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INTEGRAL REPRESENTATIONS OF THE g-DRAZIN INVERSE IN C^*-ALGEBRAS

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Abstract. The paper gives new integral representations of the g-Drazin inverse of an element a of a C^*-algebra that require no restriction on the spectrum of a. The representations involve powers of a and of its adjoint.

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1. Introduction

Let $\mathcal{A}$ be a complex unital C^*-algebra with unit 1. Following [4], we define the *g-Drazin inverse* (or the *generalized Drazin inverse*) of $a \in \mathcal{A}$ as the element $b \in \mathcal{A}$ such that

$$ab = ba, \quad b^2 a = b, \quad a^2 b = a + u, \tag{1.1}$$

where u is quasinilpotent, that is, $\lambda 1 - u$ is invertible in $\mathcal{A}$ for all $\lambda \neq 0$. The g-Drazin inverse b of a is unique when it exists, and is denoted by a^{D}. The *g-Drazin index* $i(a)$ of a is defined to be 0 if a is invertible, k if the element u in (1.1) is nilpotent of index k, and ∞ if u is quasinilpotent but not nilpotent. The g-Drazin inverse of finite index will be called the *Drazin inverse* since Drazin's original definition [3] is consistent with u nilpotent.

We note that if $i(a)$ is finite, then for any $m \geqslant i(a)$,

$$a^{m+1} a^{\mathrm{D}} = a^m. \tag{1.2}$$

According to [4, Theorem 4.2], an element $a \in \mathcal{A}$ is g-Drazin invertible if and only if there exists an idempotent $p \in \mathcal{A}$ such that

$$ap = pa \text{ is quasinilpotent}, \quad a + p \text{ is invertible}. \tag{1.3}$$

The element p is the *spectral idempotent* of a denoted by a^{π}. We have

$$a^{\mathrm{D}} = (a+p)^{-1}(1-p) \text{ and } a^{\pi} = 1 - a^{\mathrm{D}}a. \tag{1.4}$$

An element $c \in \mathcal{A}$ is a *Moore-Penrose inverse* of $a \in \mathcal{A}$ if

$$aca = a, \quad cac = c, \quad (ac)^* = ac, \quad (ca)^* = ca. \tag{1.5}$$

The Moore-Penrose inverse is unique, whenever it exists; we will denote it by $a^{\dagger}$. According to [6, Theorem 2.5], the following is true.

Lemma 1.1. *An element $a \in \mathcal{A}$ is Moore-Penrose invertible if and only if a^*a (respectively aa^*) is Drazin invertible. The Moore-Penrose inverse of a is then expressed by*

$$a^{\dagger} = (a^*a)^{\mathrm{D}}a^* = a^*(aa^*)^{\mathrm{D}}. \tag{1.6}$$

In this case 0 is at most a simple pole of the resolvent of a^*a (respectively aa^*).

The reverse relation between the Drazin and Moore-Penrose inverse is well known for matrices. We give an analogous result for elements of C^*-algebras, which improves on [6, Proposition 3.7]:

Lemma 1.2. *If $a \in \mathcal{A}$ is Drazin invertible and $m \geqslant i(a)$, then a^{2m+1} is Moore-Penrose invertible, and*

$$a^{\mathrm{D}} = a^m (a^{2m+1})^{\dagger} a^m. \tag{1.7}$$

Proof. If $k \geqslant i(a)$, then $a^{k+j}(a^{\mathrm{D}})^j = a^k$ for any $j \geqslant 0$. In particular, $a^k(a^{\mathrm{D}})^k a^k = a^{2k}(a^{\mathrm{D}})^k = a^k$, which means that a^k is (von Neumann) regular. By [6, Theorem 2.8], a^k is Moore-Penrose invertible. Consequently, a^{2m+1} is Moore-Penrose invertible whenever $m \geqslant i(a)$. Then

$$\begin{aligned} a^m(a^{2m+1})^{\dagger}a^m &= (a^{\mathrm{D}})^{m+1}a^{2m+1}(a^{2m+1})^{\dagger}a^{2m+1}(a^{\mathrm{D}})^{m+1} \\ &= (a^{\mathrm{D}})^{m+1}a^{2m+1}(a^{\mathrm{D}})^{m+1} \\ &= (a^{\mathrm{D}})^{2m+2}a^{2m+1} = a^{\mathrm{D}}. \end{aligned}$$

