OSCILLATORY BEHAVIOUR OF SOLUTIONS OF NONLINEAR HIGHER ORDER NEUTRAL DIFFERENTIAL EQUATIONS

N. Parhi, Bhubaneswar, R. N. Rath, Berhampur

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Abstract. Necessary and sufficient conditions are obtained for oscillation of all bounded solutions of

$$(*) \qquad [y(t) - y(t-\tau)]^{(n)} + Q(t)G(y(t-\sigma)) = 0,\ t \geqslant 0,$$

where $n \geqslant 3$ is odd. Sufficient conditions are obtained for all solutions of $(*)$ to oscillate. Further, sufficient conditions are given for all solutions of the forced equation associated with $(*)$ to oscillate or tend to zero as $t \to \infty$. In this case, there is no restriction on n.

Keywords: oscillation, nonoscillation, neutral differential equations

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1. Introduction

In recent years, nonlinear higher order neutral differential equations of the form

$$(1) \qquad [y(t) - p(t)y(t-\tau)]^{(n)} + Q(t)G(y(t-\sigma)) = 0,\ t \geqslant 0,$$

are the subject of study for many authors [1]–[7]. The nonhomogeneous equations associated with (1) are given by

$$(2) \qquad [y(t) - p(t)y(t-\tau)]^{(n)} + Q(t)G(y(t-\sigma)) = f(t),\ t \geqslant 0,$$

where p and $f \in C([0,\infty),\mathbb{R})$, $Q \in C([0,\infty),\ [0,\infty))$, $G \in C(\mathbb{R},\mathbb{R})$ is nondecreasing, and $xG(x) > 0$ for $x \neq 0$, $\tau > 0$, and $\sigma \geqslant 0$. While studying oscillatory/nonoscillatory and asymptotic behaviour of solutions of (1)/(2), various ranges of $p(t)$ are considered. However, few authors (see [7], [8]) have dealt with critical cases, viz, $p(t) \equiv 1$ or $p(t) \equiv -1$. In [8], the authors have considered these critical cases for (2) with $n = 1$. Suppose that

(H$_1$) $\liminf\limits_{|u|\to\infty} G(u)/u > \alpha > 0$

(H$_2$) $\int_0^\infty t^{n-1} Q(t)\,\mathrm{d}t = \infty$

(H$_3$) $\int_0^\infty Q(t)\,\mathrm{d}t = \infty$

(H$_4$) there exists $F \in C^n([0,\infty),\mathbb{R})$ such that $F^{(n)}(t) = f(t)$ and $\lim\limits_{t\to\infty} F(t) = 0$

(H$_5$) $\int_0^\infty t^{n-2} Q^*(t)\,\mathrm{d}t = \infty$, where $Q^*(t) = \min\{Q(t), Q(t-\tau)\}$

(H$_6$) $G(u\nu) \leqslant G(u)G(\nu)$ for $u > 0$, $\nu > 0$

(H$_7$) for $u > 0$, $\nu > 0$, there exists a $\delta > 0$ such that $G(u) + G(\nu) \geqslant \delta G(u+\nu)$

(H$_8$) for every sequence $\{\sigma_i\} \subset (0,\infty)$, $\sigma_i \to \infty$ as $i \to \infty$, and for every $\gamma > 0$ such that the intervals $(\sigma_i - \gamma, \sigma_i + \gamma)$, $i = 1, 2, \ldots$, are nonoverlapping, $\sum\limits_{i=1}^{\infty} \int_{\sigma_i-\gamma}^{\sigma_i+\gamma} Q(t)\,\mathrm{d}t = \infty$

(H$_9$) $f(t) \leqslant 0$, $-\infty < \sum\limits_{k=0}^{\infty} \int_{k\tau}^{\infty} (t-k\tau)^{n-1} f(t)\,\mathrm{d}t \leqslant 0$

(H$_{10}$) $0 < \sum\limits_{k=0}^{\infty} \int_{k\tau}^{\infty} (t-k\tau)^{n-1} Q(t)\,\mathrm{d}t < \infty$

(H$_{11}$) $G(-u) = -G(u)$.

The following results are particular cases of some results in [9]:

Theorem A (Corollary 3.2, [9])**.** *Let $n \geqslant 3$ be odd. If* (H$_1$), (H$_2$) *hold, then every bounded solution of* (1) *with $p(t) \equiv 1$ oscillates.*

Theorem B (Theorem 3.7, [9])**.** *Suppose that* (H$_1$), (H$_4$)–(H$_7$) *and* (H$_{11}$) *hold. Then every solution of* (2) *with $p(t) \equiv -1$ oscillates or tends to zero as $t \to \infty$.*

The following results are particular cases of some results in [10]:

Theorem C (Theorem 2.9, [10])**.** *Let $p(t) \equiv 1$. Suppose that* (H$_4$) *and* (H$_8$) *hold. Then every unbounded solution of* (2) *oscillates or tends to $\pm\infty$ as $t \to \infty$ and every bounded solution of* (2) *oscillates or tends to zero as $t \to \infty$.*

Theorem D (Corollary 2.8, [10])**.** *Let $p(t) \equiv 1$ and n be odd. If* (H$_3$) *holds, then every bounded solution of* (1) *oscillates.*

Note: In Theorem 2.7 of [10] there is a misprint. That is: $p(t)$ should satisfy (A$_5$) instead of (A$_3$).

Theorem E (Theorem 2.5, [10]). *Let $p(t) \equiv -1$. Suppose that* (H_3), (H_4) *and* (H_7) *hold. Further, suppose there exists $\beta > 0$ such that $G(u)+G(\nu) \leqslant \beta G(u+\nu)$ for $u < 0$ and $\nu < 0$. If Q is monotonic decreasing, then every solution of* (2) *oscillates or tends to zero as $t \to \infty$.*

Theorem F (Theorem 2.6, [10]). *Let $p(t) \equiv -1$. Let* (H_4), (H_6), (H_{11}) *and* (H_7) *hold. If $\int_0^\infty Q^*(t)\,\mathrm{d}t = \infty$, then every solution of* (2) *oscillates or tends to zero as $t \to \infty$, where $Q^*(t) = \min\{Q(t), Q(t-\tau)\}$.*

We may observe that (H_{11}) is needed in Theorem F only if we assume strict inequality in (H_6). Further, the superlinearity condition (H_1) is not assumed in Theorems C–F. Moreover, we may note that (H_8) $\Rightarrow$ (H_3) $\Rightarrow$ (H_2) and $\int_0^\infty Q^*(t)\,\mathrm{d}t = \infty \Rightarrow$ (H_5).

