ON MAGIC AND SUPERMAGIC LINE GRAPHS

J. Ivančo, Z. Lastivková, A. Semaničová, Košice

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Abstract. A graph is called magic (supermagic) if it admits a labelling of the edges by pairwise different (consecutive) positive integers such that the sum of the labels of the edges incident with a vertex is independent of the particular vertex. We characterize magic line graphs of general graphs and describe some class of supermagic line graphs of bipartite graphs.

Keywords: magic graphs, supermagic graphs, line graphs

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1. Introduction

We consider finite undirected graphs without loops, multiple edges and isolated vertices. If G is a graph, then $V(G)$ and $E(G)$ stand for the vertex set and edge set of G, respectively. Cardinalities of these sets, denoted by $|V(G)|$ and $|E(G)|$, are called the *order* and the *size* of G.

Let a graph G and a mapping f from $E(G)$ into positive integers be given. The *index-mapping* of f is the mapping f^* from $V(G)$ into positive integers defined by

$$f^*(v) = \sum_{e \in E(G)} \eta(v,e) f(e) \quad \text{for every } v \in V(G),$$

where $\eta(v,e)$ is equal to 1 when e is an edge incident with a vertex v, and 0 otherwise. An injective mapping f from $E(G)$ into positive integers is called a *magic labelling* of G for *index* λ if its index-mapping f^* satisfies

$$f^*(v) = \lambda \quad \text{for all } v \in V(G).$$

A magic labelling f of G is called a *supermagic labelling* of G if the set $\{f(e)\colon e \in E(G)\}$ consists of consecutive positive integers. We say that a graph G is *supermagic* (*magic*) if and only if there exists a supermagic (magic) labelling of G. Note that any supermagic regular graph G admits a supermagic labelling into the set $\{1, \dots, |E(G)|\}$. In the sequel we will consider only such supermagic labellings.

The concept of magic graphs was introduced by Sedláček [8]. The regular magic graphs are characterized in [2]. Two different characterizations of all magic graphs are given in [6] and [5].

Supermagic graphs were introduced by M. B. Stewart [9]. It is easy to see that the classical concept of a magic square of n^2 boxes corresponds to the fact that the complete bipartite graph $K_{n,n}$ is supermagic for every positive integer $n \neq 2$ (see also [9]). Stewart [10] characterized supermagic complete graphs. In [7] and [1] supermagic labellings of the Möbius ladders and two special classes of 4-regular graphs are constructed. In [4] supermagic regular complete multipartite graphs and supermagic cubes are characterized. Some constructions of supermagic labellings of various classes of regular graphs are described in [3] and [4].

The *line graph* $L(G)$ of a graph G is the graph with vertex set $V(L(G)) = E(G)$, where $e, e' \in E(G)$ are adjacent in $L(G)$ whenever they have a common end vertex in G. In the paper we deal with magic and supermagic line graphs.

2. Magic line graphs

In this section we characterize magic line graphs of connected graphs. Since, except for the complete graph of order 2, no graph with less than 5 vertices is magic, we consider connected graphs of size at least 5.

We say that a graph G is of *type* $\mathcal{A}$ if it has two edges e_1, e_2 such that $G - \{e_1, e_2\}$ is a balanced bipartite graph with a partition V_1, V_2, and the edge e_i joins two vertices of V_i $(i = 1, 2)$. A graph G is of *type* $\mathcal{B}$ if it has two edges e_1, e_2 such that $G - \{e_1, e_2\}$ has a component H which is a balanced bipartite graph with partition V_1, V_2, and e_i joins a vertex of V_i with a vertex of $V(G) - V(H)$ $(i = 1, 2)$. As usual, for $S \subset V(G)$, $\Gamma(S)$ denotes the set of vertices adjacent to a vertex in S.

Proposition 1 (Jeurissen [5]). *A connected non-bipartite graph G is magic if and only if G is neither of type $\mathcal{A}$ nor of type $\mathcal{B}$, and $|\Gamma(S)| > |S|$ for every independent non-empty subset S of $V(G)$.*

Denote by $\mathcal{F}_1$ the family of connected graphs which contain an edge uv such that $\deg(u) + \deg(v) = 3$. By $\mathcal{F}_2$ we denote the family of all connected unicyclic graphs with a 1-factor. $\mathcal{F}_3$ denotes the family of connected graphs which contain edges vu and uw such that $\deg(v) + \deg(u) = \deg(u) + \deg(w) = 4$. $\mathcal{F}_4$ is the family of six graphs illustrated in Figure. Finally, let $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3 \cup \mathcal{F}_4$.