□

For a g-Drazin invertible element $a \in \mathcal{A}$ with an arbitrary g-Drazin index we can employ the so-called 'core-quasinilpotent decomposition', that often enables us to reduce problems to the case $i(a) \leqslant 1$. Write $c = a(1 - a^{\pi})$ and $s = aa^{\pi}$. Then c is g-Drazin invertible with $i(c) \leqslant 1$, s is quasinilpotent (nilpotent in the case of a finite index), and

$$a = c + s, \quad cs = sc = 0, \quad a^{\mathrm{D}} = c^{\mathrm{D}}. \tag{1.8}$$

We call c the *core* of a and s the *quasinilpotent part* of a. In particular, we have the following result [6, Proposition 3.8].

Lemma 1.3. *Let $a \in \mathcal{A}$ be g-Drazin invertible and let c be the core of a. Then*

$$a^{\mathrm{D}} = c(c^3)^{\dagger}c.$$

The following integral representation was given in [2, Theorem 2.2] as a generalization of [4, Theorem 6.3], and is valid for a g-Drazin invertible a with finite or infinite g-Drazin index.

Theorem 1.4. *Let $\mathcal{A}$ be a Banach algebra. If $a \in \mathcal{A}$ is g-Drazin invertible and its nonzero spectrum contained in the open right half plane, then*

$$a^{\mathrm{D}} = \int_0^{\infty} \exp(-ta)(1 - a^{\pi})\,\mathrm{d}t. \tag{1.9}$$

This representation is a useful tool in theory of singular differential equations, where it can be applied to derive conditions for the asymptotic convergence of solutions (see [1], [7]).

The purpose of the present paper is to derive integral representations of the g-Drazin inverse in a C^*-algebra valid without any restriction on the spectrum of a, based on the integral representation (1.9), and the mutual relationship between the g-Drazin and Moore-Penrose inverse as expressed in Lemmas 1.1 and 1.2.

2. The case of finite g-Drazin index

The following analogue of Showalter's integral representation [8] of the Moore-Penrose inverse for C^*-algebras was given in [6, Example 3.5]:

Lemma 2.1. *Let $b \in \mathcal{A}$ be a Moore-Penrose invertible element of a C^*-algebra $\mathcal{A}$. Then*

$$b^\dagger = \int_0^\infty \exp(-tb^*b)b^*\,\mathrm{d}t = \int_0^\infty b^*\exp(-tbb^*)\,\mathrm{d}t. \tag{2.1}$$

Our first result is a combination of Lemma 1.2 and 2.1.

Theorem 2.2. *If $a \in \mathcal{A}$ is Drazin invertible, then a^{2m+1} is Moore-Penrose invertible for any $m \geqslant i(a)$, and*

$$a^{\mathrm{D}} = \int_0^\infty a^m \exp(-t(a^{2m+1})^* a^{2m+1})(a^{2m+1})^* a^m\,\mathrm{d}t \tag{2.2}$$

$$= \int_0^\infty a^m (a^{2m+1})^* \exp(-ta^{2m+1}(a^{2m+1})^*)a^m\,\mathrm{d}t. \tag{2.3}$$

