ALMOST PERIODIC SOLUTIONS WITH A PRESCRIBED SPECTRUM OF SYSTEMS OF LINEAR AND QUASILINEAR DIFFERENTIAL EQUATIONS WITH ALMOST PERIODIC COEFFICIENTS AND CONSTANT TIME LAG (FOURIER TRANSFORM APPROACH)

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Abstract. This paper is a continuation of my previous paper in Mathematica Bohemica and solves the same problem but by means of another method. It deals with almost periodic solutions of a certain type of almost periodic systems of differential equations.

Keywords: almost periodic function, Fourier coefficient, Fourier exponent, spectrum of almost periodic function, almost periodic system of differential equations, formal almost periodic solution, distance of two spectra, time lag

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1. Introduction

1.1. Preliminaries. This article is inspired by [3], [5] and generalizes the method of Favard ([1], [4], [7]). The method used here is based on the Fourier transformation. This method in comparison with the method from [6] (based on the Cauchy integral) is more complicated and laborious but at the same time it is richer and stronger. Some results, procedures, proofs and parts are the same as in [6] and therefore be not all repeated but we shall refer to [6] only.

In what follows they will all criteria of existence and uniqueness as well as all estimates deal with complex matrix (Bohr's uniformly) almost periodic functions.

1.2. Notation and definitions. We denote: $\mathbb{N}$ the set of all positive integers, $\mathbb{N}_0$ the set of all non-negative integers, $\mathbb{R}$ the set of all real numbers (real axis), $\mathbb{C}$ the set of all complex numbers (complex plane).

If $\mathbb{E}$ is a non-void set and m, n are from $\mathbb{N}$ then $\mathbb{E}^m$ denotes the Cartesian product $\mathbb{E} \times \mathbb{E} \ldots \times \mathbb{E}$ of m factors and $\mathbb{E}^{m\times n}$ is the set of all matrices of m rows and n columns, the elements of which belong to $\mathbb{E}$; $\mathbb{E}^{1\times 1} = \mathbb{E}^1 = \mathbb{E}$. Analogously we could denote more-dimensional matrices.

If $n \in \mathbb{N}$ and $\overline{m} = (m_1, \ldots, m_n) \in \mathbb{N}_0^{1\times n}$, $\overline{m}' = (m_1', \ldots, m_n') \in \mathbb{N}_0^{1\times n}$ then the inequality $\overline{m} \leqslant \overline{m}'$ stands for the system of inequalities $m_j \leqslant m_j'$, $j = 1, 2, \ldots, n$.

If $\mathcal{M}$, $\mathcal{N}$ are non-void subsets of $\mathbb{C}$ or $\mathbb{R}$ and if ω, ξ are complex numbers then $\omega\mathcal{M} = \{\omega\lambda\colon\ \lambda \in \mathcal{M}\}$, $\xi + \mathcal{N} = \{\xi + \mu\colon\ \mu \in \mathcal{N}\}$, $\mathcal{M} + \mathcal{N} = \{\lambda + \mu\colon\ \lambda \in \mathcal{M}, \mu \in \mathcal{N}\}$, $\emptyset + \mathcal{N} = \mathcal{M} + \emptyset = \emptyset$ and $S(\mathcal{M})$ stands for the smallest additive semigroup containing $\mathcal{M}$ and $S(\emptyset) = \emptyset$.

The distance of two sets $\mathcal{M}$, $\mathcal{N}$, of a point z and a set $\mathcal{N}$ and of two points z, w in $\mathbb{C}$ or $\mathbb{R}$, respectively, is denoted by $\operatorname{dist}[\mathcal{M}, \mathcal{N}]$, $\operatorname{dist}[z, \mathcal{N}]$ and $\operatorname{dist}[z, w]$.

The boundary of a set $\mathcal{M}$ is denoted by $\partial\mathcal{M}$.

If α is a positive number then by a strip or an α-strip in the complex plane we mean the set $\pi(\alpha) = \{z \in \mathbb{C}\colon\ |\operatorname{Re} z| \leqslant \alpha\}$. If $z_0 \in \mathbb{C}$ and $R \in (0, \infty)$ then $\kappa(z_0, R)$, $\overline{\kappa}(z_0, R)$ and $K(z_0, R)$, respectively, denote an open disc, a closed disc and a circle centred at z_0 with its radius R in the complex plane.

For number vectors or matrices, even more-dimensional, we use the norm $|\cdot|$, which is equal to the sum of absolute values of all coordinates of the vectors or all elements of the matrix.

In addition to the usual symbol $\prod\limits_{j=1}^{k} = a_1 a_2 \ldots a_k$ for a product we will use the symbol $\prod\limits_{j=k}^{1} = a_k a_{k-1} \ldots a_1$ for the product with a reversed order of factors.

For a vector $\overline{m} = (m_1, \ldots, m_M) \in \mathbb{N}_0^M, M \in \mathbb{N}$, we introduce the combinatory number

$$\binom{|\overline{m}|}{\overline{m}} = \frac{|\overline{m}|!}{(m_1!)\cdot\ldots\cdot(m_M!)}, \quad \text{where } |\overline{m}| = m_1 + \ldots + m_M.$$

1.3. Spaces and the starting problem. We will deal with functions $f\colon\ \mathbb{R} \to \mathbb{X}$, where $\mathbb{X}$ is one of the spaces $\mathbb{E}$, $\mathbb{E}^m$, $\mathbb{E}^{m\times n}$ and $\mathbb{E} = \mathbb{R}$ or $\mathbb{E} = \mathbb{C}$.

We denote by $C(\mathbb{X})$, $CB(\mathbb{X})$ and $AP(\mathbb{X})$, respectively, the space of all continuous functions $f\colon\ \mathbb{R} \to \mathbb{X}$, the space of all functions from $C(\mathbb{X})$ bounded on $\mathbb{R}$ and the space of all almost periodic functions from $CB(\mathbb{X})$. The mean value of a function $f \in AP(\mathbb{X})$ is denoted by $M(f)$ or $M_t\{f(t)\}$.

The spaces $CB(\mathbb{X})$ and $AP(\mathbb{X})$ are made Banach spaces (B-spaces) with the norm defined by $|f| = \sup\{|f(t)|_{\mathbb{X}}\colon\ t \in \mathbb{R}\}$. For a positive integer k we will denote by $C^k(\mathbb{X}), CB^k(\mathbb{X})$ and $AP^k(\mathbb{X})$ the space of all functions from $C(\mathbb{X})$ with continuous derivatives up to the order k on $\mathbb{R}$, the space of all functions from $C^k(\mathbb{X})$ which are bounded on $\mathbb{R}$ and have bounded derivatives up to the order k, and the space of all functions from $CB^k(\mathbb{X})$ which are almost periodic and have almost periodic derivatives up to the order k.

The spaces $CB^1(\mathbb{X})$ and $AP^1(\mathbb{X})$ endowed with the norm

$$\|f\| = \max\{|f|, |\dot{f}|\}$$

become B-spaces. If all elements of a matrix almost periodic function $f \in AP(\mathbb{X})$ are trigonometric polynomials then f is called a trigonometric polynomial.

If $f \in AP(\mathbb{X})$ then by Λ_f we denote the set of all Fourier exponents of f and the set $\mathrm{i}\Lambda_f$ will be called the spectrum of f.

If f is an almost periodic function with the Fourier series $\sum\limits_{\lambda} \varphi(\lambda) \exp(\mathrm{i}\lambda t)$, $\lambda \in \Lambda_f$, then we denote $\sum(f) = \sum\limits_{\lambda} |\varphi(\lambda)|, \lambda \in \Lambda_f$. If the Fourier series of a function f converges absolutely then $\sum(f) < \infty$.

For any function f from $AP(\mathbb{X})$ there exists a sequence of the so-called Bochner-Fejér approximation (trigonometric) polynomials B_m, $m = 1, 2, \ldots$, of the function f with their spectra contained in $\mathrm{i}\Lambda_f$ and uniformly convergent to f on $\mathbb{R}$ and moreover $\sum(B_m) \leqslant \sum(f)$, $m = 1, 2, \ldots$, (see [1], [4], [7]).

The starting problem solved in [3], [5] is to find an almost periodic solution of the almost periodic differential equation with constant coefficients

$$\dot{x}(t) = a_0 x(t) + b_0 x(t - \tau) + f(t), \qquad t \in \mathbb{R} \tag{1.1}$$

where τ is a positive constant, the so-called time lag, a_0, b_0 belong to $\mathbb{C}^{n_0 \times n_0}$, $n_0 \in \mathbb{N}$, $f \in AP(\mathbb{C}^{n_0 \times n_0})$ and x is a function from $C^1(\mathbb{C}^{n_0 \times 1})$. An important role is played by the properties of the matrix function

$$\Phi(z) = zE - a_0 - b_0 \exp(-z\tau), \qquad z \in \mathbb{C}, \tag{1.2}$$

where $E = E_{n_0}$ is the unit matrix from $\mathbb{C}^{n_0 \times n_0}$, and by the properties of its determinant $\Delta(z) = \det \Phi(z)$, the so-called characteristic quasipolynomial, and the equation $\Delta(z) = 0$, the so-called characteristic equation of (1.1).

Under $\sigma(\Delta(z))$ we understand the set of all roots of the characteristic quasipolynomial, which is a transcendent entire function (in general) of the complex variable z. Consequently, the quasipolynomial $\Delta(z)$ has an infinite number of roots without any finite limit point. Each strip $\pi(\alpha)$, $\alpha > 0$, contains only a finite number of roots of $\Delta(z)$ because $\Phi(z)z^{-1}$ is arbitrarily close to the unit matrix E in the strip $\pi(\alpha)$ for all z sufficiently large (in absolute value). Hence the matrix $\Phi(z)$ is a regular one for such z. Therefore a positive number α can be chosen so that the finite set $\pi(2\alpha) \cap \sigma(\Delta(z))$ lies on the imaginary axis of the complex plane. If $\pi(2\alpha) \cap \sigma(\Delta(z)) \neq \emptyset$ and this set contains just the points $\mathrm{i}\xi_1, \dots, \mathrm{i}\xi_{j_0}$, $j_0 \in \mathbb{N}$, then we set $\theta = \{\xi_j - \xi_k \colon j, k = 1, \dots, j_0\}$, and if $\pi(2\alpha) \cap \sigma(\delta(z)) = \emptyset$, then we set $\theta = \emptyset$.

2. Equations with almost periodic coefficients

2.1. Basic equations. In the sequel we study the differential equation

$$\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + f(t), \ t \in \mathbb{R} \tag{2.1}$$

where τ, a_0, b_0, f, x have the same meaning as in 1.4 and further $a, b \in AP(\mathbb{C}^{n_0 \times n_0})$ with $\sum(a) < \infty, \sum(b) < \infty$. Our aim is to prove the existence and uniqueness of an almost periodic solution of Equation (2.1) the spectrum of which is contained in a certain apriori given set $\mathrm{i}\Lambda$, $\Lambda \subset \mathbb{R}$. Such a solution is called an almost periodic Λ-solution.

2.2. Formal solutions. First,we solve the given equation in a formal manner. This means that we are looking for the so-called formal solution x_f represented by a trigonometric series with coefficients from $\mathbb{C}^{n_0 \times 1}$ which formally satisfies Equation (2.1). In [6] the proof of the following theorem can be found.

Theorem 2.1. *If in Equation* (2.1) a, b *are nonconstant trigonometric polynomials* $a(t) = \sum\limits_{k=1}^{M} \alpha(\mu_k) \exp{(\mathrm{i}\mu_k t)}$, $b(t) = \sum\limits_{k=1}^{N} \beta(\nu_k) \exp{(\mathrm{i}\nu_k t)}$ *and* f *is a (non-zero) trigonometric polynomial* $f(t) = \sum\limits_{\lambda} \varphi(\lambda) \exp{(\mathrm{i}\lambda t)}$ *for* $M, N \in \mathbb{N}$, $t \in \mathbb{R}$ *and if (see at the end of* 1.3 *concerning the definition of the set* θ*)*

$$\Delta = \inf(\Lambda_a \cup \Lambda_b) > 0, \tag{2.2}$$

$$d_\theta = \begin{cases} \operatorname{dist}[\theta, S(\Lambda_a \cup \Lambda_b)] > 0 & \text{for } \theta \neq \emptyset, \\ 4 & \text{for } \theta = \emptyset, \end{cases} \tag{2.3}$$

$$d = \operatorname{dist}[\mathrm{i}\Lambda, \sigma(\Delta(z))] > 0, \tag{2.4}$$

where $\Lambda = \Lambda_f + S(\Lambda_a \cup \Lambda_b \cup \{0\})$, *then there exists a unique formal almost periodic* Λ*-solution* x_f *of Equation* (2.1).

