x,y) \in L^2$. We use $\theta_{\mathrm{lat}}(x,y)$ to denote the principal lattice congruence generated by (x,y).

We are interested in several subvarieties of the variety of semi-De Morgan algebras, introduced by Sankappanavar in [15]. A *semi-De Morgan algebra* is an algebra $\langle L, \wedge, \vee, ^*, 0, 1\rangle$ satisfying:

$\langle L, \wedge, \vee, 0, 1\rangle$ is a distributive lattice with 0 and 1,

(C1) $(x \vee y)^* = x^* \wedge y^*$,

(C2) $(x \wedge y)^{**} = x^{**} \wedge y^{**}$,

(C3) $0^* = 1$,

(C4) $1^* = 0$,

(C5) $x^{***} = x^*$.

A semi-De Morgan algebra L is a *distributive demi-pseudocomplemented lattice (demi-p-lattice,* for brevity) if L satisfies $x^* \wedge x^{**} = 0$. We refer the reader to [14] for the basic properties of demi-p-lattices. A demi-p-lattice L is a *distributive p-algebra* if L satisfies the condition $x \wedge a = 0$ iff $a \leqslant x^*$. A distributive p-algebra is a Stone algebra if it satisfies the identity $x^* \vee x^{**} = 1$.

On the other hand, a semi-De Morgan algebra L is an *Ockham lattice with De Morgan skeleton* [2] if it satisfies the equation $(x \wedge y)^* = x^* \vee y^*$. This variety is denoted by $\mathcal{K}_{1,1}$. If, in addition, L satisfies $x \leqslant x^{**}$, then L is called an MS-algebra. We refer the reader to [3] and [4] for the basic properties of MS-algebras. An MS-algebra is a De Morgan algebra if $x = x^{**}$ for every $x \in L$. We recall that the subdirectly irreducible De Morgan algebras are $\mathbf{2}$, $\mathbf{3}$, and $\mathbf{M}_0$, where $\mathbf{n}$ is the chain of n elements for $n = 2, 3$, and $\mathbf{M}_0$ is the algebra with universe $\mathbf{2} \times \mathbf{2}$ and $(0,0)^* = (1,1)$, $(0,1)^* = (0,1)$, $(1,0)^* = (1,0)$, $(1,1)^* = (0,0)$.

Let L be a semi-De Morgan algebra. Let $S(L) = \{x^* \colon x \in L\} = \{x \colon x = x^{**}\}$, the *skeleton* of L, and also let $G_z = \{x \in L \colon x^* = z^*\}$ for $z \in L$. The skeleton $S(L)$ can be made into a De Morgan algebra $\langle S(L), \dot{\vee}, \wedge, ^*, 0, 1\rangle$ by defining $a \dot{\vee} b = (a \vee b)^{**}$. Note that the map $x \to x^{**}$ from L to $S(L)$ is an onto homomorphism. We denote by γ^L the kernel of this homomorphism. Note that if θ is a lattice-congruence and $\theta \subseteq \gamma^L$ then $\theta \in \mathrm{Con}(L)$.

If $|G_x| \leqslant 2$ for every $x \in L$, then we say that L is 2-*regular*. The following definition was given in [6, Section 5]. If $C \subseteq L$ is the finite chain $x_1 < x_2 < \ldots < x_n$, then let

$$n^{L,C} = |\{1 \leqslant i < n \colon x_i^* = x_{i+1}^*\}|.$$

If $\{n^{L,C}\colon\ C \subseteq L \text{ is a finite chain}\}$ has a maximum, then we denote it by n^L. Clearly, if $n^L = 1$, then L is 2-regular and $n^L = 0$ if and only if $\gamma^L = \Delta^L$.

Lemma 1. *Let L be a (finitely) subdirectly irreducible semi-De Morgan algebra. Then $n^L \leqslant 1$.*

P r o o f. If $x < y \leqslant v < z$ and (x, y), $(v, z) \in \gamma^L$, we claim that $\sigma = \theta_{\mathrm{lat}}(x, y) \cap \theta_{\mathrm{lat}}(v, z)$ is the trivial congruence Δ^L. (Note that since $\sigma \subseteq \gamma^L$ we have $\sigma \in \mathrm{Con}(L)$). Let $(a, b) \in \sigma$. Then

$$a \wedge x = b \wedge x,\ \ a \vee y = b \vee y \text{ and } a \wedge v = b \wedge v,\ \ a \vee z = b \vee z$$

(see [7, II. 3, Theorem 3]). Since $y \leqslant v$ we have $a \wedge y = b \wedge y$. By the cancelation property for distributive lattices, $a = b$. □

We conclude the section by observing that if L is a semi-De Morgan algebra satisfying $n^L \leqslant 1$ then for every $x \in L$ we have that either $x^{**} \geqslant x$ or $x^{**} \leqslant x$.