The purpose of this paper is to obtain necessary and sufficient conditions for oscillation of all bounded solutions of

$$\text{(E)} \qquad [y(t) - y(t-\tau)]^{(n)} + Q(t)G(y(t-\sigma)) = 0,\ t \geqslant 0$$

where $n \geqslant 3$, and Q, G, τ and σ are the same as in (2). Sufficient conditions are obtained for all solutions of (E) to oscillate. The present results improve Theorems A, C and D and extend some results in [7].

By a solution of (E) we mean a real-valued continuous function y on $[T_y - \varrho, \infty)$ for some $T_y \geqslant 0$, where $\varrho = \max\{\tau, \sigma\}$, such that $y(t) - y(t-\tau)$ is n-times continuously differentiable and (E) is satisfied for $t \in [T_y, \infty)$. A solution of (E) is said to be oscillatory if it has arbitrarily large zeros; otherwise, it is called nonoscillatory.

2. Some lemmas

Lemma 2.1. *If $\int_0^\infty t^{n-1}|f(t)|\,\mathrm{d}t < \infty$, then there exists $F \in C^n([0,\infty), \mathbb{R})$ such that $F^{(n)}(t) = f(t)$ and $F(t) \to 0$ as $t \to \infty$. If $f(t) \geqslant 0$, then $F(t) < 0$ or > 0 according as n is odd or even. If $f(t) \leqslant 0$, then $F(t) > 0$ or < 0 according as n is odd or even.*

The lemma follows if we define

$$F(t) = \frac{(-1)^n}{(n-1)!} \int_t^\infty (s-t)^{n-1} f(s)\,\mathrm{d}s,\ t \geqslant 0.$$

Lemma 2.2 (See [6, p. 17]). *Let $f, g \in C([0,\infty), \mathbb{R})$ be such that $f(t) = g(t) + pg(t-\tau)$, $t \geqslant \tau$, where $p \in \mathbb{R}$ and $p \neq 1$. Let $\lim\limits_{t\to\infty} f(t) = \ell \in \mathbb{R}$ exist. Then* (i) *$\ell = (1+p)a$, if $\liminf\limits_{t\to\infty} g(t) = a \in \mathbb{R}$ and* (ii) *$\ell = (1+p)b$, if $\limsup\limits_{t\to\infty} g(t) = b \in \mathbb{R}$.*

Lemma 2.3. *Let $Q \in C([0,\infty),[0,\infty))$ and $Q(t) \not\equiv 0$ on any interval of the form $[T,\infty)$, $T \geqslant 0$, and $G \in C(\mathbb{R},\mathbb{R})$ with $uG(u) > 0$ for $u \neq 0$. Let $y \in C([0,\infty),\mathbb{R})$ with $y(t) > 0$ for $t \geqslant t_0 \geqslant 0$. If $w \in C^{(n)}([0,\infty),\mathbb{R})$, $w^{(n)}(t) = -Q(t)G(y(t-\sigma))$, $t \geqslant t_0+\sigma$, $\sigma \geqslant 0$, and there exists an integer $n^* \in \{0,1,2,\dots,n-1\}$ such that $\lim\limits_{t\to\infty} w^{(n^*)}(t)$ exists and $\lim\limits_{t\to\infty} w^{(i)}(t) = 0$ for $i \in \{n^*+1,\dots,n-1\}$, then*

$$w^{(n^*)}(t) = w^{(n^*)}(\infty) - \frac{(-1)^{n-n^*}}{(n-n^*-1)!}\int_t^\infty (s-t)^{n-n^*-1}Q(s)G(y(s-\sigma))\,\mathrm{d}s.$$

Brief outline of the proof. Let $y(t) > 0$ for $t \geqslant t_0 > 0$. Setting, for $t \geqslant t_1 > t_0+\sigma$,

$$\nu(t) = \frac{(-1)^{n-n^*+1}}{(n-n^*)!}\int_t^\infty (s-t)^{n-n^*}Q(s)G(y(s-\sigma))\,\mathrm{d}s$$

we obtain $\nu(t) > 0$, $\nu'(t) < 0$, $\nu''(t) > 0$ and so on. Hence $\lim\limits_{t\to\infty} \nu^{(i)}(t) = 0$, $i = 1,2,\dots,n-n^*$ and $\nu^{(n-n^*+1)}(t) = -w^{(n)}(t)$. Since $w^{(n^*+1)}(t) = -\nu''(t)$ for $t \geqslant t_1$, then integrating it from t to θ ($t_1 \leqslant t < \theta$) and taking limit as $\theta \to \infty$ we obtain

$$\begin{aligned} w^{(n^*)}(t) &= w^{(n^*)}(\infty) - \nu'(t) \\ &= w^{(n^*)}(\infty) - \frac{(-1)^{n-n^*}}{(n-n^*-1)!}\int_t^\infty (s-t)^{n-n^*-1}Q(s)G(y(s-\sigma))\,\mathrm{d}s. \end{aligned}$$

Lemma 2.4. *Suppose that* (H_1), (H_2) *and* (H_4) *hold. Let $y(t)$ be a solution of*

$$\text{(NE)} \qquad [y(t)-y(t-\tau)]^{(n)} + Q(t)G(y(t-\sigma)) = f(t),\ t \geqslant 0$$

such that $y(t) > 0$ for $t \geqslant t_0 > 0$ and let

$$w(t) = y(t) - y(t-\tau) - F(t)$$

for $t \geqslant t_0+\varrho$. Then either $\lim\limits_{t\to\infty} w(t) = -\infty$ or $\lim\limits_{t\to\infty} w^{(i)}(t) = 0$, $i = 0,1,2,\dots,n-1$, and $(-1)^{n+k}w^{(k)}(t) < 0$ for $k = 0,1,2,\dots,n-1$ and $w^{(n)}(t) \leqslant 0$ for $t \geqslant t_0+\varrho$. If $y(t) < 0$ for $t \geqslant t_0 > 0$, then either $\lim\limits_{t\to\infty} w(t) = \infty$ or $\lim\limits_{t\to\infty} w^{(i)}(t) = 0$, $i = 0,1,2,\dots,n-1$, $(-1)^{n+k}w^{(k)}(t) > 0$ for $k = 0,1,2,\dots,n-1$, and $w^{(n)}(t) \geqslant 0$ for $t \geqslant t_0+\varrho$.