In [6] in the proof of this theorem the unique formal almost periodic Λ-solution $x_f = x_f(t) \sim \sum\limits_{\tau} c(\tau) \exp(\mathrm{i}\tau t)$ for $\sigma \in \Lambda$ is expressed in the form

$$x_f(t) = \sum_{\lambda} x_\lambda \sim \sum_{\lambda} \sum_{\bar{s} \geqslant \bar{0}} \sum_{P} \Phi_P(\mathrm{i}\lambda) \varphi(\lambda) \exp{(\mathrm{i}(\lambda + \bar{s}\bar{\varpi})t)} \tag{2.5}$$

for $\lambda \in \Lambda_f$, $t \in \mathbb{R}$. Here $P = P(\bar{s})$ denotes an increasing sequence

$$\bar{0} = \overline{P}_0 \leqslant \overline{P}_1 \leqslant \ldots \leqslant \overline{P}_{|\bar{s}|} = \bar{s}$$

of vectors from ${\mathbb{N}_0}^{1\times(M+N)}$, which satisfies $|\overline{P}_j - \overline{P}_{j-1}| = 1$, $j = 1, \ldots, |\bar{s}|$. With every such sequence $P = P(\bar{s})$ for a fixed $\lambda \in \Lambda_f$ we can associate in a unique manner a sequence $p = p(\bar{s})$ of vectors $\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_{|\bar{s}|}$ from $\mathbb{N}_0^{1\times(M+N)}$ satisfying $\bar{p}_0 = \bar{0}$, $|\bar{p}_j| = 1$, $j = 1, \ldots, |\bar{s}|$, and $\overline{P}_k = \sum\limits_{j=0}^{k} \bar{p}_j$, $k = 0, 1, \ldots, |\bar{s}|$, while $\bar{p}_j = (\bar{q}_j, \bar{r}_j)$, $\bar{q}_j \in \mathbb{N}_0^{1\times M}$, $\bar{r}_j \in \mathbb{N}_0^{1\times N}$, $j = 1, \ldots, |\bar{s}|$. The function Φ_P is given by the formula

$$\Phi_P(z) = \prod_{j=|\bar{s}|}^{0} \Phi^{-1}(z + \mathrm{i}\overline{P}_j\bar{\omega})\gamma(\bar{p}_j\bar{\omega})$$

with $\gamma(0) = 1$, $\gamma(\bar{p}_j\bar{\omega}) = \alpha(\bar{q}_j\bar{\mu}) + \beta(\bar{r}_j\bar{\nu})\exp(-\mathrm{i}\overline{P}_{j-1}\bar{\omega}\tau)$, $j = 1, \ldots, |\bar{s}|$, while $\alpha(\mu) = 0$ for $\mu \notin \Lambda_a$, $\beta(\nu) = 0$ for $\nu \notin \Lambda_b$ ($\alpha(0) = \beta(0) = 0$).

Remark 2.2. Every almost periodic Λ-solution of Equation (2.1) is at the same time a formal almost periodic Λ-solution of Equation (2.1). The contrary is not true in general.

Every $\sigma \in \Lambda$ can be presented in the form $\sigma = \lambda + \bar{s}\bar{\omega} = \lambda + \overline{m}\bar{\mu} + \bar{n}\bar{\nu}$, where $\lambda \in \Lambda_f$,

$$\bar{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_M \end{pmatrix} \quad \bar{\nu} = \begin{pmatrix} \nu_1 \\ \vdots \\ \nu_N \end{pmatrix}, \quad \bar{\omega} = \begin{pmatrix} \bar{\mu} \\ \bar{\nu} \end{pmatrix},$$
$$\overline{m} = (m_1, \ldots, m_M) \in \mathbb{N}_0^{1\times M}, \quad \bar{n} = (n_1, \ldots, n_N) \in \mathbb{N}_0^{1\times N}, \quad \bar{s} = (\overline{m}, \bar{n}).$$

The number of all possible different "descents" from $\lambda + \bar{s}\bar{\omega}$ to λ represented by all sequences $P = P(\bar{s})$ is

$$\binom{|\bar{s}|}{\bar{s}} = \frac{|\bar{s}|!}{(m_1!)\ldots(m_M!)(n_1!)\ldots(n_N!)}.$$

2.3. Closed regions G_j, G_P. The positive number α was chosen such that $\mathrm{i}B(\alpha) = \pi(2\alpha) \cap \sigma(\Delta(z)) \subset \mathrm{i}\mathbb{R}$ ($B(\alpha) \subset \mathbb{R}$). If $B(\alpha)$ contains at least two points then we define $d_\xi = \min\{|\xi - \tilde{\xi}|\colon\ \xi, \tilde{\xi} \in B(\alpha), \xi \neq \tilde{\xi}\}$ and if $B(\alpha)$ contains one or no point then we define $d_\xi = 4$. Further, we shall assume that the number $v_0 = \inf \Lambda_f$ is positive. We pick a positive number $\delta = \frac{1}{4}\min\{\alpha, \Delta, d_\xi, d_\theta, d, \tau, v_0, 4\}$, where we suppose $d_\theta > 0$.

Further, unless stated otherwise, we assume that we are given a fixed vector $\bar{s}$ and a fixed sequence $P = P(\bar{s})$ of vectors. Recall that $\kappa(z,\delta)$ and $\bar{\kappa}(z,\delta)$ are the open disc and the closed disc centred at z with radius δ in the complex plane $\mathbb{C}$. In $\mathbb{C}$ we construct closed regions

$$G_j = \pi(\alpha) \setminus \bigcup_{\xi \in B(\alpha)} \kappa(\mathrm{i}\xi - \mathrm{i}\overline{P}_j\varpi; \delta), \quad j = 0, 1, \ldots, |\bar{s}|, \tag{2.6}$$

and we denote

$$G_P = \bigcap_{j=0}^{|\bar{s}|} G_j = \pi(\alpha) \setminus \bigcup_{j=0}^{|\bar{s}|} \bigcup_{\xi \in B(\alpha)} \kappa(\mathrm{i}\xi - \mathrm{i}\overline{P}_j\varpi;\, \delta). \tag{2.7}$$

Since the matrix function $\Phi(z)$ introduced in (1.2) is analytic and regular on G_0, the matrix function $\Phi(z + \mathrm{i}\overline{P}_j\varpi)$ is analytic and regular on G_j and the same property is possessed also by $\Phi^{-1}((z + \mathrm{i}\overline{P}_j\varpi)$, $j = 0, \ldots, |\bar{s}|$. It follows that the matrix function $\Phi_P(z)$ is analytic on the closed region $G(P)$.

Now we define the set $\mathrm{i}B_j(\alpha) = \pi(2\alpha) \cap \sigma(\Delta(z + \mathrm{i}\overline{P}_j\varpi))$ which lies also on $\mathrm{i}\mathbb{R}$, $j = 0, \ldots, |\bar{s}|$. We have $B_j(\alpha) \cap B_k(\alpha) = 0$ for $j \neq k$, $j, k \in \{0, \ldots, |\bar{s}|\}$ provided we suppose $d_\theta > 0$.

If for an integer j from $\{0, \ldots, |\bar{s}|\}$ there exists $\tilde{\xi} \in B_j(\alpha)$ then there exists $\xi \in B(\alpha) = B_0(\alpha)$ such that $\tilde{\xi} + \overline{P}_j\varpi = \xi$. It means that $\tilde{\xi} = \xi - \overline{P}_j\varpi$. We define the set $B_j = \langle v_0, \infty) \cap B_j(\alpha)$, $j = 0, \ldots, |\bar{s}|$. If there exists $\tilde{\xi} \in B_j \subset B_j(\alpha)$ then we have $\tilde{\xi} + \overline{P}_j\varpi = \xi$ for a point $\xi \in B_0(\alpha)$ and $\tilde{\xi} \geqslant v_0$, $\overline{P}_j\varpi \geqslant 4\delta j$, $v_0 \leqslant \tilde{\xi} + \overline{P}_j\varpi = \xi$ so that $\xi \geqslant v_0$ and $\xi \in B_0$.

In the sequel we will take up the case $B_0 \neq \emptyset$ but the case $B_0 = \emptyset$ would be even easier. We set $\hat{\xi} = \sup B_0$ so that for any $\xi \in B_0$ the inequality $v_0 \leqslant \xi \leqslant \hat{\xi}$ holds ($\hat{\xi} < \infty$ because B_0 is a non-void and finite set of real numbers). By virtue of the relation $\tilde{\xi} = \xi - \overline{P}_j\varpi \geqslant v_0$ we get $0 \leqslant 4\delta j \leqslant \overline{P}_j\varpi \leqslant \xi - v_0 \leqslant \hat{\xi} - v_0$ and $0 \leqslant j \leqslant j_0 = j(P) \leqslant \hat{j}_0 = [(\hat{\xi} - v_0)/(4\delta)]$ (the entire part [a] of a real number a is an integer for which the inequality $[a] \leqslant a < [a] + 1$ holds), where j_0 is the smallest integer such that $B_j = \emptyset$ for $j > j_0$ (or B_j do not exist). Finally, we define the set $B = \bigcup\limits_{j=0}^{j_0} B_j$. According to the preceding assumption we have $B \neq \emptyset$. Because $v_0 = \inf \Lambda_f \geqslant 4\delta$ and $d = \operatorname{dist}[\mathrm{i}\Lambda,\, \sigma(\Delta(z))] \geqslant 4\delta$, the inequality $|\xi - \delta - v_0| \geqslant |\xi - v_0| - \delta \geqslant d - \delta \geqslant 3\delta$ holds for any $\xi \in B$. If there exist $\xi, \tilde{\xi} \in B$, $\xi \neq \tilde{\xi}$, then $|(\xi \pm \delta) - (\tilde{\xi} \pm \delta)| \geqslant |\xi - \tilde{\xi}| - 2\delta \geqslant d_\xi - 2\delta \geqslant 2\delta$ owing to $|\xi - \tilde{\xi}| \geqslant d_\xi \geqslant 4\delta$.

At this time we construct the real number set

$$J_0 = \langle v_0, \infty) \setminus \bigcup_{\xi \in B} (\xi - \delta, \xi + \delta). \tag{2.8}$$

There exist $k_0 \in \mathbb{N}$ and a finite sequence of real numbers $0 = u_0 < v_0 < u_1 < \ldots < u_{k_0} < v_{k_0} < \infty$ such that

$$J_0 = \langle v_0, u_1\rangle \cup \ldots \cup \langle v_{k_0-1}, u_{k_0}\rangle \cup \langle v_{k_0}, \infty) \tag{2.9}$$

while $v_0 - u_0 = v_0 \geqslant 4\delta$, $v_j - u_j = 2\delta$, $j = 1, \ldots, k_0$, $u_{j+1} - v_j \geqslant 2\delta$, $j = 0, \ldots, k_0 - 1$. (The number k_0 is equal at most to the product $\hat{j}_0 m_0$, where m_0 is the number of all mutually different points from the finite set B_0.)

3. Trigonometric integrals

3.1. Auxiliary relations and calculations. Let us recall and derive some properties of a few trigonometric integrals which we will use in what follows. In the sequel we shall deal with functions $H = H(t)$ of the real variable t and with functional values from a linear space.

(i) If H is a linear function on a real interval $\langle u, v\rangle$, $-\infty < u < v < \infty$, it means that $H(t) = H(u) + ((t-u)/(v-u))(H(v) - H(u))$, $t \in \langle u, v\rangle$, then for any non-zero real number s the equality

$$\begin{aligned} \int_u^v H(t)\sin ts\,\mathrm{d}t = {} & -\frac{1}{s}(H(v)\cos vs - H(u)\cos us) \\ & + \frac{\sin vs - \sin us}{(v-u)s^2}(H(v) - H(u)) \end{aligned} \tag{3.1}$$

is valid.

(ii) If a function H has derivatives $\dot{H}$, $\ddot{H}$ absolutely integrable on a real interval $\langle u, v\rangle$, $-\infty < u < v < \infty$, then for any non-zero real number s the equality

$$\begin{aligned} \int_u^v H(t)\sin ts\,\mathrm{d}t = {} & -\frac{1}{s}(H(v)\cos vs - H(u)\cos us) \\ & + \frac{1}{s^2}((\dot{H}(v)\sin vs - \dot{H}(u)\sin us) - \frac{1}{s^2}\int_u^v \ddot{H}(t)\sin ts\,\mathrm{d}t \end{aligned} \tag{3.2}$$

holds.

(iii) If a function H has derivatives $\dot{H}$, $\ddot{H}$ absolutely integrable on a real interval $\langle v, \infty)$, $v \in \mathbb{R}$, and if $\lim H(t) = 0$ for $t \to \infty$ then for any non-zero real number s the equality

$$\begin{aligned} \int_v^\infty H(t)\sin ts\,\mathrm{d}t = {} & \frac{1}{s}H(v)\cos vs - \frac{1}{s^2}\dot{H}(v)\sin vs \\ & - \frac{1}{s^2}\int_v^\infty \ddot{H}(t)\sin ts\,\mathrm{d}t \end{aligned} \tag{3.3}$$

holds.