3. A simple quadruple construction

A 2-*quadruple* will be a tuple (M, I, F, η), where
(1) M is a De Morgan algebra.
(2) I (resp. F) is an ideal (filter) of M satisfying $I \cup F = M$ and $I \cap F \neq \emptyset$.
(3) η is a proper filter of M satisfying $M - I \subseteq \eta \subseteq F$.

We associate to each 2-quadruple $T = (M, I, F, \eta)$ an algebra of type $(2, 2, 1, 0, 0)$, denoted by L_T, with universe

$$L_T = ((I - F) \times \{0\}) \cup ((I \cap F) \times \mathbf{2}) \cup ((F - I) \times \{1\}),$$

defined by the following conditions:
(a) L_T is a sublattice of $M \times \mathbf{2}$,
(b) $(a, i)^* = \begin{cases} (a^*, 1) & a^* \in \eta \\ (a^*, 0) & a^* \notin \eta. \end{cases}$

For simplicity we also denote by η the map from M to $\mathbf{2}$ defined by the condition $\eta(a) = 1 \Leftrightarrow a \in \eta$. So, $(a, i)^*$ can be defined as $(a^*, \eta(a^*))$. If we conceive of η as a map, then the condition (3) takes the following form, which will be occasionally used:

(3′) η is a $\{0, 1, \wedge\}$-preserving map from M to $\mathbf{2}$ satisfying $\eta(x) = 0$ for every $x \in I - F$, and $\eta(x) = 1$, for every $x \in F - I$.

Lemma 2. *Let $T = (M, I, F, \eta)$ be a 2-quadruple. Then L_T is a semi-De Morgan algebra satisfying $n^{L_T} = 1$ and $S(L_T) \cong M$.*

Proof. That L_T is a distributive lattice follows from (a). The conditions (C1) and (C2) can be easily checked. Conditions (C3) and (C4) hold because η is a $\{0,1\}$-preserving map. (C5) follows from the fact that M is a De Morgan algebra. We prove now that $S(L_T) \cong M$. Clearly $S(L_T) = \{(a, \eta(a)) : a \in M\}$. We check now that $a \to (a, \eta(a))$ is a De-Morgan isomorphism from M to $S(L_T)$. It is trivial that this map is a one-to-one $\{0, 1, \wedge\}$-homomorphism. By the definition of L_T, the map is onto. Moreover,

$$a \vee b \to (a \vee b, \eta(a \vee b)) = ((a \vee b)^{**}, \eta((a \vee b)^{**})),$$

because M is a De Morgan algebra. But

$$((a \vee b)^{**}, \eta((a \vee b)^{**})) = ((a \vee b)^{**}, i)^{**}$$

for every i such that $((a \vee b)^{**}, i) \in L_T$. Take $i = \eta(a) \vee \eta(b)$ to conclude that the above map preserves $\vee$.

Finally, $n^{L_T} = 1$ follows from the fact that $((a, i), (b, j)) \in \gamma^{L_T}$ iff $a = b$. □

Note that different 2-quadruples can generate isomorphic algebras. For instance, consider the 2-quadruples

$$T_1 = (\mathbf{M}_0, \mathbf{M}_0, \{(0,1), (1,1)\}, \{(0,1), (1,1)\}),$$
$$T_2 = (\mathbf{M}_0, \mathbf{M}_0, \{(1,0), (1,1)\}, \{(1,0), (1,1)\}).$$

Both the 2-quadruples generate the only (up to isomorphism) MS-algebra having lattice reduction $\mathbf{2} \times \mathbf{3}$ and skeleton $\mathbf{M}_0$. However, the representation is unique up to *isomorphisms of* 2-*quadruples*.

Definition 3. The 2-quadruples $T = (M, I, F, \eta)$ and $T' = (M', I', F', \eta')$ are isomorphic if there exists an isomorphism $\alpha\colon M \to M'$ such that $I' = \alpha(I)$, $F' = \alpha(F)$ and $\eta' = \alpha(\eta)$.

Lemma 4. *Let L be a semi-De Morgan algebra satisfying $n^L = 1$. Then there exists (up to isomorphism) only one 2-quadruple $T = (M, I, F, \eta)$ such that $L_T \cong L$.*

Proof. Let $T = (S(L), (S], [S), \eta)$[1], $S = \{a \in S(L)\colon |G_a| = 2\}$ and

$$\eta = \{a \in S(L)\colon a \text{ is the top element of } G_a \text{ and } a \geqslant b \text{ for some } b \in S\}.$$

[1] $[S)$ is the set of x such that $x \geqslant s$ for some $s \in S$. In a similar manner $(S]$ is defined.