The proof of Lemma 2.4 is given in [9].

3. Necessary and sufficient conditions for oscillation

In this section, we obtain necessary and sufficient conditions for all bounded solutions of (E) to oscillate and sufficient conditions for all solutions of (E) to oscillate.

Remark 1. Let $0 < t_0 < t$. Hence $\lim\limits_{t\to\infty} (t-t_0)/t = 1$. If $h(t) \geqslant 0$, $t \geqslant 0$, then

$$\int_T^\infty (t-T)^n h(t)\,\mathrm{d}t < \infty \text{ if and only if } \int_T^\infty t^n h(t)\,\mathrm{d}t < \infty \text{ for large } T.$$

Theorem 3.1. *Let $n \geqslant 3$ be odd and* (H_9) *hold. Then Eq.* (NE) *admits a positive bounded solution if and only if* (H_{10}) *holds.*

Proof. From (H_9) we obtain

$$-\infty < \int_{k\tau}^\infty (t-k\tau)^{n-1} f(t)\,\mathrm{d}t \leqslant 0$$

for every integer $k \geqslant 0$. Hence $\int_0^\infty t^{n-1}|f(t)|\,\mathrm{d}t < \infty$. There exists $F \in C^n([0,\infty),\mathbb{R})$ such that $F^{(n)}(t) = f(t)$, $\lim\limits_{t\to\infty} F(t) = 0$ and $F(t) > 0$, $t \geqslant 0$, by Lemma 2.1. Let $y(t)$ be a positive bounded solution of (NE) such that $y(t) > 0$ for $t \geqslant t_0 > 0$. Setting

$$z(t) = y(t) - y(t-\tau),\; w(t) = z(t) - F(t),\; t \geqslant t_0 + \varrho, \tag{3}$$

we obtain $w^{(n)}(t) = -Q(t)G(y(t-\sigma)) \leqslant 0$. Hence each of $w, w', w'', \ldots, w^{(n-1)}$ is monotonic and is of constant sign for large t. Since $w(t)$ is bounded, then $\lim\limits_{t\to\infty} w(t) = \ell$ exists. Hence $(-1)^{n+k} w^{(k)}(t) < 0$ for $k = 1, 2, \ldots, n-1$ for large t and $\lim\limits_{t\to\infty} w^{(i)}(t) = 0$ for $i = 1, 2, \ldots, n-1$. Further, $\lim\limits_{t\to\infty} z(t) = \ell$ exists. From Lemma 2.2 it follows that $\ell = 0$. Since n is odd, $w'(t) < 0$ for large t and hence $w(t) > 0$ for $t \geqslant t_1 > t_0$. From (3) we obtain $y(t) > y(t-\tau)$, $t \geqslant t_1$ because $F(t) > 0$. Hence $\liminf\limits_{t\to\infty} y(t) > 0$. Thus there exists $\beta > 0$ such that $y(t) > \beta$, for $t \geqslant t_2 > t_1$. Lemma 2.3 yields, for $t \geqslant t_3 > t_2 + \sigma$,

$$w(t) = \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1} Q(s) G(y(s-\sigma))\,\mathrm{d}s > \frac{G(\beta)}{(n-1)!}\int_t^\infty (s-t)^{n-1} Q(s)\,\mathrm{d}s,$$

that is,

$$y(t-\tau) < y(t) - \frac{G(\beta)}{(n-1)!}\int_t^\infty (s-t)^{n-1} Q(s)\,\mathrm{d}s + \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1} f(s)\,\mathrm{d}s,$$

since n is odd. Hence

$$y(t) < y(t+\tau) - \frac{G(\beta)}{(n-1)!}\int_{t+\tau}^\infty (s-t-\tau)^{n-1} Q(s)\,\mathrm{d}s + \frac{1}{(n-1)!}\int_{t+\tau}^\infty (s-t-\tau)^{n-1} f(s)\,\mathrm{d}s.$$

Repeating the process we obtain, for $t \geqslant t_3$,

$$y(t-\tau) < y(t+m\tau) - \frac{G(\beta)}{(n-1)!}\sum_{k=0}^{m}\int_{t+k\tau}^{\infty}(s-t-k\tau)^{n-1}Q(s)\,\mathrm{d}s$$
$$+\frac{1}{(n-1)!}\sum_{k=0}^{m}\int_{t+k\tau}^{\infty}(s-t-k\tau)^{n-1}f(s)\,\mathrm{d}s.$$

For $t = \ell\tau$, where ℓ is a positive integer large enough such that $\ell\tau > t_3$, we have

$$\frac{G(\beta)}{(n-1)!}\sum_{k=0}^{m}\int_{(k+\ell)\tau}^{\infty}(s-\ell\tau-k\tau)^{n-1}Q(s)\,\mathrm{d}s < y(\ell\tau+m\tau)-y(\ell\tau-\tau)$$
$$+\frac{1}{(n-1)!}\sum_{k=0}^{m}\int_{(k+\ell)\tau}^{\infty}(s-\ell\tau-k\tau)^{n-1}f(s)\,\mathrm{d}s. \tag{4}$$

However, the use of (H_9) yields

$$0 > \sum_{k=0}^{\infty}\int_{(k+\ell)\tau}^{\infty}(s-\ell\tau-k\tau)^{n-1}f(s)\,\mathrm{d}s = \sum_{i=\ell}^{\infty}\int_{i\tau}^{\infty}(s-i\tau)^{n-1}f(s)\,\mathrm{d}s$$
$$= \sum_{i=0}^{\infty}\int_{i\tau}^{\infty}(s-i\tau)^{n-1}f(s)\,\mathrm{d}s - \sum_{i=0}^{\ell-1}\int_{i\tau}^{\infty}(s-i\tau)^{n-1}f(s)\,\mathrm{d}s > -\infty,$$

because,

$$0 > \sum_{i=0}^{\ell-1}\int_{i\tau}^{\infty}(s-i\tau)^{n-1}f(s)\,\mathrm{d}s > \sum_{i=0}^{\infty}\int_{i\tau}^{\infty}(s-i\tau)^{n-1}f(s)\,\mathrm{d}s > -\infty.$$