(iv) If on the real interval $\langle 0,\infty)$ a real function H is nonnegative, monotone and if $\lim H(t)=0$ for $t\to\infty$ and the integral $\int_0^{\pi} H(t)\sin t\,\mathrm{d}t$ exists then for any nonnegative number v

$$S(v)=\left|\int_v^{\infty} H(t)\sin t\,\mathrm{d}t\right| \leqslant \int_0^{\pi} H(t)\sin t\,\mathrm{d}t = a_0 \tag{3.4}$$

holds.

P r o o f. First we recall the well-known Leibniz criterion for alternating number series: If a sequence of nonnegative real numbers $\{a_k\}_{k=0}^{\infty}$ is monotone and $\lim a_k=0$ for $k\to\infty$ is true then the series $\sum\limits_{k=0}^{\infty}(-1)^k a_k$ is convergent and for any nonnegative integer m the estimate $a_m \geqslant \bigl|\sum\limits_{k=m}^{\infty}(-1)^k a_k\bigr|$ holds.

To verify the validity of (3.4) we denote

$$\begin{aligned} a_k &= \left|\int_{k\pi}^{(k+1)\pi} H(t)\sin t\,\mathrm{d}t\right|,\ k=0,1,\ldots,\\ S_m &= \left|\int_{m\pi}^{\infty} H(t)\sin t\,\mathrm{d}t\right| = \sum_{k=m}^{\infty}(-1)^{k-m}a_k,\ m=0,1,\ldots. \end{aligned}$$

Evidently $a_k \geqslant a_{k+1}$, $k=0,1,\ldots$, and at the same time $\lim a_k=0$ for $k\to\infty$. Hence for the series S_m, $m=0,1,\ldots$, the conditions of the Leibniz convergence criterion are fulfilled. Consequently, $S_m \leqslant a_m \leqslant a_0$ holds for $m=0,1,\ldots$. This proves (3.4) for $v=m\pi$, $m=0,1,\ldots$. If a nonnegative number v is not an integer multiple of the number π then there exists such an integer $m=m(v)$ that $0\leqslant(m-1)\pi<v<m\pi$ ($m=[v/\pi]+1$, where $[v/\pi]$ is the entire part of the real number v/π). We denote $V=V(v)=\bigl|\int_v^{m\pi} H(t)\sin t)\,\mathrm{d}t\bigr|$ for $m=m(v)$. Evidently the inequality $0\leqslant V\leqslant a_{m-1}$ is valid. For $v\notin\mathbb{N}_0\pi$ the inequality (3.4) is split into four cases (α), (β), (γ), (δ).

(α) If $0\leqslant V<a_m-a_{m+1}\leqslant S_m$ then $S(v)=S_m-V\leqslant S_m\leqslant a_m\leqslant a_0$.

(β) If $a_m-a_{m+1}\leqslant V<S_m$ then $S(v)=S_m-V=a_m-a_{m+1}-V+S_{m+2}\leqslant S_{m+2}\leqslant a_{m+2}\leqslant a_0$.

(γ) If $S_m\leqslant V<a_m$ then $S(v)=V-S_m=V-a_m+S_{m+1}\leqslant S_{m+1}\leqslant a_{m+1}\leqslant a_0$.

(δ) If $a_m\leqslant V\leqslant a_{m-1}$ then $S(v)=V-S_m\leqslant a_{m-1}-S_m=S_{m-1}\leqslant a_{m-1}\leqslant a_0$. $\square$

(v) If a real function H is monotone and $0\leqslant H(t)\leqslant C/t$ with a positive constant C on the real interval $(0,\infty)$ then

$$\left|\int_v^{\infty} H(t)\sin ts\,\mathrm{d}t\right| \leqslant C\pi \tag{3.5}$$

holds for any positive number s, v.

P r o o f. From (3.4) we get relations $\left|\int_v^\infty H(t)\sin ts\,\mathrm{d}t\right| = \frac{1}{s}\left|\int_{vs}^\infty H(\frac{t}{s})\sin t\,\mathrm{d}t\right| \leqslant \frac{1}{s}\int_0^\pi H(\frac{t}{s})\sin t\,\mathrm{d}t \leqslant \frac{C}{s}\int_0^\pi \frac{s}{t} t\,\mathrm{d}t = C\pi$. □

(vi) If a real function H is defined on the real interval $\langle v,\infty)$ with the positive number v and if H converges to zero for $t \to \infty$ and its derivative $\dot{H}$ exists and is absolutely integrable on the interval $\langle v,\infty)$ and if the inequality $|\dot{H}(t)| \leqslant C/t^2$ with a positive constant C holds then the inequality

$$\left|\int_v^\infty H(t)\sin ts\,\mathrm{d}t\right| \leqslant 2\pi C \tag{3.6}$$

is valid for any real number s.

P r o o f. It will be enough to consider positive numbers s. The function H can be expressed in the form $H = H_1 - H_2$, where $H_j = H_j(t) = \frac{1}{2}\int_t^\infty (|\dot{H}(w)| - (-1)^j \dot{H}(w))\,\mathrm{d}w$, $j = 1,2$. The functions H_j; $j = 1,2$, are nonnegative, monotone on the interval $\langle v,\infty)$ and the inequality $H_j(t) \leqslant C/t$ holds for $t \in \langle v,\infty)$ and $j = 1,2$. If we extend these functions by the formula $H_j(t) = \frac{v}{t}H_j(v)$ for $t \in (0,v)$, $j = 1,2$, then with the same notation for the extended functions we have on $(0,\infty)$ two functions H_1, H_2 with the properties demanded in (v). Hence we get the inequalities $\left|\int_v^\infty H_j(t)\sin ts\,\mathrm{d}t\right| \leqslant C\pi$, $j = 1,2$, and the validity of (3.6). □

3.2. Fourier integrals and transformations. For real numbers a and α we define three trigonometric integrals

$$\begin{aligned} C(\alpha) = C(\alpha,f,a) &= \int_a^\infty f(x)\cos\alpha x\,\mathrm{d}x, \\ S(\alpha) = S(\alpha,f,a) &= \int_a^\infty f(x)\sin\alpha x\,\mathrm{d}x, \\ E(\alpha) = E(\alpha,f,a) &= \int_a^\infty f(x)\exp(\mathrm{i}\alpha x)\,\mathrm{d}x = C(\alpha) + \mathrm{i}S(\alpha), \end{aligned}$$

where a complex function f is defined on the real interval (a,∞).

Theorem 3.1. *If on the interval (a,∞) a complex function $f = f(x)$ is defined and locally integrable and if for $x \to \infty$ either*

1. *f is absolutely integrable or*
2. *the real part and the imaginary part of f converges to zero*

then the integrals $C(\alpha)$, $S(\alpha)$, $E(\alpha)$ exist respectively

1. *for any real α or*
2. *for any real non-zero α and they converge to zero for $\alpha \to \pm\infty$, respectively.*

Theorem 3.2. *Let a complex function $H = H(t)$ be absolutely integrable on $\mathbb{R}$. We denote $h(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(t)\exp(-\mathrm{i}st)\,\mathrm{d}t$ for $s \in \mathbb{R}$. If in a neighbourhood of a point $t_0 \in \mathbb{R}$ the function H has a finite variation then $\frac{1}{2}(H(t_0+) + H(t_0-)) = \lim\limits_{n\to\infty}\int_{-n}^{n} h(s)\exp(\mathrm{i}st_0)\,\mathrm{d}s$ holds. If the function h is absolutely integrable on $\mathbb{R}$ and the function H is continuous on $\mathbb{R}$ then $H(t) = \int_{-\infty}^{\infty} h(s)\exp(\mathrm{i}ts)\,\mathrm{d}s$ is valid for any $t \in \mathbb{R}$. (The function h is the Fourier transformation of the function H and H is the conjugated Fourier transformation of h.)*

Remark 3.3. The proofs of Theorems 3.1 and 3.2 can be found in [2]. The condition $\alpha \neq 0$ from Theorem 3.1 is not necessary for $S(\alpha)$ because $S(0) = 0$. For the existence of h for any real non-zero s it satisfies according to Theorem 3.1 if H is odd and continuous on $\mathbb{R}$ and if its real and imaginary parts are monotone for all sufficiently large $t \in \mathbb{R}$ (in absolute value) and $\lim H(t) = 0$ for $t \to \pm\infty$. In this case the formula $h(s) = -\frac{i}{s}\int_0^{\infty} H(t)\sin st\,\mathrm{d}t$ is valid. If moreover the function h is absolutely integrable on $\mathbb{R}$ and the function H has a finite variation then for any $t \in \mathbb{R}$ we have $H(t) = \int_{-\infty}^{\infty} h(s)\exp(\mathrm{i}ts)\,\mathrm{d}s = 2\mathrm{i}\int_0^{\infty} h(s)\sin ts\,\mathrm{d}s$.

3.3. Convolution of Fourier transformation with almost periodic functions. In the sequel we suppose that functions H, h satisfy the conditions from Remark 3.3.

Theorem 3.4. *For a complex almost periodic function f with its Fourier series $\sum\limits_{\lambda}\varphi(\lambda)\exp(\mathrm{i}\lambda t)$, $t \in \mathbb{R}$, $\lambda \in \Lambda_f$, we define a function $F = F(t) = \int_{-\infty}^{\infty} h(s)f(t+s)\,\mathrm{d}s$, $t \in \mathbb{R}$, where h is the Fourier transformation of a given complex function H defined on $\mathbb{R}$. The function F is almost periodic with its Fourier series $\sum\limits_{\lambda} H(\lambda)\varphi(\lambda)\exp(\mathrm{i}\lambda t)$, $t \in \mathbb{R}$, $\lambda \in \Lambda_f$.*

Proof. For any real numbers t, v the inequality $|F(t+v) - F(t)| = |\int_{-\infty}^{\infty} h(s)(f(t+v+s) - f(t+s))\,\mathrm{d}s| \leqslant \sup\limits_{t\in\mathbb{R}} |f(t+v) - f(t)| \int_{-\infty}^{\infty} |h(s)|\,\mathrm{d}s$ holds. This immediately implies that F is uniformly continuous on $\mathbb{R}$ and an almost periodic function. If we denote by $b(\lambda)$ the Fourier coefficient of the function F for a real number λ then

$$\begin{aligned}
b(\lambda) &= M_t\{F(t)\exp(-\mathrm{i}\lambda t)\} \\
&= M_t\Big\{\int_{-\infty}^{\infty} h(s)\exp(\mathrm{i}\lambda s)f(t+s)\exp(-\mathrm{i}\lambda(t+s))\,\mathrm{d}s\Big\} \\
&= \int_{-\infty}^{\infty} h(s)\exp(\mathrm{i}\lambda s)M_{t+s}\{f(t+s)\exp(-\mathrm{i}\lambda(t+s))\}\,\mathrm{d}s \\
&= \int_{-\infty}^{\infty} h(s)\exp(\mathrm{i}\lambda s)\,\mathrm{d}s \cdot \varphi(\lambda).
\end{aligned}$$

For $\lambda \notin \Lambda_f$ we have $\varphi(\lambda) = 0$, consequently $b(\lambda) = 0$. $\square$

R e m a r k 3.5. The consideration in 3.2 and 3.3 can be easily generalized to matrix functions.

4. Almost periodic Λ-solutions

4.1. Modifications of the functions Φ_P in functions H_P. In view of the definition of Φ and in view of $\Phi' = E-(-\tau)b_0\exp(-z\tau)$, $\Phi^{(m)}(z) = -(-\tau)^m b_0 \exp(-z\tau)$ for $z \in \mathbb{C}$, $m = 2, 3, \ldots$, and the relations $(\Phi^{-1}(z))' = -\Phi^{-1}(z)\Phi'(z)\Phi^{-1}(z)$, $(\Phi^{-1}(z))'' = 2\Phi^{-1}(z)\Phi'(z)\Phi^{-1}(z)\Phi'(z)\Phi^{-1}(z) - \Phi^{-1}(z)\Phi''(z)\Phi^{-1}(z) = 2\Phi^{-1}(z) \times \Phi'(z)\Phi^{-1}(z)\Phi'(z)\Phi^{-1}(z) - \Phi^{-1}(z)\Phi''(z)\Phi^{-1}(z)\Phi(z)\Phi^{-1}(z)$ it is possible to choose the already defined constant C_1 (see [6]) large enough so that besides the estimates

$$\begin{cases} |\Phi^{-1}(z)| \leqslant C_1 \text{ for } z \in G_0, \\ |\Phi^{-1}(z)| \leqslant C_1/|z| \text{ for } z \in G_0 \setminus \{0\} \end{cases} \tag{4.1}$$

also the following ones are true:

$$\begin{cases} |(\Phi^{-1}(z))^{(m)}| \leqslant C_1 \quad \text{for } z \in G_0, \\ |(\Phi^{-1}(z))^{(m)}| \leqslant C_1/|z| \quad \text{for } z \in G_0 \setminus \{0\}, \\ |(\Phi^{-1}(z))^{(m)}| \leqslant C_1/|z|^2 \quad \text{for } z \in G_0 \setminus \{0\} \end{cases} \tag{4.2}$$

for $m = 1, 2$.