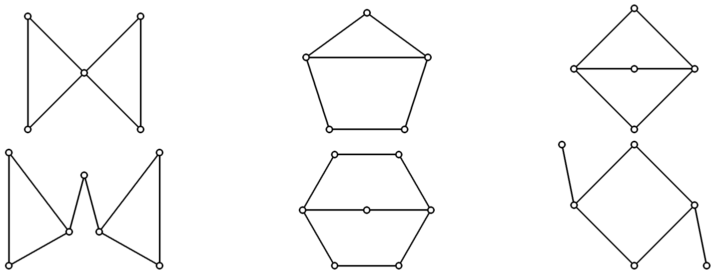

Figure. The family $\mathcal{F}_4$

The main result of this section is

Theorem 1. *Let G be a connected graph of size at least 5. The line graph $L(G)$ is magic if and only if $G \notin \mathcal{F}$.*

P r o o f. Assume that the line graph of G is not magic. If each vertex of G has degree at most 2, then G is either a path or a cycle, i.e., $G \in \mathcal{F}_1 \cup \mathcal{F}_3$. Next, we suppose that the maximum degree of G is at least 3. So, $L(G)$ is non-bipartite. According to Proposition 1, we consider the following cases.

A. There is an independent set $S \subset V(L(G))$ such that $|\Gamma(S)| \leqslant |S|$. Suppose that $S = \{e_1, \ldots, e_k\}$ is minimal possible. If $|S| = 1$, then $|\Gamma(\{e_1\})| = 1$, i.e., e_1 is a terminal edge of G with end vertices of degree 1 and 2. Thus, $G \in \mathcal{F}_1$.

If $|S| > 1$, then any edge of G is adjacent to at least two others. The edges $e_1, \ldots, e_k$ are independent, thus any edge of G is adjacent to at most two of them. Therefore,

$$|S| \geqslant |\Gamma(S)| = |\Gamma(\{e_1\}) \cup \ldots \cup \Gamma(\{e_k\})| \geqslant \tfrac{1}{2}(|\Gamma(\{e_1\})| + \ldots + |\Gamma(\{e_k\})|) \geqslant \tfrac{1}{2}2k = |S|.$$

It means $|\Gamma(S)| = |S|$ and any edge of $\Gamma(S)$ is adjacent to exactly two edges of S. As G is a connected graph, $|E(G)| = |S \cup \Gamma(S)| = 2|S| = |V(G)|$. So, G is unicyclic and S is its 1-factor, i.e., $G \in \mathcal{F}_2$.

B. Suppose that $L(G)$ is of type $\mathcal{B}$. Then there is a set $E' \subset E(G)$ such that the subgraph L' of $L(G)$ induced by E' is a balanced bipartite graph connected by a pair of edges to another subgraph. Since L' is bipartite, every vertex of the subgraph G' of G induced by E' is of degree at most two, i.e., every component of G' is either a path or an even cycle. Moreover, the set $E(G) - E'$ contains either one edge incident with a 2-vertex (i.e., vertex of degree 2) of G', or a pair of edges incident with two 1-vertices of G'. Consider the following subcases.

B1. G' contains an even cycle. Then only one edge of $E(G) - E'$ is incident with its vertex. Thus, some two adjacent edges of this cycle have both end vertices of degree 2 in G, i.e., $G \in \mathcal{F}_3$.

B2. G' consists of two paths. Then a pair of edges of $E(G) - E'$ is incident with its terminal vertices. The other terminal vertices of G' are terminal in G, too. Evidently, in this case $G \in \mathcal{F}_1$.

B3. G' is a path connected by one edge to another subgraph. Then either $|E'| > 2$ and $G \in \mathcal{F}_1$, or $|E'| = 2$ and $G \in \mathcal{F}_3$, because both edges of E' have end vertices of degree 1 and 3 in G.

B4. G' is a path connected by a pair of edges to another subgraph. Then any two adjacent edges of this path have both end vertices of degree 2 in G, i.e., $G \in \mathcal{F}_3$.

