A REMARK ON BRANCH WEIGHTS IN COUNTABLE TREES

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Abstract. Let T be a tree, let u be its vertex. The branch weight $b(u)$ of u is the maximum number of vertices of a branch of T at u. The set of vertices u of T in which $b(u)$ attains its minimum is the branch weight centroid $B(T)$ of T. For finite trees the present author proved that $B(T)$ coincides with the median of T, therefore it consists of one vertex or of two adjacent vertices. In this paper we show that for infinite countable trees the situation is quite different.

Keywords: branch weight, branch weight centroid, tree, path, degree of a vertex

MSC 2000: 05C05

In this paper we study the branch weight in countable infinite trees. We define it analogously to the case of finite trees.

Let T be a tree with the vertex set $V(T)$, let $u \in V(T)$. On the set $V(T) - \{u\}$ consider the binary relation β such that $(x, y) \in \beta$ if and only if the path connecting x and y in T does not contain u. The relation β is evidently an equivalence relation on $V(T) - \{u\}$. Each equivalence class of β induces a subtree of T which is called a branch of T at u. The set of all branches of T at u will be denoted by $\mathcal{B}(T, u)$.

The vertex set of every branch B will be denoted by $V(B)$, its cardinality by $|V(B)|$. If T is a countable tree, then $|V(B)|$ is either a positive integer, or the cardinal number $\aleph_0$.

For each $u \in V(T)$ let $b(u) = \max(|V(B)|\colon B \in \mathcal{B}(T, u))$. The number $b(u)$ is called the branch weight of u in T.

The set of the vertices u of T in which $b(u)$ attains its minimum is called the branch weight centroid $B(T)$ of T (see e.g. [2], [3]). For finite trees the present author [4] proved, solving one problem from the book [1], that $B(T)$ coincides with the median of T; therefore it consists either of one vertex, or of two adjacent vertices.

Here we will study countably infinite trees.

Theorem 1. *Let T be a countable infinite tree. If T contains an infinite path or at least two vertices of infinite degree, then $B(T) = V(T)$.*

P r o o f. Suppose that the condition is satisfied. Let $u \in V(T)$. If T contains at least two vertices of infinite degree, then at least one of them, say w, is in $V(T) - \{u\}$. As the vertex sets of branches of T at u form a partition of $V(T) - \{u\}$, the vertex w lies on some branch $B^* \in \mathcal{B}(T, u)$ and together with it an infinite number of neighbours of w lie in B^*. Therefore $|V(B^*)| = \aleph_0$. If T contains an infinite path P, then evidently there exists $B^{**} \in \mathcal{B}(T, u)$ which contains either the whole path P, or at least its one-way infinite subpath which does not contain u. Again $|V(B^{**})| = \aleph_0$. We have proved that $b(u) = \aleph_0$ for each vertex $u \in V(T)$. This is also the minimum of $b(u)$ over all $u \in V(T)$ and thus the branch weight centroid $B(T)$ is equal to the whole vertex set $V(T)$. □

Theorem 2. *Let T be a countable infinite tree. Suppose that the following two conditions are satisfied:*

(i) *The tree T contains no infinite path.*
(ii) *The tree T contains exactly one vertex w of infinite degree.*

Then one of the following two assertions is true:

(iii) *The branch weight centroid $B(T) = \{w\}$.*
(iv) *The branch weight centroid $B(T)$ is undefined.*

P r o o f. Consider $\mathcal{B}(T, w)$ and let $B^0 \in \mathcal{B}(T, w)$. Suppose that B^0 is infinite. Then it is a locally finite subtree of T, because $V(B^0) \subseteq V(T) - \{w\}$ and w is the unique vertex of T of infinite degree. However, as is well-known, in that case B^0 contains an infinite path, which is a contradiction with (i). Hence all branches of T at w are finite. If there exists a finite upper bound of the values $|V(B)|$ for $B \in \mathcal{B}(T, w)$, then there exists a finite maximum of those values and this is $b(w)$. If that upper bound does not exist, that maximum does not exist, either and $b(w)$ is undefined. Analogously as in the proof of Theorem 1 we prove that $b(u) = \aleph_0$ for each $u \in V(T) - \{w\}$. Thus in the first case w is the unique vertex with the finite branch weight and $B(T) = \{w\}$. In the other the branch weight is not defined for all vertices of T and consequently $B(T)$ is undefined. □

Corollary. *Let T be a countable infinite tree. Then either $B(T) = V(T)$, or T can be obtained by the following simple construction:*

Take a sequence $(T_n)_{n=1}^{\infty}$ of pairwise vertex-disjoint finite trees and a vertex w outside of all of them. In each tree T_n choose a vertex v_n. Join each vertex v_n by an edge with w (for $n = 1, 2, 3, \ldots$).

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