We claim that S is a subalgebra of $S(L)$. Suppose that $|G_a| = |G_b| = 2$, $a \neq b$. If $a < b$ then $|G_{a\wedge b}| = |G_a| = 2$ and $|G_{a\vee b}| = |G_b| = 2$. Suppose that a and b are not comparable. It is left as an exercise to the reader to check that $|G_{a\wedge b}| = 1$ or $|G_{a\vee b}| = 1$ implies that the sublattice generated by $G_a \cup G_b$ is not distributive, which is a contradiction. Thus the claim follows. Consequently, $(S]$ (resp. $[S)$) is an ideal (filter) of $S(L)$. Let $\varphi = [\eta)_L$ be the filter on L generated by η. Define $f\colon L \to L_T$ by $f(x) = (x^{**}, 1)$ if $x \in \varphi$, and $f(x) = (x^{**}, 0)$ if $x \notin \varphi$. That f is $1-1$ follows from the fact that if $(a, b) \in \gamma^L$ and $a, b \in \varphi$ then $a = b$. This is a consequence of the fact that $\bigwedge G_x \notin \varphi$ if $|G_x| = 2$. Finally, f is onto because $f(\bigwedge G_a) = (a, 0)$ and $f(\bigvee G_b) = (b, 1)$ for every $a \in I$ and $b \in F$. In order to check the case $a \in I$, use the fact that if $|G_x| = 1$, $|G_y| = 2$ and $x < y$ then $x \notin \varphi$ (if not we contradict the fact that $n^L = 1$). The rest of the lemma is easy to check. $\square$

Lemma 5. *Let $T = (M, I, F, \eta)$ be a 2-quadruple. Then*
(i) *L_T is an Ockham lattice with De Morgan skeleton iff η is a prime filter.*
(ii) *L_T is an MS-algebra iff η is a prime filter and $F = \eta$.*
(iii) *L_T is a demi-p-lattice iff M is a Boolean algebra.*
(iv) *L_T is a p-algebra iff M is a Boolean algebra, $I = M$ and $\eta = F$.*
(v) *L_T is a Stone algebra iff $M = I$ is a Boolean algebra and $\eta = F$ is a prime filter.*

Proof. Use the following facts: (a) η is a prime filter iff L_T satisfies the De-Morgan laws. (b) $F = \eta$ iff L_T satisfies $x \leqslant x^{**}$ for every $x \in L_T$. (c) M is a Boolean algebra iff L_T satisfies the equation $x^* \wedge x^{**} = 0$. In order to prove (iv), note that a p-algebra is a demi-p-lattice satisfying $x \leqslant x^{**}$ for every x. $\square$

Notation. The 2-quadruples of the right-hand side of (i) of Lemma 5 (resp. (ii), (iii), (iv) and (v)) will be called $\mathcal{K}_{1,1}$-tuples (resp. MS-tuples, demi-p-tuples, p-tuples and Stone tuples).

For instance, in the following table we show a set of non isomorphic 2-quadruples generating all MS-algebras with $S(L) \in \{\mathbf{2}, \mathbf{3}, \mathbf{M}_0\}$ and $n^L = 1$. Here a is the middle element of the chain $\mathbf{3}$. In the next section we will show that these 2-quadruples are exactly those corresponding to the subdirectly irreducible MS-algebras that are not De Morgan algebras.

2-*quadruple* T	*lattice reduction of* L_T
$(\mathbf{2}, \mathbf{2}, \{1\}, \{1\})$	$\mathbf{3}$
$(\mathbf{3}, \mathbf{3}, \{1\}, \{1\})$	$\mathbf{4}$
$(\mathbf{3}, \{0, a\}, \{a, 1\}, \{a, 1\})$	$\mathbf{4}$
$(\mathbf{3}, \mathbf{3}, \{a, 1\}, \{a, 1\})$	$\mathbf{1} \oplus (\mathbf{2} \times \mathbf{2})$
$(\mathbf{M}_0, \mathbf{M}_0, \{(0,1), (1,1)\}, \{(0,1), (1,1)\})$	$\mathbf{3} \times \mathbf{2}$

On the other hand, if $\mathbf{n}$ represents the chain of n elements and $n > 4$, then an algebra L with lattice reduction $\mathbf{n}$ and skeleton $\mathbf{3}$ can not be represented by a 2-quadruple, because $n^L > 1$.