Now we begin with trigonometric polynomials a, b, f from (2.1) fulfilling the conditions (2.2), (2.3), (2.4) and in addition we suppose that $v_0 = \inf \Lambda_f > 0$ and again $B_0 \neq \emptyset$. We denote $S = \sum(a)$, $T = \sum(b)$. For a fixed $\bar{s} \in {\mathbb{N}_0}^{1\times(M+N)}$ and a given sequence $P = P(\bar{s})$ we construct an odd square matrix function H_P by the formula

$$H_P(t) = \begin{cases} = 0 \quad \text{for } t = 0, \\ = \Phi_P(\mathrm{i}t) \quad \text{for } t \in J_0, \\ = H_P(u_j) + \frac{t-u_j}{v_j-u_j}(H_p(v_j) - H_P(u_j)) \quad \text{for } t \in (u_j, v_j), \\ \qquad j = 0, 1, \ldots, k_0, \\ = -H_P(-t) \quad \text{for } t < 0, \end{cases}$$

$(u_0 = 0,\ J_0 = \langle v_0, u_1\rangle \cup \ldots \cup \langle v_{k_0-1}, u_{k_0}\rangle \cup \langle v_{k_0}, \infty))$. The function $H_P = H_P(t)$ is defined, continuous and piecewise smooth up to any order on $\mathbb{R}$. We shall prove that H_P has absolutely integrable derivatives on $\mathbb{R}$, which is necessary for us in the sequel. Denote by h_P the Fourier transformation of the function H_P, which means $h_P(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H_P(t)\exp(-\mathrm{i}ts)\,\mathrm{d}t$, $s \in \mathbb{R}$.

This improper integral converges owing to the properties of H_P on J_0 (more exactly: Φ_P on G_0) except for the case $\bar{s} = \bar{0}$ when for $s = 0$ the improper integral does not converge.

Therefore, for $\bar{s} = \bar{0}$ we denote H_P, h_P also by H_0, h_0. Since the function H_P is odd on $\mathbb{R}$ we get $h_P(s) = -\frac{\mathrm{i}}{\pi}\int_0^\infty H_P(t)\sin ts\,\mathrm{d}t$, $s \in \mathbb{R}\setminus\{0\}$.

4.2. Estimates of H_P. In what follows we will show that for $\bar{s} \neq \bar{0}$ the Fourier transformation h_P of H_P is absolutely integrable on $\mathbb{R}$ and therefore

$$H_P(t) = \int_{-\infty}^{\infty} h_P(s)\exp \mathrm{i}ts\,\mathrm{d}s;\ t \in \mathbb{R}. \tag{4.3}$$

To this aim we need to prove a few assertions.

Lemma 4.1. *For all $t \in J_0$ the inequalities*

$$|H_P(t)| \leqslant \prod_{j=0}^{|\bar{s}|}\frac{C_1|\gamma_j|}{t+\overline{P}_j\varpi} \leqslant \frac{C_1}{(|\bar{s}|+1)!4\delta}\Big[\prod_{k=1}^{M}\Big(\frac{C_1|\alpha_k|}{4\delta}\Big)^{m_k}\Big]\prod_{k=1}^{N}\Big(\frac{C_1|\beta_k|}{4\delta}\Big)^{n_k}, \tag{4.4}$$

$$\begin{aligned}|\dot{H}_P(t)| &\leqslant \sum_{k=0}^{|\bar{s}|}\prod_{j=0}^{|\bar{s}|}\frac{C_1|\gamma_j|}{t+\overline{P}_j\varpi}\\ &\leqslant \frac{C_1}{(|\bar{s}|+1)!4\delta}\Big[\prod_{k=1}^{M}\Big(\frac{C_1|\alpha_k|}{4\delta}\Big)^{m_k}\Big]\prod_{k=1}^{N}\Big(\frac{C_1|\beta_k|}{4\delta}\Big)^{n_k},\end{aligned} \tag{4.5}$$

$$\begin{aligned}|\ddot{H}_P(t)| &\leqslant \sum_{k=0}^{|\bar{s}|}\sum_{k=0}^{|\bar{s}|}\prod_{j=0}^{|\bar{s}|}\frac{C_1|\gamma_j|}{t+\overline{P}_j\varpi}\\ &\leqslant \frac{C_1}{(|\bar{s}|+1)!4\delta}\Big[\prod_{k=1}^{M}\Big(\frac{C_1|\alpha_k|}{2\delta}\Big)^{m_k}\Big]\prod_{k=1}^{N}\Big(\frac{C_1|\beta_k|}{2\delta}\Big)^{n_k}\end{aligned} \tag{4.6}$$

are valid for $\alpha_k = \alpha(\mu_k)$, $k = 1,\dots,M$; $\beta_k = \beta(\nu_k)$, $k = 1,\dots,N$; $\gamma_0 = \gamma(0) = 1$, $\gamma_j = \gamma(\bar{p}_j\varpi) = \alpha(\bar{q}_j\varpi) + \beta(\bar{r}_j\varpi)\exp(-\mathrm{i}\bar{p}_{j-1}\varpi)$, $\bar{p}_j = (\bar{q}_j, \bar{r}_j)$, $1,\dots,|\bar{s}|$. (The derivatives $\dot{H}_P$, $\ddot{H}_P$ at the boundary points of J_0 are the corresponding one-sided derivatives on J_0.)

P r o o f. Owing to $|\bar{p}_j| = |\bar{q}_j| + |\bar{r}_j| = 1$, $j = 1,\dots,|\bar{s}|$, the equality

$$\prod_{j=1}^{|\bar{s}|}|\gamma_j| = \Big[\prod_{k=1}^{M}|\alpha_k|^{m_k}\Big]\prod_{k=1}^{N}|\beta_k|^{n_k} \tag{4.7}$$

is true. Further, for $t \in J_0$ in accord with the definition of H_P we have

$$H_P(t) = \Phi_P(\mathrm{i}t) = \prod_{j=|\bar{s}|}^{0} \Phi^{-1}\left(\mathrm{i}t + \mathrm{i}\overline{P}_j\varpi\right)\gamma\left(\bar{p}_j\varpi\right),$$

$$\dot{H}_P(t) = \mathrm{i}\Phi'_P(\mathrm{i}t) = \mathrm{i}\sum_{k=0}^{|\bar{s}|}\prod_{j=|\bar{s}|}^{0}\left(\Phi^{-1}\left(\mathrm{i}t + \mathrm{i}\overline{P}_j\varpi\right)\right)^{(\delta_{jk})}\gamma\left(\bar{p}_j\varpi\right),$$

$$\ddot{H}_P(t) = -\Phi''_P(\mathrm{i}t) = -\sum_{l=0}^{|\bar{s}|}\sum_{k=0}^{|\bar{s}|}\prod_{j=|\bar{s}|}^{0}\left(\Phi^{-1}\left(\mathrm{i}t + \mathrm{i}\overline{P}_j\varpi\right)\right)^{(\delta_{jk}+\delta_{jl})}\gamma\left(\bar{p}_j\varpi\right),$$

where $\delta_{jk} = 0$ for $j \neq k$ and $\delta_{jk} = 1$ for $j = k$ (analogously for δ_{jl}). Hence, the validity of (4.4), (4.5), (4.6) already follows by means of (4.1) and (4.2) for $t \in J_0$.

□

Corollary 4.2. *The consequences of the estimates* (4.4), (4.5), (4.6)

$$(4.4') \qquad |H_P(t)| \leqslant \frac{C_1}{(|\bar{s}|+1)!4\delta}\left[\prod_{k=1}^{M}\left(\frac{C_1|\alpha_k|}{4\delta}\right)^{m_k}\right]\prod_{k=1}^{N}\left(\frac{C_1|\beta_k|}{4\delta}\right)^{n_k},$$

$$(4.5') \qquad |\dot{H}_P(t)| \leqslant \frac{C_1}{|\bar{s}|!4\delta}\left[\prod_{k=1}^{M}\left(\frac{C_1|\alpha_k|}{4\delta}\right)^{m_k}\right]\prod_{k=1}^{N}\left(\frac{C_1|\beta_k|}{4\delta}\right)^{n_k},$$

$$(4.6') \qquad |\ddot{H}_P(t)| \leqslant \frac{C_1}{|\bar{s}|!4\delta}\left[\prod_{k=1}^{M}\left(\frac{C_1|\alpha_k|}{2\delta}\right)^{m_k}\right]\prod_{k=1}^{N}\left(\frac{C_1|\beta_k|}{2\delta}\right)^{n_k}$$

are valid for any $t \in \mathbb{R}$.

Proof. The validity of these estimates follows at once from (4.4), (4.5), (4.6), because H_P and its derivatives are continuous on $J_0 \cup (-J_0)$, odd or even functions on $\mathbb{R}$ and linear on each component of the open set $\mathbb{R} \setminus J_0 \cup (-J_0)$. □

For $\bar{s} \neq \bar{0}$ the absolute integrability of H_P on $\mathbb{R}$ follows from Lemma 4.1 and Corollary 4.2.

4.3. Estimates of h_P. Now we shall estimate the function h_P.

Lemma 4.3. *For any $m \in \mathbb{N}$ the inequality*

$$(4.8) \qquad \int_0^\infty \frac{\mathrm{d}t}{\prod_{j=0}^{m}(t+j+1)} \leqslant \frac{1}{m!}$$

holds.

P r o o f. First we define the converging series

$$Z_m = \sum_{k=0}^{\infty} \frac{1}{(k+1)\dots(k+m+1)}, \qquad m = 1, 2, \dots,$$
$$Z_1 = \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+2)} = \sum_{k=0}^{\infty} \Bigl(\frac{1}{k+1} - \frac{1}{k+2}\Bigr) = 1 \leqslant \frac{1}{1!}.$$

If for a number $m \in \mathbb{N}$ the inequality $Z_m \leqslant \frac{1}{m!}$ is true then $Z_{m+1} \leqslant \frac{1}{m+1} Z_m \leqslant \frac{1}{(m+1)!}$. This means that the inequality $Z_m \leqslant \frac{1}{m!}$ is true for any $m \in \mathbb{N}$. Hence, we have

$$\int_0^{\infty} \frac{\mathrm{d}t}{\prod\limits_{j=0}^{m} (t+j+1)} = \sum_{k=0}^{\infty} \int_k^{k+1} \frac{\mathrm{d}t}{\prod\limits_{j=0}^{m} (t+j+1)}$$
$$\leqslant \sum_{k=0}^{\infty} \frac{1}{(k+1)\dots(k+m+1)} = Z_m \leqslant \frac{1}{m!}$$

for any $m \in \mathbb{N}$. □

Lemma 4.4. *There exists a positive constant C_2 independent of $\bar{s}$ and P such that for all non-zero real numbers s the inequality*

$$|h_P(s)| \leqslant \frac{1}{s^2} \frac{C_1 C_2}{|\bar{s}|!} \Bigl[\prod_{k=1}^{M} \Bigl(\frac{C_1 |\alpha_k|}{\delta} \Bigr)^{m_k} \Bigr] \prod_{k=1}^{N} \Bigl(\frac{C_1 |\beta_k|}{\delta} \Bigr)^{n_k} \tag{4.9}$$

holds for $\bar{s} \neq \bar{0}$.