P r o o f. According to Lemma 1.2, $b = a^{2m+1}$ is Moore-Penrose invertible. By Lemma 2.1, we obtain

$$\begin{aligned} a^{\mathrm{D}} = a^m b^\dagger a^m &= a^m\Big(\int_0^\infty \exp(-tb^*b)b^*\,\mathrm{d}t\Big)a^m \\ &= \int_0^\infty a^m \exp(-tb^*b)b^*a^m\,\mathrm{d}t, \end{aligned}$$

which is (2.2); equation (2.3) is proved similarly. $\square$

R e m a r k 2.3. It is of interest to observe that we can give an alternative proof of the preceding theorem independent of Showalter's representation, using our Theorem 1.4 and the mutual relationship between the Drazin and Moore-Penrose inverse as expressed in Lemma 1.1 and 1.2. Indeed, we show that the element $q = (a^{2m+1})^* a^{2m+1}$ is Drazin invertible with the nonzero spectrum in the interval $(0, \infty)$ and apply Theorem 1.4:

$$\begin{aligned} &\int_0^\infty a^m \exp(-t(a^{2m+1})^* a^{2m+1})(a^{2m+1})^* a^m\,\mathrm{d}t \\ &\quad = \int_0^\infty a^m \exp(-tq)q(a^{\mathrm{D}})^{m+1}\,\mathrm{d}t \\ &\quad = a^m\Big(\int_0^\infty \exp(-tq)(1-q^\pi)\,\mathrm{d}t\Big)q(a^{\mathrm{D}})^{m+1} \\ &\quad = a^m q^{\mathrm{D}} q(a^{\mathrm{D}})^{m+1}. \end{aligned}$$

We show that the last expression is equal to a^{D} using Lemma 1.1 and 1.2.

We give a direct proof of our second representation.

Theorem 2.4. *If $a \in \mathcal{A}$ is Drazin invertible, then a^{2m+1} is Moore-Penrose invertible for any $m \geqslant i(a)$, and*

$$a^{\mathrm{D}} = \int_0^\infty \exp(-ta^m (a^{2m+1})^* a^{m+1}) a^m (a^{2m+1})^* a^m \, \mathrm{d}t \tag{2.4}$$

$$= \int_0^\infty a^m (a^{2m+1})^* a^m \exp(-ta^{m+1} (a^{2m+1})^* a^m) \, \mathrm{d}t. \tag{2.5 }$$

Proof. In this proof we use the core-nilpotent decomposition (1.8) of a. Write $c = a(1 - a^{\pi})$ for the core of a. For any $k \geqslant i(a)$, $a^k = a^k(1 - a^{\pi}) = c^k$. Hence

$$a^m (a^{2m+1})^* a^{m+1} = c^m (c^{2m+1})^* c^{m+1}.$$

For brevity write $s = c^m (c^{2m+1})^* c^{m+1}$ and $q = (c^{2m+1})^* c^{2m+1}$. Then

$$\begin{aligned}&\int_0^t \exp(-\tau a^m (a^{2m+1})^* a^{m+1}) a^m (a^{2m+1})^* a^m \, \mathrm{d}\tau \\ &\qquad = \int_0^t \exp(-\tau s) s c^{\mathrm{D}} \, d\tau = c^{\mathrm{D}} - \exp(-ts) c^{\mathrm{D}}.\end{aligned}$$

Since $c^{\mathrm{D}} = a^{\mathrm{D}}$, it is enough to show that $\lim\limits_{t \to \infty} \exp(-ts) c^{\mathrm{D}} = 0$. We have

$$s^i = c^m q^i (c^{\mathrm{D}})^m, \quad i = 1, 2, \ldots$$

by induction, and

$$\exp(-ts) c^{\mathrm{D}} = c^m \exp(-tq) (c^{\mathrm{D}})^{m+1}.$$

Since c^{2m+1} is Moore-Penrose invertible by Lemma 1.2, q is simply polar with the nonzero spectrum in $(0, \infty)$. Hence by [5, Example 2.3],