4. Subdirectly irreducible algebras

Let $\mathcal{V}$ be any of the varieties from Lemma 5. By Lemmas 1 and 3.3, in order to give a complete characterization of the (finitely) subdirectly irreducible members of $\mathcal{V}$, it is sufficient to determine the class of $\mathcal{V}$-tuples T such that L_T is (finitely) subdirectly irreducible. In this section we do this for each of those varieties, and also for the class of all semi-De Morgan algebras.

The classes of subdirectly irreducible of the above mentioned varieties were determined by the following authors:

p-algebras: Lakser, [13],

demi-p-lattices: Sankappanavar, [14],

MS-algebras: Blyth and Varlet, [3],

semi-De Morgan algebras (topological characterization): Hobby, [9].

The following theorem follows from [6, Theorem 5.6].

Theorem 6. *Let $T = (M, I, F, \eta)$ be a 2-quadruple. L_T is a (finitely) subdirectly irreducible semi-De Morgan algebra if and only if for every chain $a < b$ in M there exist $c, d \in M$ satisfying either $(c \vee a) \wedge d \notin \eta$ and $(c \vee b) \wedge d \in \eta$, or $(c \vee b^*) \wedge d \notin \eta$ and $(c \vee a^*) \wedge d \in \eta$.*

Corollary 7. (i) *Let $T = (M, I, F, \eta)$ be a 2-quadruple such that η is a prime filter. Then L_T is (finitely) subdirectly irreducible if and only if $M \in \{\mathbf{2}, \mathbf{3}, \mathbf{M}_0\}$.*

(ii) *Let $T = (M, I, F, \eta)$ be a 2-quadruple such that M is a Boolean algebra. Then L_T is (finitely) subdirectly irreducible if and only if $\eta = \{1\}$.*

P r o o f. (i) The "if" part follows from a simple computation, by checking that γ^{L_T} is a monolite of the congruence lattice of L_T. Suppose that L_T is a finitely subdirectly irreducible semi-De Morgan algebra. Using Theorem 6 and the fact that η is a prime filter, it can be checked that

(a) for every $a \leqslant b$, if $\eta(a) = \eta(b)$ and $\eta(a^*) = \eta(b^*)$ then $a = b$. Let $a \in M - \{0, 1\}$. We claim that $\eta(a \wedge a^*) = \eta(a \vee a^*)$. Suppose not. Thus $\eta(a \wedge a^*) = 0$ and $\eta(a \vee a^*) = 1$. Thus, for some $b \in \{a, a^*\}$, we have $\eta(b) = 0$ and $\eta(b^*) = 1$. By (a), $b = 0$, which is a contradiction. Thus we have the claim. Consequently, $a = a^*$. Since the map $a \to a^*$ from $S(L)$ to $S(L)$ is an anti-isomorphism, we have $S(L) \in \{\mathbf{2}, \mathbf{3}, \mathbf{M}_0\}$.

(ii) Let L_T be subdirectly irreducible and suppose that $a \in \eta$. Since $a \wedge a^* = 0$ and η is a proper filter, we have $a^* \notin \eta$. If $a \neq 1$, then, by Theorem 6, there exist

$c, d \in M$ such that either $(c \vee a) \wedge d \notin \eta$ and $(c \vee 1) \wedge d \in \eta$, or $(c \vee 1^*) \wedge d \notin \eta$ and $(c \vee a^*) \wedge d \in \eta$. That $(c \vee a) \wedge d \notin \eta$ and $(c \vee 1) \wedge d \in \eta$ is impossible, because η is a filter and $a \in \eta$. Suppose that $(c \vee 1^*) \wedge d \notin \eta$ and $(c \vee a^*) \wedge d \in \eta$. Then $c \wedge d \geqslant c \wedge d \wedge a = a \wedge (c \vee a^*) \wedge d \in \eta$, which is a contradiction. Hence $a = 1$.