C. Suppose that $L(G)$ is of type $\mathcal{A}$. Moreover, assume that $G \notin \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$. For $d \leqslant 2$, every d-vertex of G is adjacent to some vertex of degree at least 3, because $G \notin \mathcal{F}_1 \cup \mathcal{F}_3$. As $L(G)$ is a balanced bipartite graph with two added edges, $6 \leqslant |E(G)| \equiv 0 \pmod 2$ and G contains either one 4-vertex or two 3-vertices. One can easily see that $G \in \mathcal{F}_4$ in this case.

The converse implication is obvious. □

It is easy to see that the complexity of deciding whether the graph G belongs to the family F_i $(i = 1, 2, 3, 4)$ is polynomial. Using the Even-Kariv algorithm for finding 1-factor in G we get that testing whether the line graph of a given graph is magic has computational complexity $O(n^{5/2})$. Moreover, each graph of the family $\mathcal{F}$ contains a vertex of degree at most two. Thus, we immediately obtain

Corollary 1. *Let G be a connected graph with minimum degree at least 3. Then $L(G)$ is a magic graph.*

3. Supermagic line graphs

The problem of characterizing supermagic line graphs of general graphs seems to be difficult. It is solved in this section for regular bipartite graphs.

Let $K_{k[n]}$ denote the complete k-partite graph whose every part has n vertices. As usual, the union of m disjoint copies of a graph G is denoted by mG. In the sequel we will use the following assertions proved in [4].

Proposition 2 ([4]). *Let $F_1, F_2, \ldots, F_k$ be mutually edge-disjoint supermagic factors of a graph G which form its decomposition. Then G is supermagic.*

Proposition 3 ([4]). *The graph $mK_{k[n]}$ is supermagic if and only if one of the following conditions is satisfied:*

(1) $n = 1$, $k = 2$, $m = 1$;

(2) $n = 1$, $k = 5$, $m \geqslant 2$;

(3) $n = 1$, $5 < k \equiv 1 \pmod 4$, $m \geqslant 1$;
(4) $n = 1$, $6 \leqslant k \equiv 2 \pmod 4$, $m \equiv 1 \pmod 2$;
(5) $n = 1$, $7 \leqslant k \equiv 3 \pmod 4$, $m \equiv 1 \pmod 2$;
(6) $n = 2$, $k \geqslant 3$, $m \geqslant 1$;
(7) $3 \leqslant n \equiv 1 \pmod 2$, $2 \leqslant k \equiv 1 \pmod 4$, $m \geqslant 1$;
(8) $3 \leqslant n \equiv 1 \pmod 2$, $2 \leqslant k \equiv 2 \pmod 4$, $m \equiv 1 \pmod 2$;
(9) $3 \leqslant n \equiv 1 \pmod 2$, $2 \leqslant k \equiv 3 \pmod 4$, $m \equiv 1 \pmod 2$;
(10) $4 \leqslant n \equiv 0 \pmod 2$, $k \geqslant 2$, $m \geqslant 1$.

Note that all edges of a graph G incident with a vertex v induce a subgraph $K(v)$ of $L(G)$, which is isomorphic to the complete graph of order $\deg(v)$. Subgraphs $K(v)$, for all $v \in V(G)$, are edge-disjoint and form a decomposition of $L(G)$. If vertices u and v of G are not adjacent, then $K(u)$ and $K(v)$ are vertex-disjoint subgraphs of $L(G)$. So, for a bipartite graph G with parts V_1 and V_2, the subgraph $R_1(G) = \bigcup\limits_{v\in V_1} K(v)$ $\big(R_2(G) = \bigcup\limits_{v\in V_2} K(v)\big)$ consists of mutually disjoint complete subgraphs of $L(G)$. Moreover, $R_1(G)$ and $R_2(G)$ are spanning subgraphs of $L(G)$ which form its decomposition.