Since $y(t)$ is bounded, we get from (4)

$$0 < \sum_{k=0}^{\infty}\int_{(k+\ell)\tau}^{\infty}(s-\ell\tau-k\tau)^{n-1}Q(s)\,\mathrm{d}s < \infty. \tag{5}$$

In particular,

$$0 < \int_{\ell\tau}^{\infty}(s-\ell\tau)^{n-1}Q(s)\,\mathrm{d}s < \infty.$$

From Remark 1 it follows that

$$0 < \int_{\ell\tau}^{\infty}s^{n-1}Q(s)\,\mathrm{d}s < \infty.$$

Hence

$$\int_0^\infty s^{n-1}Q(s)\,\mathrm{d}s = \int_0^{\ell\tau} s^{n-1}Q(s)\,\mathrm{d}s + \int_{\ell\tau}^\infty s^{n-1}Q(s)\,\mathrm{d}s < \infty,$$

$$\begin{aligned}\int_\tau^\infty (s-\tau)^{n-1}Q(s)\,\mathrm{d}s &= \int_\tau^{\ell\tau} (s-\tau)^{n-1}Q(s)\,\mathrm{d}s + \int_{\ell\tau}^\infty (s-\tau)^{n-1}Q(s)\,\mathrm{d}s\\ &< \int_\tau^{\ell\tau} (s-\tau)^{n-1}Q(s)\,\mathrm{d}s + \int_{\ell\tau}^\infty s^{n-1}Q(s)\,\mathrm{d}s < \infty\end{aligned}$$

and so on. Hence from (5) we have (H_{10}).

Next we assume that (H_{10}) holds. It is possible to choose $N > 0$, sufficiently large, such that

$$\sum_{k=N}^\infty \int_{k\tau}^\infty (t-k\tau)^{n-1}Q(t)\,\mathrm{d}t < \frac{(n-1)!}{2G(1)}$$

and

$$\sum_{k=N}^\infty \int_{k\tau}^\infty (t-k\tau)^{n-1}|f(t)|\,\mathrm{d}t < \frac{(n-1)!}{2},$$

that is,

$$\sum_{k=0}^\infty \int_{T+k\tau}^\infty (t-T-k\tau)^{n-1}Q(t)\,\mathrm{d}t < \frac{(n-1)!}{2G(1)}$$

and

$$\sum_{k=0}^\infty \int_{T+k\tau}^\infty (t-T-k\tau)^{n-1}|f(t)|\,\mathrm{d}t < (n-1)!/2,$$

where $T = N\tau$. Define

$$L(t) = \begin{cases} 0, & -\infty < t < T\\ \dfrac{G(1)}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{n-1}Q(s)\,\mathrm{d}s - \dfrac{1}{(n-1)!}\displaystyle\int_t^\infty (s-t)^{n-1}f(s)\,\mathrm{d}s, & t \geqslant T.\end{cases}$$

Then $L(t) > 0$ for $t \geqslant T$. Let

$$u(t) = \begin{cases} 0, & 0 \leqslant t < T\\ \displaystyle\sum_{i=0}^\infty L(t-i\tau), & t \geqslant T.\end{cases}$$

We may note that $\sum_{i=0}^\infty L(t-i\tau)$ contains only finitely many terms, $u(t) > 0$ for $t \geqslant T$ and $u(t) - u(t-\tau) = L(t)$, for $t \geqslant T$. Further, for $t \geqslant T$, it is possible to choose an integer $k \geqslant 0$ such that $T + k\tau \leqslant t < T + (k+1)\tau$. Hence, for $t \geqslant T$,

$$\begin{aligned}
u(t) &= \sum_{i=0}^{k} L(t - i\tau) = \frac{G(1)}{(n-1)!} \sum_{i=0}^{k} \int_{t-i\tau}^{\infty} (s - t + i\tau)^{n-1} Q(s)\,\mathrm{d}s \\
&\quad - \frac{1}{(n-1)!} \sum_{i=0}^{k} \int_{t-i\tau}^{\infty} (s - t + i\tau)^{n-1} f(s)\,\mathrm{d}s \\
&\leqslant \frac{G(1)}{(n-1)!} \sum_{j=0}^{k} \int_{T+(k-j)\tau}^{\infty} (s - T - (k-j)\tau)^{n-1} Q(s)\,\mathrm{d}s \\
&\quad + \frac{1}{(n-1)!} \sum_{j=0}^{k} \int_{T+(k-j)\tau}^{\infty} (s - T - (k-j)\tau)^{n-1} |f(s)|\,\mathrm{d}s \\
&= \frac{G(1)}{(n-1)!} \sum_{i=0}^{k} \int_{T+i\tau}^{\infty} (s - T - i\tau)^{n-1} Q(s)\,\mathrm{d}s \\
&\quad + \frac{1}{(n-1)!} \sum_{i=0}^{k} \int_{T+i\tau}^{\infty} (s - T - i\tau)^{n-1} |f(s)|\,\mathrm{d}s \\
&< \frac{G(1)}{(n-1)!} \sum_{i=0}^{\infty} \int_{T+i\tau}^{\infty} (s - T - i\tau)^{n-1} Q(s)\,\mathrm{d}s \\
&\quad + \frac{1}{(n-1)!} \sum_{i=0}^{k} \int_{T+i\tau}^{\infty} (s - T - i\tau)^{n-1} |f(s)|\,\mathrm{d}s \\
&< 1/2 + 1/2 = 1.
\end{aligned}$$

Let $X = BC([T,\infty),\mathbb{R})$, the Banach space of all real-valued bounded continuous functions on $[T,\infty)$ with supremum norm. Let $K = \{x \in X\colon\ x(t) \geqslant 0 \text{ for } t \geqslant T\}$. For $x, z \in X$, we define $x \leqslant z$ if and only if $z - x \in K$. Thus X is a partially ordered Banach space. Let $M = \{x \in X\colon\ 0 \leqslant x(t) \leqslant u(t),\ t \geqslant T\}$. Clearly, $x(t) \equiv 0$ for $t \geqslant T$ belongs to M and it is the infimum of M. Let $\varphi \subset M^* \subseteq M$. Define $x_0(t) = \sup\{x(t)\colon\ x \in M^*\}$, $t \geqslant T$. Hence x_0 is the supremum of M^* and $x_0 \in M$. For $y \in M$, define