Suppose now that $\eta = \{1\}$, and let $a < b$. We will check the condition of Theorem 6. Suppose that $a < b$. Since $a \vee b^* \neq 1$, we have $(b^* \vee a) \wedge 1 \notin \eta$ and $(b^* \vee b) \wedge 1 \in \eta$. □

Corollary 8. *Let $T = (M, I, F, \eta)$ be a 2-quadruple.*

(i) *L_T is a (finitely) subdirectly irreducible Ockham lattice with De Morgan skeleton iff η is a prime filter and $M \in \{\mathbf{2}, \mathbf{3}, \mathbf{M}_0\}$.*

(ii) *L_T is a (finitely) subdirectly irreducible MS-algebra iff η is a prime filter, $F = \eta$ and $M \in \{\mathbf{2}, \mathbf{3}, \mathbf{M}_0\}$.*

(iii) *L_T is a (finitely) subdirectly irreducible demi-p-lattice iff M is a Boolean algebra and $\eta = \{1\}$.*

(iv) *L_T is a (finitely) subdirectly irreducible p-algebra iff M is a Boolean algebra, $I = M$ and $\eta = F = \{1\}$.*

(v) *L_T is a (finitely) subdirectly irreducible Stone algebra iff $M = I = \mathbf{2}$ and $\eta = F = \{1\}$*[2].

5. A general construction

In this section we extend the construction from Section 4 to the class of all semi-De Morgan algebras such that n^L is finite. In this Section we omit the proofs of the results. If N is a finite distributive lattice, then by $\mathcal{I}(N)$ we denote the set of intervals of N, that is,

$$\mathcal{I}(N) = \{I_{a,b} \colon\ a, b \in N,\ a \leqslant b\},$$

where $I_{a,b} = \{x \in N \colon\ a \leqslant x \leqslant b\}$. The set $\mathcal{I}(N)$ can be made into a lattice by defining $I_{u,v} \leqslant I_{w,r}$ iff for every x,

$$x \in I_{u,v} \text{ implies } x \vee w \leqslant r.$$

It is easy to check that

$$I_{a,b} \wedge I_{c,d} = I_{a \wedge c, b \wedge d},$$
$$I_{a,b} \vee I_{c,d} = I_{a \vee c, b \vee d}.$$

[2] This determines the chain of three elements, the only s.i. proper 2-regular Stone algebra.

A *quadruple* will be a tuple (M, N, φ, η), where

(1) M is a De Morgan algebra and N is a finite distributive lattice.

(2) $\varphi\colon M \to \mathcal{I}(N)$ is a lattice homomorphism satisfying $0 \in \varphi(0)$, $1 \in \varphi(1)$ and that for every $s, t \in N$ such that t covers s there exists $a \in M$ with $s, t \in \varphi(a)$.

(3) η is a $\{0, 1, \wedge\}$-preserving map from M to N such that $\eta(a) \in \varphi(a)$ for every $a \in M$.

We will occasionally use $l(a)$ and $u(a)$ to denote the elements of N satisfying that $\varphi(a) = I_{l(a),u(a)}$. We associate with each quadruple $T = (M, N, \varphi, \eta)$ an algebra of type $(2, 2, 1, 0, 0)$, denoted by L_T, with universe $L_T = \{(a, t)\colon\ t \in \varphi(a)\}$, defined by the following conditions:

(a) L_T is a sublattice of $M \times N$,

(b) $(a, i)^* = (a^*, \eta(a^*))$.

Definition 9. Two quadruples $T = (M, N, \varphi, \eta)$, $T' = (M', N', \varphi', \eta')$ are *isomorphic* if there exist isomorphisms $\alpha\colon M \to M'$ and $\beta\colon N \to N'$ such that $\varphi'(\alpha(a)) = \beta(\varphi(a))$ and $\eta'(\alpha(a)) = \beta(\eta(a))$.

Lemma 10. (i) *Let $T = (M, N, \varphi, \eta)$ be a quadruple. Then L_T is a semi-De Morgan algebra satisfying $S\,(L_T) \cong M$ and $n^{L_T} < \infty$.*

(ii) *Let L be a semi-De Morgan algebra satisfying $n^L < \infty$ and $S(L) \cong M$, and let*

$$N = L/(\gamma^L)^*,$$
$$\varphi(a) = \{[x](\gamma^L)^*\colon\ x^{**} = a\},\ \text{and}\ \eta(a) = [a](\gamma^L)^*,$$

where $(\gamma^L)^*$ is the pseudocomplement of γ^L in the congruence lattice of L. Then $T = (M, N, \varphi, \eta)$ is (up to isomorphism) the only quadruple satisfying $L_T \cong L$.

Lemma 11. *Let $T = (M, N, \varphi, \eta)$ be a quadruple. Then*

(i) *L_T is a Ockham lattice with De Morgan skeleton iff η preserves $\vee$.*

(ii) *L_T is a MS-algebra iff η preserves $\vee$ and $\eta(a) = u(a)$ for every $a \in M$.*

(iii) *L_T is a demi-p-lattice iff M is a Boolean algebra.*

(iv) *L_T is a p-algebra iff M is a Boolean algebra and $\eta(a) = u(a)$ for every $a \in M$.*

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