Let d_1, d_2, q be positive integers and let $\mathcal{G}(q; d_1, d_2)$ be the family of all bipartite graphs of size q whose every edge joins a d_1-vertex to a d_2-vertex. Clearly, there is a vertex partition $\{V_1, V_2\}$ of $G \in \mathcal{G}(q; d_1, d_2)$ where V_i consists of d_i-vertices of G $(i = 1, 2)$. Then $|V_i|d_i = q$ and $R_i(G) = \frac{q}{d_i} K_{d_i}$ is a factor of $L(G)$ for $i \in \{1, 2\}$. So, combining Proposition 2 and Proposition 3 we immediately obtain

Corollary 2. *Let $d_1 \geqslant 5$, $d_2 \geqslant 5$ and q be positive integers such that one of the following conditions is satisfied:*
(1) $d_1 \equiv 1 \pmod 4$, $d_2 \equiv 1 \pmod 4$;
(2) $d_1 \equiv 1 \pmod 4$, $d_2 \equiv 2 \pmod 4$, $q \equiv 2 \pmod 4$;
(3) $d_1 \equiv 1 \pmod 4$, $d_2 \equiv 3 \pmod 4$, $q \equiv 1 \pmod 2$;
(4) $d_1 \equiv 2 \pmod 4$, $d_2 \equiv 2 \pmod 4$, $q \equiv 2 \pmod 4$;
(5) $d_1 \equiv 3 \pmod 4$, $d_2 \equiv 3 \pmod 4$, $q \equiv 1 \pmod 2$.
If $G \in \mathcal{G}(q; d_1, d_2)$, then $L(G)$ is a supermagic graph.

For regular bipartite graphs we are able to extend this result. First, we prove an auxiliary assertion.

Lemma 1. *Let m and $d \geqslant 3$ be positive integers. Suppose $v_{i,1}, v_{i,2}, \ldots, v_{i,d}$ are vertices of the ith component of mK_d for $i \in \{1, \ldots, m\}$. Then there is a bijective mapping $f\colon E(mK_d) \to \{1, \ldots, m\binom{d}{2}\}$ such that*

$$f^*(v_{1,j}) = f^*(v_{2,j}) = \ldots = f^*(v_{m,j}) \quad \text{for all} \quad j \in \{2, \ldots, d\}.$$

P r o o f. Evidently, it is sufficient to consider $m \geqslant 2$. If mK_d is supermagic, then its supermagic labelling has the desired properties. So, according to Proposition 3 it remains to consider the following cases.

A. $d = 3$. Define a mapping $f\colon\ E(mK_3) \to \{1, \dots, 3m\}$ by

$$f(v_{i,j}v_{i,k}) = \begin{cases} i & \text{if } \{j,k\} = \{1,2\}, \\ 1 + 2m - i & \text{if } \{j,k\} = \{2,3\}, \\ 2m + i & \text{if } \{j,k\} = \{1,3\}. \end{cases}$$

Clearly, f is the desired mapping because

$$f^*(v_{i,j}) = \begin{cases} 2m + 2i & \text{if } j = 1, \\ 1 + 2m & \text{if } j = 2, \\ 1 + 4m & \text{if } j = 3. \end{cases}$$

B. $d = 4$. In this case we define a bijection $f\colon\ E(mK_4) \to \{1, \dots, 6m\}$ by

$$f(v_{i,j}v_{i,k}) = \begin{cases} i & \text{if } \{j,k\} = \{1,2\}, \\ m + i & \text{if } \{j,k\} = \{3,4\}, \\ 1 + 4m - 2i & \text{if } \{j,k\} = \{2,3\}, \\ 2 + 4m - 2i & \text{if } \{j,k\} = \{1,4\}, \\ 4m + i & \text{if } \{j,k\} = \{1,3\}, \\ 5m + i & \text{if } \{j,k\} = \{2,4\}. \end{cases}$$

For its index-mapping we get

$$f^*(v_{i,j}) = \begin{cases} 2 + 8m & \text{if } j = 1, \\ 1 + 9m & \text{if } j = 2, \\ 1 + 9m & \text{if } j = 3, \\ 2 + 10m & \text{if } j = 4. \end{cases}$$

C. $4 < d \equiv 0 \pmod 4$. Then there is an integer $p \geqslant 2$ such that $d = 4p$. The subgraph $H_{i,s}$ of mK_d induced by $\{v_{i,4s-3}, v_{i,4s-2}, v_{i,4s-1}, v_{i,4s}\}$ is a complete graph for all $i \in \{1, \dots m\}$ and $s \in \{1, \dots, p\}$. Therefore, the spanning subgraph $H := \bigcup_{i=1}^{m} \bigcup_{s=1}^{p} H_{i,s}$ of mK_d is isomorphic to mpK_4. As is proved in the case B, there is a bijection $h\colon\ E(H) \to \{1, \dots, 6mp\}$ such that $h^*(v_{1,j}) = \dots = h^*(v_{m,j})$ for all $j \in \{1, \dots, d\}$. Similarly, the spanning subgraph $B := mK_d - E(H)$ of mK_d is isomorphic to $mK_{p[4]}$. By Proposition 3, $mK_{p[4]}$ is a supermagic graph. Thus, there exists a supermagic labelling $g\colon E(B) \to \{1, \ldots, |E(B)|\}$ of B for an index λ, i.e., $g^*(v_{i,j}) = \lambda$ for all $i \in \{1, \ldots m\}$ and $j \in \{1, \ldots, d\}$. Since H and B form a decomposition of mK_d, we can define a mapping $f\colon E(mK_d) \to \{1, \ldots, m\binom{d}{2}\}$ by