$$Sy(t) = \begin{cases} y(t-\tau) + \dfrac{1}{(n-1)!} \displaystyle\int_t^{\infty} (s-t)^{n-1} Q(s) G(y(s-\sigma))\,\mathrm{d}s \\ \qquad\qquad + \dfrac{1}{(n-1)!} \displaystyle\int_t^{\infty} (s-t)^{n-1} |f(s)|\,\mathrm{d}s,\ t \geqslant T_1 \\ \dfrac{tu(t)}{T_1 u(T_1)} Sy(T_1) + (1 - \dfrac{t}{T_1}) u(t),\ T \leqslant t \leqslant T_1 \end{cases} \tag{6}$$

where $T_1 = T + \varrho$. From (6) we obtain, for $t \geqslant T_1$,

$$0 < Sy(t) \leqslant u(t-\tau) + \frac{G(1)}{(n-1)!} \int_t^\infty (s-t)^{n-1} Q(s)\,\mathrm{d}s + \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} |f(s)|\,\mathrm{d}s \leqslant u(t-\tau) + L(t) = u(t)$$

and, for $t \in [T, T_1]$, $0 < Sy(t) \leqslant (t/T_1)u(t) + (1 - (t/T_1))u(t) = u(t)$. Hence, $SM \subseteq M$. Further, for $y_1, y_2 \in M$, $y_1 \leqslant y_2$ implies that $Sy_1 \leqslant Sy_2$. From the Knaster-Tarski theorem (see [6, p. 30]) it follows that S has a fixed point y_0 in M. Hence, for $t \geqslant T_1$,

$$y_0(t) - y_0(t-\tau) = \frac{1}{(n-1)!} \int_t^\infty (s-t)^{n-1} [Q(s)G(y_0(s-\sigma)) + |f(s)|]\,\mathrm{d}s, \tag{7}$$

that is, $y_0(t)$ is a solution of (NE) for $t \geqslant T_1$ because n is odd. If $y_0(T_1) = 0$, then from (7) we have $y_0(T_1 - \tau) < 0$ since $y_0(t) = (1 - (t/T_1))u(t) \geqslant 0$, $t \in [T, T_1]$. On the other hand, we obtain $y_0(T_1 - \tau) = (1 - (T_1 - \tau)/T_1)u(T_1 - \tau) > 0$. This contradiction shows that $y_0(T_1) > 0$. For $T \leqslant t \leqslant T_1$,

$$y_0(t) = \frac{tu(t)}{T_1 u(T_1)} y_0(T_1) + \Big(1 - \frac{1}{T_1}\Big) u(t) > 0$$

implies that $y_0(t) > 0$ for $t \in [T_1, T_1 + r)$, where $r = \min\{\tau, \sigma\}$. Hence $y_0(t) > 0$ for $t \geqslant T_1$. Thus the theorem is proved. $\square$

E x a m p l e 1. The equation

$$(y(t) - y(t-1))''' + 6(t-2)^{-3}(t-1)^{-1}t^{-4}(t^4 - (t-1)^4)y^3(t-1) = 0,$$

$t \geqslant 3$, admits a positive bounded solution by Theorem 3.1. Indeed, $y(t) = 1 - t^{-1}$ is a positive bounded solution of the equation.

Theorem 3.2. *Let $n \geqslant 3$ be odd and (H_9') hold, where*

$$f(t) \geqslant 0, \quad 0 \leqslant \sum_{k=0}^{\infty} \int_{k\tau}^{\infty} (t - k\tau)^{n-1} f(t)\,\mathrm{d}t < \infty. \tag{H_9'}$$

Then Eq. (NE) *admits a negative bounded solution if and only if* (H_{10}) *holds.*

The proof is similar to that of the previous Theorem and hence is omitted.

Corollary 3.3. *Let $n \geqslant 3$ be odd. Then every bounded solution of* (E) *oscillates if and only if*

$$\sum_{k=0}^{\infty} \int_{k\tau}^{\infty} (t-k\tau)^{n-1} Q(t)\,\mathrm{d}t = \infty.$$

This follows from Theorem 3.1.

Theorem 3.4. *Consider*

$$x^{(m)}(t) + Q(t)x(t-\sigma) = 0,\ t \geqslant 0, \tag{8}$$

where $m \geqslant 2$ is an even integer, $Q \in C([0,\infty),[0,\infty))$ and $\sigma \geqslant 0$. Then the following statements are equivalent:

(a) *every bounded solution of* (8) *oscillates*
(b) $\int_0^\infty t^{m-1} Q(t)\,\mathrm{d}t = \infty$
(c) $\sum\limits_{k=0}^{\infty} \int_{k\tau}^{\infty} (t-k\tau)^{m-2} Q(t)\,\mathrm{d}t = \infty$, *where $\tau > 0$.*

Remark 2. (c) implies that

$$I \equiv \sum_{k=0}^{\infty} \int_{t_0+k\tau}^{\infty} (t-t_0-k\tau)^{m-2} Q(t)\,\mathrm{d}t = \infty$$

for every $t_0 > 0$. Indeed, there exists $r \geqslant 0$ such that $r\tau \leqslant t_0 < (r+1)\tau$. If possible, let $I < \infty$. Then

$$\sum_{k=0}^{\infty} \int_{(r+1)\tau+k\tau}^{\infty} (t-(r+1)\tau-k\tau)^{m-2} Q(t)\,\mathrm{d}t < \infty. \tag{9}$$

Hence

$$\int_{(r+1)\tau}^{\infty} (t-(r+1)\tau)^{m-2} Q(t)\,\mathrm{d}t < \infty.$$

From Remark 1 it follows that

$$\int_{(r+1)\tau}^{\infty} t^{m-2} Q(t)\,\mathrm{d}t < \infty.$$

Then

$$\int_{0}^{\infty} t^{m-2} Q(t)\,\mathrm{d}t < \infty$$

and hence

$$\int_{\tau}^{\infty} (t-\tau)^{m-2} Q(t)\,\mathrm{d}t < \infty,\ \ldots,\ \int_{r\tau}^{\infty} (t-r\tau)^{m-2} Q(t)\,\mathrm{d}t < \infty.$$

Consequently,

$$\sum_{k=0}^{\infty}\int_{k\tau}^{\infty}(t-k\tau)^{m-2}Q(t)\,\mathrm{d}t=\sum_{k=0}^{r}\int_{k\tau}^{\infty}(t-k\tau)^{m-2}Q(t)\,\mathrm{d}t$$
$$+\sum_{k=r+1}^{\infty}\int_{k\tau}^{\infty}(t-k\tau)^{m-2}Q(t)\,\mathrm{d}t<\infty,$$

by using (9), a contradiction. Hence the claim holds.