P r o o f. Since the function H_P is odd on $\mathbb{R}$, the equality

$$h_P(s) = -\frac{\mathrm{i}}{\pi} \int_0^{\infty} H_P(t) \sin ts \, \mathrm{d}t$$
$$= -\frac{\mathrm{i}}{\pi} \Bigl(\sum_{j=0}^{k_0} \int_{u_j}^{v_j} + \int_{J_0} \Bigr) H_P(t) \sin ts \, \mathrm{d}t$$

holds. By double integration by parts and using (3.1) and (3.2) we get for any $\bar{s} \geqslant \bar{0}$

$$
\begin{aligned}
h_P(s) = -\frac{\mathrm{i}}{\pi}\Big[&- \frac{1}{s}\sum_{j=0}^{k_0}(H_P(v_j)\cos v_j s - H_P(u_j)\cos u_j s)\\
&+ \frac{1}{s^2}\sum_{j=0}^{k_0}\frac{\sin v_j s - \sin u_j s}{v_j - u_j}(H_P(v_j) - H_P(u_j))\\
&- \frac{1}{s}\sum_{j=0}^{k_0-1}(H_P(u_{j+1})\cos u_{j+1}s - H_P(v_j)\cos v_j s)\\
&+ \frac{1}{s^2}\sum_{j=0}^{k_0-1}(\dot{H}_P(u_{j+1})\sin u_{j+1}s - \dot{H}_P(v_j)\sin v_j s)\\
&+ \frac{1}{s}H_P(v_{k_0})\cos v_{k_0}s - \frac{1}{s^2}\dot{H}_P(v_{k_0})\sin v_{k_0}s - \frac{1}{s^2}\int_{J_0}\ddot{H}_P(t)\sin ts\,\mathrm{d}t\Big]\\
= \frac{\mathrm{i}}{\pi s^2}\Big[&- \sum_{j=0}^{k_0}\frac{\sin v_j s - \sin u_j s}{v_j - u_j}(H_P(v_j) - H_P(u_j))\\
&- \sum_{j=0}^{k_0-1}(\dot{H}_P(u_{j+1})\sin u_{j+1}s - \dot{H}_P(v_j)\sin v_j s)\\
&+ \dot{H}_P(v_{k_0})\sin v_{k_0}s + \int_{J_0}\ddot{H}_P(t)\sin ts\,\mathrm{d}t\Big].
\end{aligned}
$$

Recall that $\frac{1}{\delta} \leqslant \frac{1}{\delta^2}$, $v_0 - u_0 = v_0 \geqslant 4\delta$, $v_j - u_j = 2\delta$, $j = 1, \ldots, k_0$, where $0 = u_0 < v_1 < u_1 < \ldots < u_{k_0} < v_{k_0} < \infty$ so that $\sum\limits_{j=0}^{k_0}\frac{2}{v_j - u_j} \leqslant \frac{k_0+1}{\delta}$, $\sum\limits_{j=0}^{k_0-1}(u_{j+1} - v_j) \leqslant u_{k_0} - v_0$. By means of further modifications we get

$$
\begin{aligned}
|h_P(s)| \leqslant \frac{1}{\pi s^2}\Big[&\sum_{j=0}^{k_0}\frac{2}{v_j - u_j}(|H_P(v_j)| + |H_P(u_j)|) + \sum_{j=0}^{k_0}|\dot{H}_P(v_j)|\\
&+ \sum_{j=1}^{k_0}|\dot{H}_P(u_j)| + \sum_{j=0}^{k_0-1}\int_{v_j}^{u_{j+1}}|\ddot{H}_P(t)|\,\mathrm{d}t + \int_{v_{k_0}}^{\infty}|\ddot{H}_P(t)|\,\mathrm{d}t\Big].
\end{aligned}
$$

Because for $\bar{s} \neq \bar{0}$ the estimate

$$
\begin{aligned}
\int_{v_{k_0}}^{\infty}\frac{\mathrm{d}t}{\prod\limits_{j=0}^{|\bar{s}|}(t + \overline{P}_j\varpi)} &= \int_0^{\infty}\frac{\mathrm{d}t}{\prod\limits_{j=0}^{|\bar{s}|}(t + v_{k_0} + \overline{P}_j\varpi)}\\
&\leqslant \frac{1}{(4\delta)^{|\bar{s}|}}\int_0^{\infty}\frac{\mathrm{d}t}{\prod\limits_{j=0}^{|\bar{s}|}(t + 1 + j)} \leqslant \frac{1}{|\bar{s}|!(4\delta)^{|\bar{s}|}}
\end{aligned}
$$

holds according to Lemma 4.3, by using (4.6) we obtain

$$\int_{v_{k_0}}^{\infty} |\ddot{H}_P(t)|\,\mathrm{d}t \leqslant (|\bar{s}|+1)^2 \Biggl[\prod_{j=0}^{|\bar{s}|} C_1|\gamma_j|\Biggr] \int_{v_{k_0}}^{\infty} \frac{\mathrm{d}t}{\prod\limits_{j=0}^{|\bar{s}|}(t+\overline{P}_j\varpi)} \leqslant \frac{C_1}{|\bar{s}|!}\prod_{j=1}^{|\bar{s}|}\frac{C_1|\gamma_j|}{\delta}$$

by virtue of the correct inequalities $m+1 \leqslant 2^m$ and $(m+1)^2 \leqslant 4^m$, $m = 0, 1, \ldots$. Owing to (4.4′), (4.5′), (4.6′) and the previous results we conclude that

$$\begin{aligned} |h_P(s)| \leqslant \frac{C_1}{\pi s^2}\Biggl[&\frac{2}{(|\bar{s}|+1)!4\delta}\sum_{j=0}^{k_0}\frac{2}{v_j-u_j} + \frac{2k_0+1}{|\bar{s}|!4\delta} \\ &+ \sum_{j=0}^{k_0-1}(u_{j+1}-v_j)\frac{1}{|\bar{s}|!4\delta} + \frac{1}{|\bar{s}|!}\Biggr]\prod_{j=1}^{|\bar{s}|}\frac{C_1|\gamma_j|}{\delta} \leqslant \frac{C_1C_2}{|\bar{s}|!s^2}\prod_{j=1}^{|\bar{s}|}\frac{C_1|\gamma_j|}{\delta}, \end{aligned}$$

where $C_2 = (4k_0 + 4 + u_{k_0} - v_0)/(4\pi\delta^2)$. □

Lemma 4.5. *For all $\bar{s} \neq \bar{0}$ and P there exists a positive constant C_3 independent of $\bar{s}$ and P such that the following inequality*

$$|h_P(s)| \leqslant \frac{C_1C_3}{|\bar{s}|!}\Biggl[\prod_{k=1}^{M}\Bigl(\frac{C_1|\alpha_k|}{4\delta}\Bigr)^{m_k}\Biggr]\prod_{k=1}^{N}\Bigl(\frac{C_1|\beta_k|}{4\delta}\Bigr)^{n_k} \tag{4.10}$$

is true for all real non-zero s. This means that $|h_P(s)|$ is uniformly bounded on $\mathbb{R}\setminus\{0\}$.

Proof. From (4.4′) and (4.4) we get

$$\begin{aligned} |h_P(s)| &\leqslant \frac{1}{\pi}\int_0^{v_{k_0}}|H_P(t)|\,\mathrm{d}t + \int_{v_{k_0}}^{\infty}|H_P(t)|\,\mathrm{d}t \\ &\leqslant \frac{v_{k_0}C_1}{(|\bar{s}|+1)!4\pi\delta}\prod_{j=1}^{|\bar{s}|}\frac{C_1|\gamma_j|}{4\delta} + \frac{C_1}{\pi}\int_{v_{k_0}}^{\infty}\frac{\mathrm{d}t}{\prod\limits_{j=0}^{|\bar{s}|}(t+\overline{P}_j\varpi)}\prod_{j=1}^{|\bar{s}|}C_1|\gamma_j| \leqslant \\ &\leqslant \frac{(v_{k_0}+1)C_1}{|\bar{s}|!4\pi\delta}\prod_{j=1}^{|\bar{s}|}\frac{C_1|\gamma_j|}{4\delta} \leqslant \frac{C_1C_3}{|\bar{s}|!}\prod_{j=1}^{|\bar{s}|}\frac{C_1|\overline{\gamma}_j|}{4\delta}, \end{aligned}$$

where $C_3 = (v_{k_0}+1)/(4\pi\delta)$, so that (4.10) is true. □

4.4. Estimates of H_0, h_0. Now we still need to verify the validity of (4.9) and (4.10) for $\bar{s} = \bar{0}$ and $s \neq 0$.

Lemma 4.6. *For $s \neq 0$ the estimates* (4.9) *and* (4.10) *are valid for H_0, h_0 (if necessary we increase the already defined positive constants C_1, C_2 and C_3).*

P r o o f. The function H_0 has an absolutely integrable derivative $\dot{H}_0$ on the interval $\langle v_{k_0}, \infty)$, since for $t \geqslant v_{k_0}$ the inequality

$$|\dot{H}_0(t)| = |(\Phi^{-1}(\mathrm{i}t))'| \leqslant C_1/t^2 \tag{4.11}$$

is valid by virtue of the third estimate from (4.2). This means that the real and imaginary parts of each element of the matrix function H_0 satisfy the conditions from the assertion (vi). Consequently, for all real non-zero s we get $\left|\int_{v_{k_0}}^{\infty} H_0(t) \sin ts \,\mathrm{d}t\right| \leqslant 4\pi n_0^2 C_1$ by using (3.6).

Owing to the form of H_0 the inequality $\left|\int_0^{v_{k_0}} H_0(t) \sin ts \,\mathrm{d}t\right| \leqslant \int_0^{v_{k_0}} |H_0(t)| \,\mathrm{d}t \leqslant v_{k_0} C_1$ is correct, so that $|h_0(s)| \leqslant (v_{k_0} + 4n_0^2)C_1/(4\delta)$. If we choose $C_3 = (v_{k_0} + 4n_0^2)/(4\delta)$ then (4.10) is verified. The estimate (4.9) remains valid also for h_0 with regard to the correct inequality $|\ddot{H}_0(t)| = |(\Phi^{-1}(\mathrm{i}t))''| \leqslant C_1/t^2$ for $t \geqslant v_{k_0}$ based on the third estimate from (4.2) and therefore by virtue of (4.11) the integral $\int_{v_{k_0}}^{\infty} H(t) \sin ts \,\mathrm{d}t$ converges. □

Lemma 4.7. *Owing to the validity of* (4.9) *and* (4.10) *for any $\bar{s}$ and P the estimate*

$$\int_{-\infty}^{\infty} |h_P(s)| \,\mathrm{d}s \leqslant \frac{C_1 C_4}{|\bar{s}|!} \left[\prod_{k=1}^{M} \Big(\frac{C_1|\alpha_k|}{\delta}\Big)^{m_k}\right] \prod_{k=1}^{N} \Big(\frac{C_1|\beta_k|}{\delta}\Big)^{n_k} \tag{4.12}$$

is true with a positive constant C_4 independent of $\bar{s}$ and P.

P r o o f. With regard to $\int_0^{v_{k_0}} |h_P(s)| \,\mathrm{d}s \leqslant \frac{C_1 C_2}{|\bar{s}|!} \prod_{j=1}^{|\bar{s}|} \frac{C_1|\gamma_j|}{\delta}$ and

$$\int_{v_{k_0}}^{\infty} |h_P(s)| \,\mathrm{d}s \leqslant \frac{C_1 C_3}{|\bar{s}|!} \int_{v_{k_0}}^{\infty} \frac{\mathrm{d}s}{s^2} \prod_{j=1}^{|\bar{s}|} \frac{C_1|\gamma_j|}{\delta} = \frac{C_1 C_3}{|\bar{s}|! v_{k_0}} \prod_{j=1}^{|\bar{s}|} \frac{C_1|\gamma_j|}{\delta}$$

we obtain a true estimate

$$\int_{-\infty}^{\infty} |h_P(s)| \,\mathrm{d}s \leqslant 2 \int_0^{\infty} |h_P(s)| \,\mathrm{d}s \leqslant \frac{C_1 C_4}{|\bar{s}|!} \prod_{j=1}^{|\bar{s}|} \frac{C_1|\gamma_j|}{\delta}$$

with the positive constant $C_4 = 2(C_2 v_{k_0}^2 + C_3)/v_{k_0}$. □

4.5. Almost periodic Λ-solutions. We show that the obtained formal Λ-solution x_f from (2.5) is an almost periodic Λ-solutions. To this aim we prove directly only a certain absolute convergence of the trigonometric series ($\lambda \in \Lambda_f$)

$$\tag{4.13} \begin{aligned} &\sum_{\bar{s}\geqslant\bar{0}}\Big[\sum_{P}\sum_{\lambda}\Phi_P(\mathrm{i}\lambda)\varphi(\mathrm{i}\lambda)\varphi(\lambda)\exp{(\mathrm{i}\lambda t)}\Big]\exp{(\mathrm{i}\bar{s}\bar{\varpi}t)} \\ &\quad=\sum_{\bar{s}\geqslant\bar{0}}\Big[\sum_{P}\sum_{\lambda}H_p(\lambda)\varphi(\mathrm{i}\lambda)\varphi(\lambda)\exp{(\mathrm{i}\lambda t)}\Big]\exp{(\mathrm{i}\bar{s}\bar{\varpi}t)} \end{aligned}$$

which arises by a rearrangement of the trigonometric series x_f. Namely, the convergence of the series

$$\tag{4.14} \begin{aligned} &\sum_{\bar{s}\geqslant\bar{0}}\sum_{P}\Big|\sum_{\lambda}H_P(\lambda)\varphi(\lambda)\exp{(\mathrm{i}\lambda t)}\Big| \\ &=\sum_{\bar{s}\geqslant\bar{0}}\sum_{P}\Big|\int_{-\infty}^{\infty}h_p(s)\sum_{\lambda}\varphi(\lambda)\exp{(\mathrm{i}\lambda(s+t))}\,\mathrm{d}s\Big| \\ &=\sum_{\bar{s}\geqslant\bar{0}}\sum_{P}\Big|\int_{-\infty}^{\infty}h_p(s)f(s+t)\,\mathrm{d}s\Big|\leqslant|f|\sum_{\bar{s}\geqslant\bar{0}}\sum_{P}\Big|\int_{-\infty}^{\infty}|h_p(s)|\,\mathrm{d}s\Big| \\ &\leqslant|f|C_1C_4\sum_{\bar{s}\geqslant\bar{0}}\sum_{P}|\frac{1}{|\bar{s}|!}\prod_{j=1}^{|\bar{s}|}\frac{C_1|\gamma_j|}{\delta}\leqslant|f|C_1C_4\sum_{\bar{s}\geqslant\bar{0}}\binom{|\bar{s}|}{\bar{s}}\frac{1}{|\bar{s}|!}\prod_{j=1}^{|\bar{s}|}\frac{C_1|\gamma_j|}{\delta} \\ &=|f|C_1C_4\sum_{\bar{s}\geqslant\bar{0}}\Big[\prod_{k=1}^{M}\Big(\frac{C_1|\alpha_k|}{\delta}\Big)^{m_k}/m_k!\Big]\prod_{k=1}^{N}\Big(\frac{C_1|\beta_k|}{\delta}\Big)^{n_k}/n_k! \\ &=|f|C_1C_4\Big[\prod_{k=1}^{M}\Big(\sum_{m=0}^{\infty}\Big(\frac{C_1|\alpha_k|}{\delta}\Big)^{m}/m!\Big)\Big]\prod_{k=1}^{N}\Big(\sum_{n=0}^{\infty}\Big(\frac{C_1|\beta_k|}{\delta}\Big)^{n}/n!\Big) \\ &=|f|C_1C_4\exp{(C_1(S+T)/\delta)}=\tilde{A}|f|\ \text{for}\ t\in\mathbb{R}, \end{aligned}$$

where $\tilde{A}=C_1C_4\exp{(C_1(S+T)/\delta)}$, will be considered in the sequel. In the case of the one-point spectrum for $f(t)=\varphi(\lambda)\exp{(\mathrm{i}\lambda t)}$, $t\in\mathbb{R}$, when x_f and x_λ coincide and x_λ coincides with (4.13), the convergence of the series (4.14) ensures the absolute and consequently uniform convergence of x_λ on $\mathbb{R}$ for every $\lambda\in\Lambda_f$. Hence, the trigonometric series $x_f=\sum x_\lambda$, $\lambda\in\Lambda_f$, converges absolutely and uniformly on $\mathbb{R}$ and satisfies the estimate

$$\tag{4.15} |x_f|\leqslant\tilde{A}|f|.$$

(Recall that f is a trigonometric polynomial.)

