BASIC SUBGROUPS IN COMMUTATIVE MODULAR GROUP RINGS

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Abstract. Let $S(RG)$ be a normed Sylow p-subgroup in a group ring RG of an abelian group G with p-component G_p and a p-basic subgroup B over a commutative unitary ring R with prime characteristic p. The first central result is that $1 + I(RG; B_p) + I(R(p^i)G; G)$ is basic in $S(RG)$ and $B[1 + I(RG; B_p) + I(R(p^i)G; G)]$ is p-basic in $V(RG)$, and $[1 + I(RG; B_p) + I(R(p^i)G; G)]G_p/G_p$ is basic in $S(RG)/G_p$ and $[1 + I(RG; B_p) + I(R(p^i)G; G)]G/G$ is p-basic in $V(RG)/G$, provided in both cases G/G_p is p-divisible and R is such that its maximal perfect subring R^{p^i} has no nilpotents whenever i is natural. The second major result is that $B(1+I(RG; B_p))$ is p-basic in $V(RG)$ and $(1+I(RG; B_p))G/G$ is p-basic in $V(RG)/G$, provided G/G_p is p-divisible and R is perfect.

In particular, under these circumstances, $S(RG)$ and $S(RG)/G_p$ are both starred or algebraically compact groups. The last results offer a new perspective on the long-standing classical conjecture which says that $S(RG)/G_p$ is totally projective.

The present facts improve the results concerning this topic due to Nachev (Houston J. Math., 1996) and others obtained by us in (C. R. Acad. Bulg. Sci., 1995) and (Czechoslovak Math. J., 2002).

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1. Introduction

Throughout this work, RG denotes the group algebra of an abelian group G over a commutative ring R with identity in prime characteristic, for instance, p. As usual, $S(RG) = V_p(RG)$ will denote the p-primary component (often called the Sylow p-subgroup) of the group $V(RG)$ of all normalized units in RG. For L a subring of R and A a subgroup of G, we denote by $I(LG; A)$ the relative augmentation ideal of the ring LG with respect to A; $1 + I(LG; A) = V(LG; A)$ for simpleness. All other notation used and the terminology from the abelian group theory and abelian group ring theory are standard and follow essentially those in the monographs of [6] and [9].

The attempts at obtaining basic subgroups of $V(RG)$ were first started in [17] and independently and more generally in [1]. After this, we have developed in [2] and [5] this branch by making use of modern ideas. New results in this way are presented in the next paragraph.

2. Main results

p-basic subgroups of $V(RG)$ and $V(RG)/G$. The main aim of the current section is to give a description of the p-basic subgroups of $V(RG)$ and $V(RG)/G$ under minimal restrictions on R and G. This can be made by the following assertions which are useful for applications. But, before proving the global goals, we need some preliminaries to begin with.

Lemma 1. *Let R^{p^i} have no nilpotents for some positive integer i. Then, for each natural n,*

$$[R(p^i)]^{p^n} = R^{p^n}(p^i).$$

P r o o f. Observe that $n \geqslant i$ implies $R^{p^n}(p^i) = 0 = [R(p^i)]^{p^n}$. That is why we restrict our attention to $n < i$. It is clear that the right hand side contains the left hand side. For the converse, take an arbitrary element x from the right hand side. Hence $x = y^{p^n}$ and $y^{p^{n+i}} = 0$ for some $y \in R$. But $(y^{p^i})^{p^n} = 0$ and so $y^{p^i} = 0$, i.e. in other words $y \in R(p^i)$. We conclude that x lies in the left hand side, as well. The statement is proved. $\square$

The next technical claim is crucial for our further investigations.