$$f(e) = \begin{cases} h(e) & \text{if} \quad e \in E(H), \\ 6mp + g(e) & \text{if} \quad e \in E(B). \end{cases}$$

As $f^*(v_{i,j}) = h^*(v_{i,j}) + 6mp(d-4) + \lambda$, we have $f^*(v_{1,j}) = \ldots = f^*(v_{m,j})$ for all $j \in \{1, 2, \ldots, d\}$.

D. $6 \leqslant d \equiv 2 \pmod 4$ and $m \equiv 0 \pmod 2$. Then there is a positive integer p such that $d = 4p + 2$. The subgraph G of mK_d induced by $\bigcup_{i=1}^{m} \bigcup_{j=3}^{d} \{v_{i,j}\}$ is isomorphic to mK_{4p}. As is proved in the case C (B, if $p = 1$), there is a bijection $t\colon E(G) \to \{1, \ldots, m\binom{4p}{2}\}$ such that $t^*(v_{1,j}) = \ldots = t^*(v_{m,j})$ for all $j \in \{3, 4, \ldots, d\}$. Consider a mapping $f\colon E(mK_d) \to \{1, \ldots, m\binom{d}{2}\}$ given by

$$f(v_{i,j}v_{i,k}) = \begin{cases} (k-3)m + i & \text{if} \ \ j = 2,\ 3 \leqslant k,\ k \equiv 1 \pmod 2, \\ 1 + (k-2)m - i & \text{if} \ \ j = 2,\ 4 \leqslant k < d,\ k \equiv 0 \pmod 2, \\ 1 + (k-1)m - 2i & \text{if} \ \ j = 2,\ k = d, \\ (k-3)m + 2i & \text{if} \ \ j = 1,\ k = d, \\ (2d-k-2)m + i & \text{if} \ \ j = 1,\ 4 \leqslant k < d,\ k \equiv 0 \pmod 2, \\ 1 + (2d-k-1)m - i & \text{if} \ \ j = 1,\ 3 \leqslant k,\ k \equiv 1 \pmod 2, \\ 2(d-2)m + i & \text{if} \ \ j = 1,\ k = 2, \\ (2d-3)m + t(v_{i,j}v_{i,k}) & \text{if} \ \ 2 < j < k \leqslant d. \end{cases}$$

It is not difficult to check that f is a bijection and for its index-mapping we have

$$f^*(v_{i,j}) = \begin{cases} 2p + (8p(3p+1) - 1)m + 2i & \text{if} \ \ j = 1, \\ 2p + (8p(p+1) + 1)m & \text{if} \ \ j = 2, \\ 1 + 2(d-2)m + (2d-3)m(d-3) + t^*(v_{i,j}) & \text{if} \ \ 3 \leqslant j \leqslant d. \end{cases}$$

E. $7 \leqslant d \equiv 3 \pmod 4$ and $m \equiv 0 \pmod 2$. Then the subgraph G of mK_d induced by $\bigcup_{i=1}^{m} \bigcup_{j=3}^{d} \{v_{i,j}\}$ is isomorphic to mK_{d-2}. By Proposition 3 the graph G is supermagic and so there is a supermagic labelling $t\colon E(G) \to \{1, \ldots, m\binom{d-2}{2}\}$ of G for an index