Proof of Theorem 3.4. We show that (a) $\Leftrightarrow$ (c) and (a) $\Leftrightarrow$ (b). Hence (b) $\Leftrightarrow$ (c).

Suppose (a) holds. If possible, let

$$\sum_{k=0}^{\infty}\int_{k\tau}^{\infty}(t-k\tau)^{m-2}Q(t)\,\mathrm{d}t<\infty.$$

Hence we can choose an integer $N>0$ such that $N\tau>\sigma$ and

$$\frac{\tau}{(m-2)!}\sum_{k=N}^{\infty}\int_{k\tau}^{\infty}(t-k\tau)^{m-2}Q(t)\,\mathrm{d}t<\frac{1}{3},$$

that is,

$$\frac{\tau}{(m-2)!}\sum_{k=0}^{\infty}\int_{T+k\tau}^{\infty}(t-T-k\tau)^{m-2}Q(t)\,\mathrm{d}t<\frac{1}{3},$$

where $T=N\tau$. Let $X=BC([0,\infty),\mathbb{R})$ and

$$M=\{x\in X\colon\ 1\leqslant x(t)\leqslant 3/2\}.$$

Hence M is a complete metric space. For $x\in M$, define

$$Sx(t)=\begin{cases}1, & 0\leqslant t\leqslant T\\ 1+\dfrac{1}{(m-2)!}\displaystyle\int_T^t\Big(\int_s^{\infty}(u-s)^{m-2}Q(u)x(u-\sigma)\,\mathrm{d}u\Big)\,\mathrm{d}s, & t\geqslant T.\end{cases}$$

For $t\geqslant T$, we have

$$\begin{aligned}1<Sx(t)&<1+\frac{1}{(m-2)!}\int_T^{\infty}\Big(\int_s^{\infty}(u-s)^{m-2}Q(u)x(u-\sigma)\,\mathrm{d}u\Big)\,\mathrm{d}s\\
&=1+\frac{1}{(m-2)!}\sum_{k=0}^{\infty}\int_{T+k\tau}^{T+(k+1)\tau}\Big(\int_s^{\infty}(u-s)^{m-2}Q(u)x(u-\sigma)\,\mathrm{d}u\Big)\,\mathrm{d}s\\
&<1+\frac{\tau}{(m-2)!}\sum_{k=0}^{\infty}\int_{T+k\tau}^{\infty}(u-T-k\tau)^{m-2}Q(u)x(u-\sigma)\,\mathrm{d}u\\
&<1+\frac{1}{2}=\frac{3}{2}.\end{aligned}$$

Hence $SM \subset M$. Further, it may be shown that $\|Sx_1 - Sx_2\| < \frac{1}{3}\|x_1 - x_2\|$ for x_1 and $x_2 \in M$. Hence S is a contraction. Consequently, S has a unique fixed point $x \in M$. It is a positive bounded solution of (8) on $[T, \infty)$, a contradiction. Thus (a) $\Rightarrow$ (c). Next suppose that (c) holds. Let $x(t)$ be a bounded nonoscillatory solution of (8). We may take, without any loss of generality, $x(t) > 0$ for $t \geqslant t_0 > 0$. Then $x^{(m)}(t) \leqslant 0$ for $t \geqslant t_0 + \sigma$. Hence each of $x, x', \ldots, x^{(m-1)}$ is monotonic and is of constant sign for large t. Since $x(t)$ is bounded, then $\lim\limits_{t\to\infty} x(t) = \ell > 0$ exists, $(-1)^{m+k}x^{(k)}(t) < 0$ for $t \geqslant t_1 > t_0$ and $\lim\limits_{t\to\infty} x^{(i)}(t) = 0$, $i = 1, 2, \ldots, m-1$. Let $x(t) > \alpha > 0$, for $t > t_2 > t_1$. From Lemma 2.3 we obtain, for $t \geqslant t_3 > t_2 + \sigma$,

$$x'(t) = \frac{1}{(m-2)!}\int_t^\infty (s-t)^{m-2}Q(s)x(s-\sigma)\,\mathrm{d}s.$$

Hence

$$\begin{aligned}\sum_{k=0}^{j}\int_{t_3+k\tau}^{t_3+(k+1)\tau} & x'(t)\,\mathrm{d}t \\ &= \frac{1}{(m-2)!}\sum_{k=0}^{j}\int_{t_3+k\tau}^{t_3+(k+1)\tau}\left(\int_t^\infty (s-t)^{m-2}Q(s)x(s-\sigma)\,\mathrm{d}s\right)\mathrm{d}t,\end{aligned}$$

that is,

$$\begin{aligned}x(t_3+(j+1)\tau) - x(t_3) &\geqslant \frac{\tau}{(m-2)!}\sum_{k=0}^{j}\int_{t_3+(k+1)\tau}^\infty (s-t_3-k\tau-\tau)^{m-2}Q(s)x(s-\sigma)\,\mathrm{d}s \\ &> \frac{\alpha\tau}{(m-2)!}\sum_{k=0}^{j}\int_{t_3+(k+1)\tau}^\infty (s-t_3-k\tau-\tau)^{m-2}Q(s)\,\mathrm{d}s.\end{aligned}$$

Since $x(t)$ is bounded, we obtain

$$\sum_{k=0}^{\infty}\int_{t_3+(k+1)\tau}^\infty (s-t_3-k\tau-\tau)^{m-2}Q(s)\,\mathrm{d}s < \infty,$$

a contradiction by Remark 2. Thus (c) $\Rightarrow$ (a).