Theorem 4.8. *The formal solution x_f from Theorem* 2.1 *is an almost periodic Λ-solution of Equation* (2.1). *Moreover, it is unique and satisfies the estimate*

$$\|x_f\| \leqslant A|f| \tag{4.16}$$

where the positive constant A depends only on $a_0, b_0, v_0, \Delta, d_\theta, d, d_\xi, S, T$ and $S = \sum|\alpha(\mu)| = \sum(a)$, $\mu \in \Lambda_a$; $T = \sum|\beta(\nu)| = \sum(b)$, $\nu \in \Lambda_b$.

Proof. If an almost periodic Λ-solution of Equation (2.1) exists, its uniqueness is ensured by the uniqueness of the formal almost periodic Λ-solution x_f. The function x_f satisfies (4.15). Inserting x_f into the right-hand side of (2.1) we get the formal derivative $\dot{x}_f$ of x_f and the estimate

$$|\dot{x}_f| \leqslant (|a_0| + |b_0| + |a| + |b|)\,|x_f| + |f| \leqslant [(|a_0| + |b_0| + S + T)\,\tilde{A} + 1]|f|$$

which implies the absolute and uniform convergence of the trigonometric series $\dot{x}_f$, which means that $\dot{x}_f$ is the derivative of x_f and x_f is the unique almost periodic Λ-solution of Equation (2.1). Setting

$$A = (|a_0| + |b_0| + S + T + 1)\,\tilde{A} + 1$$

we conclude that the estimate (4.16) holds. □

Corollary 4.9. *Let Λ_1, Λ_2 be two non-void sets of real numbers and let S, T be two positive constants. If a, b, f from Equation* (2.1) *are trigonometric polynomials with $\Lambda_f \subset \Lambda_1$, $\Lambda_a \subset \Lambda_2$, $\Lambda_b \subset \Lambda_2$ and $\sum(a) \leqslant S$, $\sum(b) \leqslant T$ and if*

$$v_0' = \inf \Lambda_1 > 0, \tag{4.17}$$

$$\Delta' = \inf \Lambda_2 > 0, \tag{4.18}$$

$$d_\theta' = \begin{cases} \operatorname{dist}\,[\theta;\, S(\Lambda_2)] > 0 & \text{for } \theta \neq 0, \\ 4 \text{ for } \theta = 0, \end{cases} \tag{4.19}$$

$$d' = \operatorname{dist}\,[\mathrm{i}\Lambda'; \sigma\,(\Delta\,(z))] > 0, \tag{4.20}$$

where $\Lambda' = \Lambda_1 + S\,(\Lambda_2 \cup \{0\})$, then there exists exactly one almost periodic Λ'-solution x_f of Equation (2.1). *This solution satisfies the estimate* (4.16) *where the positive constant A depends only on $a_0, b_0, v_0', \Delta', d_\theta, d', d_\xi, \tau, S, T$.*

Proof. The existence of an almost periodic Λ'-solution x_f follows from Theorem 4.8 which ensures the existence of an almost periodic Λ-solution where $\Lambda = \Lambda_f + S\,(\Lambda_a \cup \Lambda_b \cup \{0\})$, so that $\Lambda \subset \Lambda'$ and an almost periodic Λ-solution is also an almost periodic Λ'-solution. The uniqueness of an almost periodic Λ'-solution x_f follows from the facts that $\alpha(\mu) = 0$ for $\mu \in \Lambda_2 \setminus \Lambda_a$ and $\beta(\nu) = 0$ for $\nu \in \Lambda_2 \setminus \Lambda_b$ and $\varphi(\lambda) = 0$ for $\lambda \in \Lambda_1 \setminus \Lambda_f$, which means that the Λ'-solution x_f coincides with the Λ-solution x_f from Theorem 4.8. (More detailed explanation is in [6].)

The construction of the positive constant A is the same as before with the only exception that the constants v_0, Δ, d_θ, d are replaced by the constants $v_0', \Delta', d_\theta', d'$, respectively, for which it is apparent that $v_0' \leqslant v_0$, $\Delta' \leqslant \Delta$, $d_\theta' \leqslant d_\theta$, $d' \leqslant d$ so that the constant A could at worst increase. □

Remark 4.10. Corollary 4.9 ensures the validity of the estimate (4.16) with a positive constant A common for all almost periodic Λ'-solutions x_f of Equation (2.1) of the whole class of trigonometric polynomials a, b, f from Corollary 4.9.

4.6. Limit passages. The conclusions obtained under the assumption that a, b, f are trigonometric polynomials remain valid even under more general assumptions.

Theorem 4.11. *If in Equation* (2.1) a, b *are trigonometric polynomials and* f *is almost periodic and if the conditions* $v_0 = \inf \Lambda_f > 0$, (2.2), (2.3), (2.4) *are fulfilled then Equation* (2.1) *has exactly one almost periodic* Λ*-solution* x_f *and this solution satisfies the estimate* (4.16).

Remark 4.12. Equation (2.1) may admit even infinitely many almost periodic solutions but only one of them has its spectrum in $\mathrm{i}\Lambda$ (hence it is derived the name of an almost periodic Λ-solution).

Proof of Theorem 4.11. There exists a sequence of Bochner-Fejér approximation polynomials B_m, $m = 1, 2, \ldots$, of the function f (with spectra contained in Λ_f) uniformly convergent to f on $\mathbb{R}$.

If we choose $\Lambda_1 = \Lambda_f, \Lambda_2 = \Lambda_a \cup \Lambda_b$ then $\Lambda' \subset \Lambda$ and for Equation (2.1) with $f = B_m$ we have satisfied the assumptions from Corollary 4.9 which coincide in this case with the assumptions from Theorem 2.1 and $v_0' = v_0 = \inf \Lambda_f > 0$, $m = 1, 2, \ldots$. The equation

$$\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + B_m(t),\ t \in \mathbb{R}, \tag{2.1m}$$

has exactly one almost periodic Λ-solution x_m and this solution satisfies the estimate

$$\|x_m\| \leqslant A|B_m|,\ m = 1, 2, \ldots. \tag{4.16m}$$

Since the spectrum of the trigonometric polynomial $B_{m+k} - B_m$ is contained in $\mathrm{i}\Lambda_f$, the equation (for $t \in \mathbb{R}$) $\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) +$

$B_{m+k}(t) - B_m(t)$ has also exactly one almost periodic Λ-solution, namely $x_{m+k} - x_m$, and the estimate $\|x_{m+k} - x_m\| \leqslant A|B_{m+k} - B_m|$ holds for $m, k = 1, 2, \ldots$. In virtue of the uniform convergence of the sequence of the trigonometric polynomials B_m to the almost periodic function f on $\mathbb{R}$, it is readily seen that the sequences of almost periodic functions $\{x_m\}, \{\dot{x}_m\}$ converge uniformly on $\mathbb{R}$ and the limit functions $x_f = \lim x_m$, $\dot{x}_f = \lim \dot{x}_m$ satisfy Equation (2.1). Thus, x_f is an almost periodic Λ-solution of Equation (2.1) and the validity of the estimate (4.16) can be verified by using the limit passage for $m \to \infty$ in the estimates (4.16m).

It remains to check the uniqueness which could be damaged by the limit passage. So, let us suppose the existence of another almost periodic Λ-solution y of Equation (2.1). In such a case there exists a sequence y_m, $m = 1, 2, \ldots$, of Bochner-Fejér approximation polynomials of the almost periodic function y to which they converge uniformly on $\mathbb{R}$ and their derivatives $\dot{y}_m$, $m = 1, 2, \ldots$, form a sequence of Bochner-Fejér approximation polynomials of the almost periodic function $\dot{y}$ to which they converge uniformly on $\mathbb{R}$ for $m \to \infty$. It is easy to verify that the sequence of trigonometric polynomials $f_m(t) = \dot{y}(t) - a_0 y_m(t) - b_0 y_m(t-\tau) - a(t) y_m(t) - b(t) y_m(t-\tau)$, $m = 1, 2, \ldots$, converges uniformly on $\mathbb{R}$ to the almost periodic function f. Denoting $\Lambda_1 = \Lambda = \Lambda_f + S\,(\Lambda_a \cup \Lambda_b \cup \{0\})$, $\Lambda_2 = \Lambda_a \cup \Lambda_b$ we have $\Lambda' = \Lambda_1 + S\,(\Lambda_2 \cup \{0\})$ and $v_0' = \inf \Lambda_1 = \inf \Lambda_f = v_0 > 0$ and the assumptions (4.18), (4.19), (4.20) are satisfied which coincide here with the assumptions (2.2), (2.3), (2.4). The spectra of the trigonometric polynomials f_m and consequently also the spectra of the trigonometric polynomials $B_m - f_m$ are contained in $\mathrm{i}\Lambda$, $m = 1, 2, \ldots$, so that by Corollary 4.9. the equation

$$\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + B_m(t) - f_m(t)$$

has exactly one almost periodic Λ-solution, namely $w_m = x_m - y_m$, which satisfies the estimate $\|w_m\| = \|x_m - y_m\| \leqslant A|B_m - f_m|$, $m = 1, 2, \ldots$. However, $\|x_f - y\| = \lim \|x_m - y_m\| = 0$ and hence $x_f = y$. □

Corollary 4.13. *Let Λ_1, Λ_2 be two non-void sets of real numbers and let S, T be two positive constants. If the assumptions* (4.17), (4.18), (4.19), (4.20) *are satistied and if f is an almost periodic function with its spectrum contained in* $\mathrm{i}\Lambda_1$ *and if a, b are trigonometric polynomials with their spectra contained in* $\mathrm{i}\Lambda_2$ *for which* $\sum(a) \leqslant S$, $\sum(b) \leqslant T$, *then Equation* (2.1) *has exactly one almost periodic Λ'-solution x_f where $\Lambda' = \Lambda_1 + S\,(\Lambda_1 \cup \Lambda_2 \cup \{0\})$ and this solution satisfies the estimate* (4.16) *where the positive constant A depends only on $a_0, b_0, v_0', \Delta', d_\theta', d', d_\xi, \tau, S, T$.*

P r o o f. The validity of Corollary 4.13 can be verified by passing to the limit analogously as in the proof of Theorem 4.11. □

R e m a r k 4.14. Corollary 4.13 ensures the validity of the estimate (4.16) with a positive constant A common for all almost periodic Λ'-solutions x_f of Equation (2.1) of the whole class of trigonometric polynomials a, b and an almost periodic function f from Corollary 4.13.

Now we abandon the assumptions that a, b are trigonometric polynomials.