λ. Consider a mapping $f\colon\ E(mK_d) \to \{1, \ldots, m\binom{d}{2}\}$ given by

$$f(v_{i,j}v_{i,k}) = \begin{cases} (k-3)m+i & \text{if } \ j=2,\ 3 \leqslant k \equiv 1 \pmod 2, \\ 1+(k-2)m-i & \text{if } \ j=2,\ 4 \leqslant k \equiv 0 \pmod 2, \\ 1+(2d-k-1)m-i & \text{if } \ j=1,\ 3 \leqslant k \equiv 1 \pmod 2, \\ (2d-k-2)m+i & \text{if } \ j=1,\ 4 \leqslant k \equiv 0 \pmod 2, \\ 1+(2d-3)m-i & \text{if } \ j=1,\ k=2, \\ (2d-3)m+t(v_{i,j}v_{i,k}) & \text{if } \ 2<j<k \leqslant d. \end{cases}$$

It is easy to verify that f is a bijection. Moreover, for its index-mapping we get

$$f^*(v_{i,j}) = \begin{cases} \frac{1}{2}(d+1)+(\frac{1}{2}(d-3)(3d+1)+5)m-2i & \text{if } \ j=1, \\ \frac{1}{2}(d-1)+(\frac{1}{2}(d-1)(d+1)-1)m & \text{if } \ j=2, \\ 1+2(d-2)m+(d-3)(2d-3)m+\lambda & \text{if } \ 3 \leqslant j \leqslant d, \end{cases}$$

which completes the proof. □

Theorem 2. *Let G be a bipartite regular graph of degree $d \geqslant 3$. Then the line graph $L(G)$ is supermagic.*

P r o o f. Suppose that V_1, V_2 are parts of G. As G is a bipartite d-regular graph, there exist mutually edge-disjoint 1-factors $F_1, \ldots, F_d$ of G which form its decomposition. Put $m = |V_1|$ (clearly, $|V_1| = |V_2|$) and denote the vertices of G by $u_1, \ldots, u_m$, $v_1, \ldots, v_m$ in such a way that $E(F_1) = \{u_1v_1, \ldots, u_mv_m\}$, $V_1 = \{u_1, \ldots, u_m\}$ and $V_2 = \{v_1, \ldots, v_m\}$.

The subgraphs $R_1(G)$, $R_2(G)$ of the line graph $L(G)$ consist of complete graphs with d vertices. Therefore, they are isomorphic to mK_d. Denote by $a_{i,j}$ $(b_{i,j})$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, d\}$, the vertex of $R_1(G)$ $(R_2(G))$ which corresponds to the edge of G incident with u_i (v_i) and which belongs to F_j, i.e., the vertex of $L(G)$ corresponding to $u_rv_s \in E(F_j)$ is denoted by $a_{r,j}$ in $R_1(G)$ and by $b_{s,j}$ in $R_2(G)$.

By Lemma 1, there exists a bijective mapping $g_1\colon\ E(R_1(G)) \to \{1, \ldots, m\binom{d}{2}\}$ such that $g_1^*(a_{1,j}) = g_1^*(a_{2,j}) = \ldots = g_1^*(a_{m,j})$ for all $j \in \{2, \ldots, d\}$. Then a mapping $g_2\colon\ E(R_2(G)) \to \{1+m\binom{d}{2}, \ldots, 2m\binom{d}{2}\}$ given by

$$g_2(b_{i,j}b_{i,k}) = 1 + 2m\binom{d}{2} - g_1(a_{i,j}a_{i,k})$$

is bijective, too. Moreover, $g_2^*(b_{i,j}) = (d-1)(1+2m\binom{d}{2}) - g_1^*(a_{i,j})$. Consider the mapping $f\colon\ E(L(G)) \to \{1, \ldots, 2m\binom{d}{2}\}$ defined by

$$f(e) = \begin{cases} g_1(e) & \text{if } \ e \in E(R_1(G)), \\ g_2(e) & \text{if } \ e \in E(R_2(G)). \end{cases}$$

Evidently, f is a bijection. Let x be an edge of G which belongs to F_1. Then there exists $i \in \{1, \ldots, m\}$ such that $x = u_i v_i$, i.e., the vertex of $L(G)$ corresponding to x is denoted by $a_{i,1}$ in $R_1(G)$ and by $b_{i,1}$ in $R_2(G)$. Thus

$$f^*(x) = g_1^*(a_{i,1}) + g_2^*(b_{i,1}) = (d-1)\left(1 + 2m\binom{d}{2}\right).$$