Suppose that (a) holds. Let $\int_0^\infty t^{m-1}Q(t)\,\mathrm{d}t < \infty$. Hence $\int_t^\infty s^{m-1}Q(s)\,\mathrm{d}s \to 0$ as $t \to \infty$. Since $0 < \int_t^\infty (s-t)^{m-1}Q(s)\,\mathrm{d}s < \int_t^\infty s^{m-1}Q(s)\,\mathrm{d}s$, then $\int_t^\infty (s-t)^{m-1}Q(s)\,\mathrm{d}s < (m-1)!/4$ for $t \geqslant t_0 > \sigma$. Setting $M = \{x \in X\colon\ 3/4 \leqslant x(t) \leqslant 1\}$ and, for $x \in M$,

$$Sx(t) = \begin{cases} Sx(t_0), & 0 \leqslant t \leqslant t_0 \\ 1 - \dfrac{1}{(m-1)!}\displaystyle\int_t^\infty (s-t)^{m-1}Q(s)x(s-\sigma)\,\mathrm{d}s, & t \geqslant t_0 \end{cases}$$

we obtain $SM \subseteq M$ and S is a contraction. Hence S has a unique fixed point in M and it is a positive bounded solution of (8), a contradiction. Hence (a) $\Rightarrow$ (b). Next suppose that (b) holds. Let $x(t)$ be a bounded nonoscillatory solution of (8) such that $x(t) > 0$ for $t \geqslant t_0 > 0$. Proceeding as in the proof of the case (c) $\Rightarrow$ (a), we obtain $\lim\limits_{t\to\infty} x(t) = \ell > 0$ exists and $\lim\limits_{t\to\infty} x^{(i)}(t) = 0$, $i = 1, 2, \ldots, m-1$. From Lemma 2.3 we obtain

$$x'(t) = -\nu'(t)$$

because $\lim\limits_{t\to\infty} x'(t) = 0$, where

$$\nu(t) = \frac{(-1)^m}{(m-1)!} \int_t^\infty (s-t)^{m-1} Q(s) x(s-\sigma)\,\mathrm{d}s.$$

Hence

$$x(t) = \mu + \frac{1}{(m-1)!} \int_t^\infty (s-t)^{m-1} Q(s) x(s-\sigma)\,\mathrm{d}s,$$

where $\infty > \mu = \ell + \nu(\infty) > 0$ because $0 \leqslant \nu(\infty) < \infty$. Suppose that $x(t) > \ell/2$, for $t \geqslant t_2 > 0$. From the above identity, we obtain

$$\int_{t_3}^\infty (s-t_3)^{m-1} Q(s) x(s-\sigma)\,\mathrm{d}s < \infty$$

where $t_3 > t_2 + \sigma$. On the other hand, the use of (b) yields

$$\int_{t_3}^\infty (s-t_3)^{m-1} Q(s) x(s-\sigma)\,\mathrm{d}s > \ell/2 \int_{t_3}^\infty (s-t_3)^{m-1} Q(s)\,\mathrm{d}s = \infty,$$

a contradiction. Hence (b) $\Rightarrow$ (a). Thus the theorem is proved.

Remark 3. From Theorem 3.4 we have (b) $\Leftrightarrow$ (c).

Corollary 3.5. *Let $n \geqslant 3$ be odd. Then every bounded solution of* (E) *oscillates if and only if*

$$\int_0^\infty t^n Q(t)\,\mathrm{d}t = \infty. \tag{H$_{12}$}$$

This follows from Corollary 3.3 and Theorem 3.4 by taking $m = n + 1$.

Remark 4. We may note that (H$_3$) $\Rightarrow$ (H$_2$) $\Rightarrow$ (H$_{12}$). Hence Corollary 3.5 improves Theorems A and D.

Theorem 3.6. *Suppose that* (H_1) *and* (H_2) *hold. If* $n \geqslant 3$ *is odd, then every solution of* (E) *oscillates.*

P r o o f. Let $y(t)$ be a nonoscillatory solution of (E). Then $y(t) > 0$ or $y(t) < 0$ for $t \geqslant t_0 > 0$. Let $y(t) > 0$ for $t \geqslant t_0$. Setting $z(t)$ as in (3) for $t \geqslant t_0 + \varrho$, we obtain $z^{(n)}(t) = -Q(t)G(y(t-\sigma))$. From Lemma 2.4 it follows that either $\lim\limits_{t\to\infty} z(t) = -\infty$ or $\lim\limits_{t\to\infty} z^{(i)}(t) = 0$, $i = 0,1,2,\ldots,n-1$ and $(-1)^k z^{(k)}(t) > 0$ for $t \geqslant t_0 + \varrho$, $k = 0,1,2,\ldots,n-1$ because n is odd. However, $\lim\limits_{t\to\infty} z(t) = -\infty$ implies that $z(t) < 0$ for $t \geqslant t_1 > t_0 + \varrho$, that is, $y(t) < y(t-\tau)$, $t \geqslant t_1$. Hence $y(t)$ is bounded. Consequently, $z(t)$ is bounded, a contradiction. Therefore $\lim\limits_{t\to\infty} z(t) = 0$ and $z'(t) < 0$, $t \geqslant t_0 + \varrho$. Then $z(t) > 0$, that is, $y(t) > y(t-\tau)$, for $t \geqslant t_0 + \varrho$. From this we obtain $\liminf\limits_{t\to\infty} y(t) > 0$. There exists $\beta > 0$ such that $y(t) > \beta$ for $t \geqslant t_3 > t_0 + \varrho$. Further, by Lemma 2.3,

$$z(t) = \frac{1}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)G(y(s-\sigma))\,\mathrm{d}s > \frac{G(\beta)}{(n-1)!}\int_t^\infty (s-t)^{n-1}Q(s)\,\mathrm{d}s$$

for $t \geqslant t_4 > t_3 + \varrho$. Hence

$$\int_{t_4}^\infty (s-t_4)^{n-1}Q(s)\,\mathrm{d}s < \infty,$$

that is,

$$\int_{t_4}^\infty s^{n-1}Q(s)\,\mathrm{d}s < \infty$$

by Remark 1, a contradiction to (H_2). If $y(t) < 0$ for $t \geqslant t_0$, then we set $x(t) = -y(t) > 0$ to obtain

$$[x(t) - x(t-\tau)]^{(n)} + Q(t)H(x(t-\sigma)) = 0,\ t \geqslant 0,$$

where $H(u) = -G(-u)$. Proceeding as above we get a contradiction. Hence the theorem is proved. □

E x a m p l e 2. Consider

$$(y(t) - y(t-\pi))''' + 2y\big(t - \tfrac{3}{2}\pi\big) = 0,\ t \geqslant 0. \tag{10}$$

Every bounded solution of (10) oscillates by Corollary 3.5 and every solution of (10) oscillates by Theorem 3.6. Indeed, $y(t) = \sin t$ is a bounded oscillatory solution of (10).