Theorem 4.15. *If a and b are almost periodic functions with absolutely convergent Fourier series and f is an almost periodic function and if the assumptions $v_0 = \inf \Lambda_f > 0$,* (2.2), (2.3), (2.4) *are satisfied, then Equation* (2.1) *has exactly one almost periodic Λ-solution x_f, where $\Lambda = \Lambda_f + S(\Lambda_a \cup \Lambda_b \cup \{0\})$, and this solution satisfies the estimate* (4.16) *in which the positive constant A depends only on $a_0, b_0, v_0, \Delta, d_\theta, d, d_\xi, \tau, S, T$, where $S = \sum(a), T = \sum(b)$.*

P r o o f. There exist sequences a_m and b_m, $m = 1, 2, \ldots$, of Bochner-Fejér approximation polynomials of the almost periodic functions a and b, respectively, to which they converge uniformly on $\mathbb{R}$. If we denote $\Lambda_2 = \Lambda_a \cup \Lambda_b$, $\Lambda_1 = \Lambda_f + S(\Lambda_2 \cup \{0\})$ then $\Lambda' = \Lambda_1 + S(\Lambda_2 \cup \{0\}) = \Lambda$, $\Lambda_{a_m} \subset \Lambda_2$, $\Lambda_{b_m} \subset \Lambda_2$, $m = 1, 2, \ldots$; $\Lambda_f \subset \Lambda_1$. Moreover, $\sum(a_m) \leqslant S$, $\sum(b_m) \leqslant T$, $m = 1, 2, \ldots$. According to the choice of Λ_1, Λ_2 the assumptions of Corollary 4.13 are satisfied for the equation $\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a_m(t)x(t) + b_m(t)x(t-\tau) + f(t)$, $t \in \mathbb{R}$. Therefore, this equation has exactly one almost periodic Λ-solution x_m and for this solution we have the estimate $\|x_m\| \leqslant A|f|$, $m = 1, 2, \ldots$. Corollary 4.13 implies that the equation $\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a_m(t)x(t) + b_m(t)x(t-\tau) + f_{m,k}(t)$, $t \in \mathbb{R}$, where $f_{m,k}(t) = (a_{m+k}(t) - a_m(t))x_{m+k}(t) + (b_{m+k}(t) - b_m(t))x_{m+k}(t-\tau)$, $t \in \mathbb{R}$, has exactly one periodic Λ-solution. It is evident that this solution is $x_{m+k} - x_m$ and for this solution the estimate $\|x_{m+k} - x_m\| \leqslant A|f_{m,k}|$ holds true, $m = 1, 2, \ldots$. Further, we get the inequality $\|x_{m+k} - x_m\| \leqslant A|f_{m,k}| \leqslant A(|a_{m+k} - a_m| + |b_{m+k} - b_m|)|x_{m+k}| \leqslant A^2(|a_{m+k} - a_m| + |b_{m+k} - b_m|)|f|$; $m, k = 1, 2, \ldots$. But this means that $\lim \|x_{m+k} - x_m\| = 0$ for $m \to \infty$ uniformly with respect to $k = 1, 2, \ldots$, so that the almost periodic function $x_f = \lim x_m$ is an almost periodic Λ-solution of Equation (2.1) and satisfies the estimate (4.16).

Again, it is necessary to verify the uniqueness of this solution, which could be lost by the passage to the limit. Let y be also an almost periodic Λ-solution of Equation (2.1). Then the almost periodic function $w = x_f - y$ is a unique almost periodic Λ-solution of the equation $\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a_m(t)x(t) + b_m(t)x(t-\tau) + F_m(t)$, $t \in \mathbb{R}$, where $F_m(t) = (a(t) - a_m(t))w(t) + (b(t) - b_m(t))w(t-\tau)$, $t \in \mathbb{R}$, and this solution satisfies the estimate $\|w\| = \|x_f - y\| \leqslant A(|a - a_m| + |b - b_m|)|w|$, $m = 1, 2, \ldots$. The right-hand side converges to zero for $m \to \infty$, so that $y = x_f$. □

Corollary 4.16. *Let Λ_1, Λ_2 be two non-void sets of real numbers and let S, T be two positive constants. If the assumptions* (4.17), (4.18), (4.19), (4.20) *are satisfied and if f is an almost periodic function with its spectrum contained in* $\mathrm{i}\Lambda_1$ *and a, b are almost periodic functions with their spectra contained in* $\mathrm{i}\Lambda_2$ *satisfying* $\sum(a) \leqslant S$, $\sum(b) \leqslant T$, *then Equation* (2.1) *has exactly one almost periodic Λ'-solution x_f where* $\Lambda' = \Lambda_1 + S(\Lambda_2 \cup \{0\})$ *and this solution satisfies the estimate* (4.16) *where the positive constant A depends only on* $a_0, b_0, v_0', \Delta', d_\theta', d', d_\xi, \tau, S, T$.

Proof. Analogous reasoning as in the proof of Theorem 4.11. □

Remark 4.17. Corollary 4.14 ensures the validity of the estimate (4.16) with a positive constant A common for all almost periodic Λ'-solution x_f of Equation (2.1) of the whole class of almost periodic functions a, b, f from Corollary 4.13.

4.7. A slight generalization. The assumption $v_0 = \inf \Lambda_f > 0$ for Equation (2.1) can be weakend.

Theorem 4.18. *If a, b, f from Equation* (2.1) *are almost periodic functions while a, b have absolutely convergent Fourier series and if in addition to the conditions* (2.2), (2.3), (2.4) *the condition* $-\infty < v_0 = \inf \Lambda_f$ *is fulfilled then there exists exactly one almost periodic Λ-solution x_f of Equation* (2.1). *This solution satisfies the estimate* (4.16), *in which the positive constant A depends only on* $a_0, b_0, \Delta, d_\theta, d, d_\xi, \tau, S, T$, *where* $S = \sum(a)$, $T = \sum(b)$.

Proof. It is sufficient to consider only the case $-\infty < v_0 \leqslant 0$. We use the substitution $x(t) = y(t)\exp(-\mathrm{i}vt)$, $t \in \mathbb{R}$. In this substitution we will choose a suitable positive constant v such that the transformed Equation (2.1) satisfies the conditions from Theorem 4.11. The substitution gives the equation

$$\dot{y}(t) = \tilde{a}_0 y(t) + \tilde{b}_0 y(t-\tau) + \tilde{a}(t)y(t) + \tilde{b}(t)y(t-\tau) + \tilde{f}(t), \ t \in \mathbb{R}, \tag{4.21}$$

where $\tilde{a}_0 = a_0 + \mathrm{i}vE$, $\tilde{b}_0 = b_0 \exp(\mathrm{i}vt), \tilde{a}(t) = a(t)$, $\tilde{b}(t) = b(t)\exp(\mathrm{i}v\tau)$, $\tilde{f}(t) = f(t)\exp(\mathrm{i}vt)$. For Equation (2.1) we denote its characteristic equation $\det \Psi(z) = 0$ where the matrix function $\Psi(z) = zE - \tilde{a}_0 - \tilde{b}_0 \exp(-z\tau) = (z - \mathrm{i}v)E - a_0 - b_0 \exp(-(z-\mathrm{i}v)\tau) = \Phi(z - \mathrm{i}v)$. It means that for the characteristic quasipolynomial $\det \Psi(z)$ we have $\det \Psi(z) = \Delta(z - \mathrm{i}v)$ and its spectrum is $\sigma(\det \Psi(z)) = \sigma(\Delta(z - \mathrm{i}v)) = \sigma(\Delta(z)) + \mathrm{i}v$. We denote $\delta = \frac{1}{4}\min\{\alpha,\ \Delta,\ d_\theta, d, d_\xi, \tau, 4\}$ where the positive constants from the composed brackets have the same meaning as before. Now we choose $v = -v_0 + 4\delta$ and for Equation (4.21) we apparently get $\tilde{\theta} = \{\tilde{\xi} - \tilde{\xi}' \colon \tilde{\xi}, \tilde{\xi}' \in \sigma(\det(\Psi(z)))\} = \theta$, $\Lambda_{\tilde{f}} = \Lambda_f + v$, $\Lambda_{\tilde{a}} = \Lambda_a, \Lambda_{\tilde{b}} = \Lambda_b$ and $\tilde{\Lambda} = \Lambda_{\tilde{f}} + S(\Lambda_a \cup \Lambda_b \cup \{0\}) = \Lambda + v$ so that $\tilde{v}_0 = \inf \Lambda_{\tilde{f}} = v + \inf \Lambda_f = -v_0 + 4\delta + v_0 = 4\delta > 0$, $d_{\tilde{\theta}} =$

$\operatorname{dist}[\tilde{\theta};\ S(\Lambda_{\tilde{a}} \cup \Lambda_{\tilde{b}})] = \operatorname{dist}[\theta,\ S(\Lambda_a \cup \Lambda_b)] = d_\theta > 0$, $\tilde{d} = \operatorname{dist}[\mathrm{i}\tilde{\Lambda};\ \sigma(\det \Psi(z))] = \operatorname{dist}[\mathrm{i}\Lambda;\ \sigma(\Delta(z))] = d > 0$. These conditions in accord with Theorem 4.15 ensure the existence and uniqueness of an almost periodic $\tilde{\Lambda}$-solution y_f of Equation (4.21). This solution satisfies the estimate $\|y_f\| \leqslant K|\tilde{f}| = K|f|$ where the positive constant K representing the constant A from (4.16) depends only on $a_0, b_0, \Delta, d_\theta, d, d_\xi, \tau, S, T$, where $S = \sum(a)$, $T = \sum(b)$. This implies that the almost periodic function $x_f(t) = y_f(t)\exp(\mathrm{i}vt)$ is a unique almost periodic Λ-solution of Equation (2.1) and satisfies estimates $|x_f| = |y_f| \leqslant K|f|$, $|\dot{x}_f| = |\dot{y}_f - \mathrm{i}vy_f| \leqslant (1+v)|y_f| \leqslant (1+v)K|f| = (1 - v_0 + 4\sigma)K|f| \leqslant (5 - v_0)K|f|$ so that the inequality $\|x_f\| \leqslant A|f|$, where $A = (5 - v_0)K$, holds. □

R e m a r k 4.19. From the estimate $\|x_f\| \leqslant (5 - v_0)K|f|$ in the proof of Theorem 4.18 it follows that a one-sided boundedness of Λ_f may not be omitted.

5. Quasilinear equations

5.1. Functions of several variables. Let $g = g(t, x)$ be a continuous function $g\colon \mathbb{R} \times D \to \mathbb{C}^{p\times q}$, where $D \subset \mathbb{C}^{m\times n}$ is a non-void set. The function g is said to be

(a) almost periodic in the variable t on $\mathbb{R} \times D$ if $g(t, x)$ is almost periodic as a function of t for any fixed $x \in D$;

(b) uniformly almost periodic in the variable t on $\mathbb{R} \times D$ if $g(t, x)$ is almost periodic in t on $\mathbb{R} \times D$ and for any $\varepsilon > 0$ there exists a set $\{\tau\} \subset \mathbb{R}$ relatively dense in $\mathbb{R}$ such that $|g(t+\tau, x) - g(t, x)| < \varepsilon$ for every $\tau \in \{\tau\}$, $t \in \mathbb{R}$, $x \in D$;

(c) localy uniformly almost periodic in the variable t on $\mathbb{R} \times D$ if for any compact set $K \subset D$ the restriction g_k of the function g on $\mathbb{R} \times K$ is uniformly almost periodic in the variable t on $\mathbb{R} \times K$.

Lemma 5.1. *Let $g\colon \mathbb{R} \times D \to \mathbb{C}^{p\times q}$ be a function almost periodic in t on $\mathbb{R} \times D$. A necessary and sufficient condition for g to be locally uniformly almost periodic in t is that it be continuous in x uniformly with respect to $t \in \mathbb{R}$ on $\mathbb{R} \times D$.*

P r o o f. The proof can be found in [6]. □

In the sequel we deal with the cases in which the conditions for the locally uniform almost periodicity of the introduced function are fulfilled.

5.2. Harmonic analysis. Let $g\colon \mathbb{R} \times D \to \mathbb{C}^{p\times q}$ be a function almost periodic in t on $\mathbb{R} \times D$. For any $x \in D$ the Bohr transformation

$$a(\lambda, x) = a(\lambda, x, g) = \lim_{T\to\infty} \frac{1}{T} \int_s^{s+T} g(t, x) \exp(-\mathrm{i}\lambda t)\,\mathrm{d}t$$

exists for each $\lambda \in \mathbb{R}$ uniformly with respect to $s \in \mathbb{R}$. If $a(\lambda, x)$ is non-zero for a given $\lambda \in \mathbb{R}$ for at least one point $x \in D$, i.e. $a(\lambda, x) \not\equiv 0$, $x \in D$, then λ is called the Fourier exponent and $a(\lambda, x), x \in D$, is called the Fourier coefficient of the function g. The set of all Fourier exponents of the function g is denoted by Λ_g. If D is a compact set, then the set Λ_g is at most countable (see [6]). If the set D is a region (an open connected non-void set), then there exists a monotone sequence of compact sets $K_1 \subset K_2 \subset \ldots \subset K_m \ldots \subset D$ for which $\lim K_m = D$. In such a case the equality $\Lambda_g = \bigcup\limits_m \Lambda_m$ holds, where Λ_m is the set of all Fourier exponents of the restriction of the function g onto $\mathbb{R} \times K_m$, $m = 1, 2, \ldots$, and thus Λ_g is an at most countable set.