$$\lim_{t \to \infty} \exp(-tq) = q^{\pi}.$$

Then by Lemma 1.1 and Lemma 1.2,

$$\begin{aligned} c^m q^{\mathrm{D}} q (c^{\mathrm{D}})^{m+1} &= c^m ((c^{2m+1})^* c^{2m+1})^{\mathrm{D}} (c^{2m+1})^* c^{2m+1} (c^{\mathrm{D}})^{m+1} \\ &= c^m (c^{2m+1})^{\dagger} c^m \\ &= c^{\mathrm{D}}, \end{aligned}$$

and

$$\begin{aligned} \lim_{t \to \infty} c^m \exp(-tq) (c^{\mathrm{D}})^{m+1} &= c^m q^{\pi} (c^{\mathrm{D}})^{m+1} = c^m (1 - q^{\mathrm{D}} q) (c^{\mathrm{D}})^{m+1} \\ &= c^{\mathrm{D}} - c^m q^{\mathrm{D}} q (c^{\mathrm{D}})^{m+1} = 0. \end{aligned}$$

This proves (2.4). Equation (2.5) is proved similarly. $\square$

3. The case of infinite g-Drazin index

We now turn our attention to a g-Drazin invertible element whose g-Drazin index may be infinite.

Theorem 3.1. *Let $a \in \mathcal{A}$ be g-Drazin invertible with the core $c = a(1 - a^{\pi})$. Then*

$$a^{\mathrm{D}} = \int_0^{\infty} c \exp(-t(c^3)^* c^3)\,(c^3)^* c \,\mathrm{d}t \tag{3.1}$$

$$= \int_0^{\infty} c(c^3)^* \exp(-tc^3(c^3)^*)\, c \,\mathrm{d}t \tag{3.2}$$

$$= \int_0^{\infty} \exp(-tc(c^3)^* c^2)\, c(c^3)^* c \,\mathrm{d}t \tag{3.3}$$

$$= \int_0^{\infty} c(c^3)^* c \exp(-tc^2(c^3)^*)\,\mathrm{d}t. \tag{3.4}$$

P r o o f. First we obtain the integral representations for c as special cases of Theorems 2.2 and 2.4, and then use equation $a^{\mathrm{D}} = c^{\mathrm{D}}$. □

Theorem 3.1 holds for a finite as well as for an infinite index; the compensation for this additional generality is that the integrand involves the core c rather than the original element a. However, we may substitute for c^k and $(c^k)^*$ using the equations

$$c^k = a^k(1 - a^{\pi}) = (1 - a^{\pi})a^k,$$
$$(c^k)^* = (a^k)^*(1 - (a^{\pi})^*) = (1 - (a^{\pi})^*)(a^k)^*.$$

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References

[1] *N. Castro González, J. J. Koliha*: Perturbation of the Drazin inverse for closed linear operators. Integral Equations Operator Theory *36* (2000), 92–106.
[2] *N. Castro González, J. J. Koliha, Yimin Wei*: On integral representation of the Drazin inverse in Banach algebras. Proc. Edinburgh Math. Soc. *45* (2002), 327–331.
[3] *M. P. Drazin*: Pseudo-inverse in associative rings and semigroups. Amer. Math. Monthly *65* (1958), 506–514.
[4] *J. J. Koliha*: A generalized Drazin inverse. Glasgow Math. J. *38* (1996), 367–381.

[5] *J. J. Koliha*: Isolated spectral points. Proc. Amer. Math. Soc. *124* (1996), 3417–4324.
[6] *J. J. Koliha*: The Drazin and Moore-Penrose inverse in C^*-algebras. Math. Proc. Roy. Irish Acad. *99A* (1999), 17–27.
[7] *J. J. Koliha, I. Straškraba*: Power bounded and exponentially bounded matrices. Appl. Math. *44* (1999), 289–308.
[8] *D. Showalter*: Representation and computation of the pseudoinverse. Proc. Amer. Math. Soc. *18* (1967), 584–586.

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