E x a m p l e 3. Every solution of the equation

$$(y(t)-y(t-2\pi))''' + 2\sqrt{2}e^{-7\pi/4}(e^{2\pi}-1)y\big(t-\tfrac{1}{4}\pi\big)=0,\ t\geqslant 0,$$

oscillates by Theorem 3.6. In particular, $y(t)=e^t\cos t$ is an unbounded oscillatory solution of the equation.

R e m a r k 5. Theorem 3.6 improves Theorem A. Since $(H_3)\Rightarrow(H_2)$, it also improves Theorem D for superlinear G in view of the assumption (H_1).

Theorem 3.7. *Let* (H_4) *and* (H_8) *hold. Then every solution of* (NE) *oscillates or tends to zero as* $t\to\infty$.

P r o o f. Let $y(t)$ be a nonoscillatory solution of (NE) such that $y(t)>0$ for $t\geqslant t_0>0$. Setting $w(t)$ as in (3) for $t\geqslant t_0+\varrho$, we have

$$w^{(n)}(t) = -Q(t)G(y(t-\sigma))\leqslant 0. \tag{11}$$

Hence each of $w, w', \dots, w^{(n-1)}$ is monotonic and is of constant sign for $t\geqslant T>t_0+\varrho$. If $\lim\limits_{t\to\infty} w^{(n-1)}(t)=-\infty$, then $\lim\limits_{t\to\infty} w(t)=-\infty$ and hence $z(t)<0$ for large t. Since $y(t)<y(t-\tau)$, then $y(t)$ is bounded and hence $w(t)$ is bounded by (H_4), a contradiction. Hence $\lim\limits_{t\to\infty} w^{(n-1)}(t)=\ell$ exists. Integrating (11) we obtain, for $t\geqslant T$,

$$\int_t^\infty Q(s)G(y(s-\sigma))\,ds = w^{(n-1)}(t)-\ell. \tag{12}$$

If $y(t)$ is unbounded, then there exists a sequence $\{t_n\}\subset[T,\infty)$ such that $t_n\to\infty$ and $y(t_n)\to\infty$ as $n\to\infty$. For $M>0$, there exists $N_1>0$ such that $y(t_n)>M$ for $n\geqslant N_1$. Since y is continuous, there exists $\delta_n>0$ with $\liminf\limits_{n\to\infty}\delta_n>0$ such that $y(t)>M$ for $t\in(t_n-\delta_n, t_n+\delta_n)$. If $\delta_n>\delta>0$ for $n\geqslant N>N_1$, then

$$\begin{aligned}\int_T^\infty Q(s)G(y(s-\sigma))\,ds &\geqslant \sum_{n=N}^\infty\int_{t_n-\delta_n+\sigma}^{t_n+\delta_n+\sigma} Q(s)G(y(s-\sigma))\,ds\\ &> G(M)\sum_{n=N}^\infty\int_{t_n-\delta_n+\sigma}^{t_n+\delta_n+\sigma} Q(s)\,ds\\ &> G(M)\sum_{n=N}^\infty\int_{t_n-\delta+\sigma}^{t_n+\delta+\sigma} Q(s)\,ds=\infty\end{aligned}$$

due to (H_8), a contradiction to (12). Hence $y(t)$ is bounded. If $\limsup\limits_{t\to\infty} y(t)=\alpha>0$, then there exists a sequence $\{t_n\}$ such that $y(t_n)>\beta>0$ for large n. Proceeding as above we obtain a contradiction due to (H_8). Hence $\limsup\limits_{t\to\infty} y(t)=0$. Consequently $\lim\limits_{t\to\infty} y(t)=0$. If $y(t)<0$ for $t\geqslant t_0$, then we set $x(t)=-y(t)$ and show that $\lim\limits_{t\to\infty} x(t)=0$. Thus the proof of the theorem is complete. $\square$

R e m a r k 6. Theorem 3.7 improves Theorem C.

E x a m p l e 4. Consider

$$(y(t)-y(t-2\pi))''' + \mathrm{e}^{-3\pi/2} y^3\big(t-\tfrac{1}{2}\pi\big) = f(t), \ t \geqslant 0, \tag{13}$$

where $f(t) = \mathrm{e}^{-3t}\sin^3 t - 2(\mathrm{e}^{2\pi}-1)\mathrm{e}^{-t}\,(\cos t - \sin t)$. Hence

$$F(t) = -\frac{1}{2}\int_t^\infty (s-t)^2[\mathrm{e}^{-3s}\sin^3 s - 2(\mathrm{e}^{2\pi}-1)\mathrm{e}^{-s}(\cos s - \sin s)]\,\mathrm{d}s.$$

Then, for $t>0$,

$$\begin{aligned} |F(t)| &\leqslant \frac{1}{2}\int_t^\infty (s-t)^2\mathrm{e}^{-3s}\,\mathrm{d}s + 2(\mathrm{e}^{2\pi}-1)\int_t^\infty (s-t)^2\mathrm{e}^{-s}\,\mathrm{d}s \\ &< \frac{1}{2}\int_t^\infty s^2\mathrm{e}^{-3s}\,\mathrm{d}s + 2(\mathrm{e}^{2\pi}-1)\int_t^\infty s^2\mathrm{e}^{-s}\,\mathrm{d}s, \text{ which } \to 0 \text{ as } t\to\infty. \end{aligned}$$

Every solution of (13) oscillates or tends to zero as $t\to\infty$ by Theorem 3.7. In particular, $y(t) = \mathrm{e}^{-t}\cos t$ is an oscillatory solution of (13) and $y(t)\to 0$ as $t\to\infty$.

E x a m p l e 5. $y(t) = \mathrm{e}^{-t}$ is a nonoscillatory solution of the equation

$$(y(t)-y(t-\log 2))^{(4)} + (\mathrm{e}^{-3}+\mathrm{e}^{2t-3})y^3(t-1) = \mathrm{e}^{-3t}, \ t\geqslant 0,$$

and $y(t)\to 0$ as $t\to\infty$. This illustrates Theorem 3.7. Here $F(t) = \mathrm{e}^{-3t}/81$.

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Authors' addresses: *N. Parhi*, Plot No. 1365/3110, Shastri Nagar, Unit-4, Bhubaneswar-751 001, Orissa, India, e-mail: `parhi2002@rediffmail.com`; *R. N. Rath*, P. G. Department of Mathematics, Khallikote Autonomous College, Berhampur-760001, Orissa, India, e-mail: `radhanathmath@yahoo.co.in`.