If g is locally uniformly almost periodic in the variable t on $\mathbb{R} \times D$ and D is a region, then the Fourier series $g(t, x) \sim \sum\limits_{\lambda} a(\lambda, x) \exp(\mathrm{i}\lambda t)$, $\lambda \in \Lambda_g$, can be uniquely determined except for its order of summation. If the function g is also analytic in the variable x on a closed ball lying in D and containing the set $\mathbb{R}_f$ of all values of an almost periodic function f, then the inclusion $\Lambda_F \subset \Lambda_g + S(\Lambda_f \cup \{0\})$ is valid for the function $F(t) = g(t, f(t))$, $t \in \mathbb{R}$.

5.3. Derivatives. Now we will deal with a function $g = g(t, u, v, \varepsilon)\colon \mathbb{R} \times D = \mathbb{R} \times \mathbb{C}^{n\times 1} \times \mathbb{C}^{n\times 1} \times \bar{\kappa} \to \mathbb{C}^{n\times 1}$, where $\bar{\kappa}_0 = \bar{\kappa}(0; \delta_0) \subset \mathbb{C}$, $\delta_0 > 0$. In order to avoid complicated expressions, we will use the symbolic records for Jacobi matrices, as for example

$$g_t = \frac{\partial g}{\partial t} = \frac{\partial(g_1, \ldots, g_n)}{\partial t} = \begin{pmatrix} \frac{\partial g_1}{\partial t} \\ \vdots \\ \frac{\partial g_n}{\partial t} \end{pmatrix},$$

$$g_u = \frac{\partial g}{\partial u} = \frac{\partial(g_1, \ldots, g_n)}{\partial u_1, \ldots, u_n} = \begin{pmatrix} \frac{\partial g_1}{\partial u_1}, \ldots, \frac{\partial g_1}{\partial u_n} \\ \ldots \; \ldots \; \ldots \\ \frac{\partial g_n}{\partial u_1}, \ldots, \frac{\partial g_n}{\partial u_n} \end{pmatrix} = \Big(\frac{\partial g_j}{\partial u_k}\Big)_{j,k=1,\ldots,n},$$

and analogously g_v. These Jacobi matrices will be called derivatives of the function g. The norm of a matrix is the sum of absolute values of all its elements.

For any given positive constant R we define the "norms" $|g|_R$, $|g_t|_R$, $|g_u|_R$, $|g_v|_R$ as the maximum value of the least upper bounds of magnitudes of functions g, g_t, g_u, g_v, respectively, on the (metric) space $\Omega_R = \mathbb{R} \times \mathbb{C} \times \mathbb{C}_R \times \bar{\kappa}_0$, where $\mathbb{C}_R^{n\times 1} = \{w \in \mathbb{C}^{n\times 1}\colon |w| \leqslant R\}$. Further, we denote $\|g\|_R = \max\{|g|_R, |g_t|_R, |g_u|_R, |g_v|_R\}$. If two points $U = [t, u, v, \varepsilon]$, $\widetilde{U} = [t, \tilde{u}, \tilde{v}, \varepsilon]$ are from Ω_R then we have the inequality $|g(U) - g(\tilde{U})| \leqslant \|g\|_R(|u - \tilde{u}| + |v - \tilde{v}|)$.

5.4. Quasilinear equations. Using the Banach contraction principle we will deal with the quasilinear (weakly nonlinear) system

$$\begin{aligned}(5.1)\qquad \dot{x}(t) &= a_0x(t) + b_0x(t-\tau) + a(t)x(t) + b(t)x(t-\tau)\\ &\quad + f(t) + \varepsilon g(t, x(t), x(t-\tau), \varepsilon),\ t \in \mathbb{R},\end{aligned}$$

where ε is a small complex parameter. For $\varepsilon = 0$ we get the generating equation (2.1) with its conditions for a_0, b_0, a, b, f. Assume that the function $g = g(t, u, v, \varepsilon)$ together with its derivatives g_u, g_v are locally uniformly almost periodic in the variable t on $\mathbb{R} \times D$, where $D = \mathbb{C}^{n\times 1} \times \mathbb{C}^{n\times 1} \times \bar{\kappa}_0$ and $\bar{\kappa}_0 = \bar{\kappa}(0, \delta_0)$, $\delta_0 > 0$, and g is analytic in the variables u, v, ε. (Lemma 5.1 implies that g is continuous in the variables u, v uniformly to $t \in \mathbb{R}$ and $\varepsilon \in \bar{\kappa}_0$ on $\mathbb{R} \times D$.)

Put $\Lambda = S(\Lambda_f \cup \Lambda_g + S(\Lambda_a \cup \Lambda_b \cup \{0\}))$. If $\Lambda_\xi \subset \Lambda$ for a function $\xi \in AP(\mathbb{C}^{n\times 1})$, then the composite function $F(t) = F(t, \xi(t)) = g(t, \xi(t),\ \xi(t-\tau), \varepsilon)$, $t \in \mathbb{R}$, is an almost periodic function whose spectrum is contained in $\mathrm{i}\Lambda$ for each $\varepsilon \in \bar{\kappa}_0$ (see the end of 5.2), as $\Lambda_F \subset \Lambda_g + S(\Lambda_f \cup \{0\}) \subset \Lambda_f \cup \Lambda_g + S(\Lambda \cup \{0\}) \subset \Lambda$ is valid due to the analyticity of the function g in the variables u, v. Thus the "spectrum" $\mathrm{i}\Lambda$ is wide enough to allow the existence of an almost periodic Λ-solution of Equation (5.1).

Theorem 5.2. *If Equation* (5.1) *satisfies the conditions* (4.17), (4.18), (4.19), (4.20) *for* $\Lambda_1 = \Lambda_f \cup \Lambda_g$, $\Lambda_2 = \Lambda_a \cup \Lambda_b$, $\Lambda' = \Lambda = S(\Lambda_f \cup \Lambda_g + S(\Lambda_a \cup \Lambda_b \cup \{0\}))$, *then for each positive number* $R > A|f|$, *where* A *is from* (4.16), *there exists such a positive number* $\varepsilon(R)$ *that Equation* (5.2) *has a unique almost periodic* Λ*-solution* x_ε *with the norm* $\|x_\varepsilon\| \leqslant R$ *for each* $\varepsilon \in \bar{\kappa}_0$ *for which* $|\varepsilon| < \varepsilon(R)$ *holds.*

Proof. Let us consider the Banach space $H(\Lambda) = \{\xi \in AP^1(\mathbb{C}^{n\times 1})\colon\ \Lambda_\xi \subset \Lambda\}$ with a norm $\|\cdot\|$. If a non-negative number R is given, then we define the metric closed subspace $H_R(\Lambda) = \{\xi \in H(\Lambda)\colon\ \|\xi\| \leqslant R\}$ of the space $H(\Lambda)$.

If $\xi \in H(\Lambda)$, $R \geqslant \|\xi\|$ and $\varepsilon \in \bar{\kappa}_0$, then the function

$$\gamma(t) = \gamma(t, \xi) = g(t, \xi(t), \xi(t-\tau), \varepsilon),\ t \in \mathbb{R},$$

is almost periodic and belongs again to $H(\Lambda)$ and $|\gamma| \leqslant |g|_R \leqslant \|g\|_R$, $|\dot{\gamma}| = |g_t + g_u\dot{\xi}(t) + g_v\dot{\xi}(t-\tau)| \leqslant (1+2R)\|g\|_R$. Thus $\|\gamma\| \leqslant (1+2R)\|g\|_R$.

Define an operator $\mathcal{A} = \mathcal{A}(\varepsilon)$ on the Banach space $H(\Lambda)$ for each $\varepsilon \in \bar{\kappa}_0$ such that the operator $\mathcal{A}$ maps any function $\xi \in H(\Lambda)$ to the function $\mathcal{A}\xi \in H(\Lambda)$ which is the unique almost periodic Λ-solution of the equation

$$\dot{x}(t) = a_0x(t) + b_0x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + f(t) + \varepsilon g(t, \xi(t), \xi(t-\tau), \varepsilon),\ t \in \mathbb{R},$$

(uniqueness is guaranteed by Theorem 4.8) and which satisfies the estimate (4.16), i.e. $\|\mathcal{A}\xi\| \leqslant A|f+\varepsilon\gamma|$. Due to Corollary 4.9 the constant A is common for all functions from $H(\Lambda)$ for $\Lambda_1 = \Lambda, \Lambda_2 = \Lambda_a \cup \Lambda_b$ as $\Lambda' = \Lambda$. Thus the final estimate reads $\|\mathcal{A}\xi\| \leqslant A[|f| + \varepsilon(1+2R)|g|_R]$. If a positive number R is chosen such that $R > A|f|$, then the operator $\mathcal{A} = \mathcal{A}(\varepsilon)$ maps the space $H_R(\Lambda)$ into itself for any $\varepsilon \in \bar{\kappa}_0$ for which $|\varepsilon| \leqslant (R - A|f|)/(1+2R)A|g|_R)$.

Further, it is necessary to find out for which $\varepsilon \in \bar{\kappa}_0$ the operator $\mathcal{A} = \mathcal{A}(\varepsilon)$ is contractive on $H_R(\Lambda)$. If two functions ξ, η belong to $H_R(\Lambda)$ and $\varepsilon \in \bar{\kappa}_0$ is given, then we put $\gamma_\xi(t) = g(t, \xi(t), \xi(t-\tau), \varepsilon)$, $\gamma_\eta(t) = g(t, \eta(t), \eta(t-\tau), \varepsilon)$, $t \in \mathbb{R}$. The function $w = \mathcal{A}\xi - \mathcal{A}\eta$ is the unique almost periodic Λ-solution of the equation

$$\dot{x}(t) = a_0 x(t) + b_0 x(t-\tau) + a(t)x(t) + b(t)x(t-\tau) + \varepsilon(\gamma_\xi(t) - \gamma_\eta(t)),\ t \in \mathbb{R},$$

and satisfies the inequality

$$\|w\| = \|\mathcal{A}\xi - \mathcal{A}\eta\| \leqslant |\varepsilon| A|\gamma_\xi - \gamma_\eta| \leqslant |\varepsilon| 2A\|g\|_R|\xi - \eta| \leqslant \varepsilon|2A\|g\|_R\|\xi - \eta\|,$$

since

$$|\gamma_\xi - \gamma_\eta| \leqslant |g_u|_R|\xi - \eta| + |g_v|_R|\xi - \eta| \leqslant 2\,\|g\|_R\,|\xi - \eta|.$$

In order to get a contractive operator $\mathcal{A}$ on $H_R(\Lambda)$ it is sufficient to put $|\varepsilon| < 1/(2A\|g\|_R)$.

The operator $\mathcal{A}$ maps the space $H_R(\Lambda)$ into itself and turns out to be a contraction on $H_R(\Lambda)$ for $|\varepsilon| < \varepsilon(R)$, where

$$\varepsilon(R) = \min\Big\{\delta_0, \frac{R - A|f|}{(1+2R)A|g|_R}, \frac{1}{2A\|g\|_R}\Big\}.$$

Consequently, there exists a unique function x_ε from $H_R(\Lambda)$ for $|\varepsilon| < \varepsilon(R)$, $R > A|f|$, such that $\mathcal{A}x_\varepsilon = x_\varepsilon$, i.e. there exists a unique almost periodic Λ-solution x_ε of Equation (5.1) for each $\varepsilon \in \bar{\kappa}_0$ if $|\varepsilon| < \varepsilon(R)$. This completes the proof of Theorem 5.2. $\square$

Conclusion. In comparison with the method of solution from [6] we have here weaker conditions for a, b, f from Equation (2.1) (we do not require the existence of first derivative of a, b, f) except the assumption $\inf \Lambda_f > 0$. Also the estimate $\|x_f\| \leqslant A|f|$ of the Λ-solution x_f of Equation (2.1) is simpler in comparison with the estimate $\|x_f\| \leqslant A\|f\|$ in [6]. The method developed in this paper for the construction of almost periodic solutions of almost periodic systems of differential equations can be used also for finding an approximative solution of this problem.

References

[1] *Amerio, L., Prouse, G.*: Almost Periodic Functions and Functional Equations. N. Y. Nostrand Reinhold Company, 1971.
[2] *Bochner, S.*: Lectures on Fourier Integrals. Princeton University Press, 1959.
[3] *Coppel, W. A.*: Almost periodic properties of ordinary differential equations. Ann. Mat. Pura Appl., IV. Ser. *76* (1967), 27–49.
[4] *Fink, A. M.*: Almost Periodic Differential Equations. Lecture Notes in Mathematics, Springer, New York, 1978.
[5] *Fischer, A.*: Existence of almost periodic solution of systems of linear and quasilinear differential equations with time lag. Čas. Pěst. Mat. *106* (1981), 256–268.
[6] *Fischer, A.*: Almost periodic solutions with a prescribed spectrum of systems of linear and quasilinear differential equations with almost periodic coefficients and constant time lag (Cauchy integral). Math. Bohem. *124* (1999), 351–379.
[7] *Levitan, B. M.*: Almost Periodic Functions. GIZTL, Moskva, 1953. (In Russian.)
[8] *Levitan, B. M., Žikov, V. V.*: Almost Periodic Functions and Differential Equations. IMU, Moskva, 1978. (In Russian.)

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