Similarly, for an edge $y \in E(F_j)$, $j \in \{2, \ldots, d\}$, there exist $r, s \in \{1, \ldots, m\}$, $r \neq s$, such that $y = u_r v_s$. Then

$$f^*(y) = g_1^*(a_{r,j}) + g_2^*(b_{s,j}) = g_1^*(a_{s,j}) + g_2^*(b_{s,j}) = (d-1)\left(1 + 2m\binom{d}{2}\right).$$

Therefore, f is a supermagic labelling of $L(G)$ for index $(d-1)(1+2m\binom{d}{2})$. □

Corollary 3. *Let k_1, k_2, q and $d \geqslant 3$ be positive integers such that one of the following conditions is satisfied:*

(1) $d \equiv 0 \pmod 2$;

(2) $d \equiv 1 \pmod 2$, $k_1 \equiv 1 \pmod 4$, $k_2 \equiv 1 \pmod 4$;

(3) $d \equiv 1 \pmod 2$, $k_1 \equiv 1 \pmod 4$, $k_2 \equiv 2 \pmod 4$, $q \equiv 2 \pmod 4$;

(4) $d \equiv 1 \pmod 2$, $k_1 \equiv 1 \pmod 4$, $k_2 \equiv 3 \pmod 4$, $q \equiv 1 \pmod 2$;

(5) $d \equiv 1 \pmod 2$, $k_1 \equiv 3 \pmod 4$, $k_2 \equiv 3 \pmod 4$, $q \equiv 1 \pmod 2$.

If $G \in \mathcal{G}(q; k_1 d, k_2 d)$, then $L(G)$ is a supermagic graph.

P r o o f. Suppose that u_i for $i \in \{1, \ldots, m\}$, where $m = \frac{q}{k_1 d}$, (v_j for $j \in \{1, \ldots, n\}$, where $n = \frac{q}{k_2 d}$) denotes a $(k_1 d)$-vertex ($(k_2 d)$-vertex) of a graph G belonging to $\mathcal{G}(q; k_1 d, k_2 d)$. Then there is a graph $G' \in \mathcal{G}(q; d, d)$ with vertex set $V(G') = \left(\bigcup_{i=1}^{m} \bigcup_{r=1}^{k_1} \{u_i^r\}\right) \cup \left(\bigcup_{j=1}^{n} \bigcup_{s=1}^{k_2} \{v_j^s\}\right)$ such that for any edge $u_i v_j \in E(G)$ there exists an edge $u_i^r v_j^s \in E(G')$, where $r \in \{1, \ldots, k_1\}$ and $s \in \{1, \ldots, k_2\}$ (i.e., G' is obtained from G by distributing every vertex into vertices of degree d).

The subgraph $K(u_i)$ ($K(v_j)$) of $L(G)$ is decomposable into $k_1 K_d$ and $K_{k_1[d]}$ ($k_2 K_d$ and $K_{k_2[d]}$). Thus, it is not difficult to see that $L(G)$ is decomposable into factors F_1, F_2, F_3, where F_1 is isomorphic to $L(G')$, F_2 is isomorphic to $mK_{k_1[d]}$ (if $k_1 > 1$) and F_2 is isomorphic to $nK_{k_2[d]}$ (if $k_2 > 1$). Combining Theorem 2, Proposition 3 and Proposition 2 we obtain the assertion. □

We conclude this paper with the following negative statement:

Theorem 3. *Let q, d_1, d_2 be positive integers such that either $d_1 + d_2 \leqslant 4$ and $q > 2$, or $4 < d_1 + d_2 \equiv 1 \pmod 2$ and $q \equiv 0 \pmod 4$. If $G \in \mathcal{G}(q; d_1, d_2)$, then the line graph $L(G)$ is not supermagic.*

P r o o f. The line graph $L(G)$ of a graph $G \in \mathcal{G}(q; d_1, d_2)$ is $(d_1 + d_2 - 2)$-regular of order q. Evidently, $L(G)$ is not magic when $d_1 + d_2 \leqslant 4$ and $q > 2$. The other case immediately follows from the fact (see [4]) that a supermagic regular graph H of odd degree satisfies $|V(H)| \equiv 2 \pmod 4$. $\square$

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Authors' addresses: *J. Ivančo*, *Z. Lastivková*, *A. Semaničová*, Department of Geometry and Algebra, P. J. Šafárik University, 041 54 Košice, Jesenná 5, Slovakia, e-mail: `ivanco@science.upjs.sk`, `semanic@science.upjs.sk`.