Lemma 2. *Assume $1 \in L \leqslant R$ and $A \leqslant G$, $H \leqslant C \leqslant G$ such that H is p-primary. Then for $n \geqslant 0$ we have*

$$\begin{aligned}(1)\qquad & [1 + I(RG; H) + I(R(p^n)G; G)] \cap S(LA) \\ &= 1 + I(LA; A \cap H) + I(L(p^n)A; A),\end{aligned}$$

$$\begin{aligned}(2)\qquad & [G_p(1 + I(RG; H) + I(R(p^n)G; G))] \cap S(LA) \\ &= A_p[1 + I(LA; A \cap H) + I(L(p^n)A; A)],\end{aligned}$$

$$\begin{aligned}(3)\qquad & [C(1 + I(RG; H) + I(R(p^n)G; G))] \cap V(LA) \\ &= (A \cap C)[1 + I(LA; A \cap H) + I(L(p^n)A; A)].\end{aligned}$$

P r o o f. For the first identity, take x in the left hand side. Thus, $x = 1 + \sum\limits_i \sum\limits_j f_{ij} g_{ij}(1 - h_i) + \sum\limits_k \sum\limits_l r_{kl} c_{kl}(1 - b_k) = \sum\limits_m \alpha_m a_m$, where $f_{ij} \in R$; $r_{kl} \in R(p^n)$,

$\alpha_m \in L$; g_{ij}, c_{kl}, $b_k \in G$; $h_i \in H$ and $a_m \in A$. Therefore, we observe that the canonical forms of the two left sums contain elements of the type g_{ij} and $g_{ij}h_i$ plus c_{kl} and $c_{kl}b_k$, and, on the other hand, their general canonical form eventually possesses elements of the present kind. Now, we examine three cases. First, if some $g_{ij} = c_{kl}$ and $f_{ij} + r_{kl} = 0$, we derive $g_{ij}h_i \in A$ and $c_{kl}b_k \in A$ when $g_{ij}h_i \neq c_{kl}b_k$. Hence $f_{ij}g_{ij}h_i + r_{kl}c_{kl}b_k = f_{ij}g_{ij}h_i\ (1 - b_k h_i^{-1}) \in I(L(p^n)A; A)$. In the second case when $g_{ij}h_i = c_{kl}b_k$ with $f_{ij} + r_{kl} = 0$ but $g_{ij} \neq c_{kl}$, we deduce $g_{ij} \in A$, $c_{kl} \in A$ and so $f_{ij}g_{ij} + r_{kl}c_{kl} = f_{ij}g_{ij}\ (1 - c_{kl}g_{ij}^{-1}) \in I(L(p^n)A; A)$. In the last third case when $g_{ij} = c_{kl}b_k$ with $f_{ij} + r_{kl} = 0$, we conclude $g_{ij}h_i \in A$, $c_{kl} \in A$ and $f_{ij}g_{ij}h_i + r_{kl}c_{kl} = f_{ij}g_{ij}h_i\ (1 - c_{kl}g_{ij}^{-1}h_i^{-1}) \in I(L(p^n)A; A)$, as well. All other situations in which there are no relations between elements of the two sums are in agreement with [2].

The second ratio may be proved similarly to the third, which will be showed below. Really, choose again an arbitrary element y from the left-hand side. So, $y = c(1 + \sum_i \sum_j f_{ij}g_{ij}(1 - h_i) + \sum_k \sum_l r_{kl}c_{kl}(1 - b_k)) = \sum_m \alpha_m a_m$, where $c \in C$ and the other letters are as above. Further, we observe that the double sums both written in canonical records contain an element that belongs to H, say $h \in H$ (even if all $h_i = 1$ because $1 \notin R(p^n)$). That is why $ch \in A \cap C$ and we can write $y = ch(1 + (h^{-1} - 1) + \sum_i \sum_j f_{ij}g_{ij}h^{-1}(1 - h_i) + \sum_k \sum_l r_{kl}c_{kl}h^{-1}(1 - b_k))$, where the sums obviously lie in $[1 + I(RG; H) + I(R(p^n)G; G)] \cap S(LA)$. However, by what we have just shown, this intersection is equal to $1 + I(LA; A \cap H) + I(L(p^n)A; A)$, thus proving the assertion since the converse is trivial. Consequently, the equalities are proved in all generality. $\square$

We will now attack a significant

Proposition 3. *Suppose H is a pure direct sum of cyclic p-subgroups of G. Then $[1 + I(RG; H) + I(R(p^i)G; G)]/H$ is a direct sum of cyclics and thus H is a direct factor of $1 + I(RG; H) + I(R(p^i)G; G)$ with a direct sum of p-cyclics as the complementary factor.*

P r o o f. According to [6], $[1 + I(RG; H) + I(R(p^i)G; G)]/H$ is a direct sum of cyclic groups if and only if $[(1 + I(RG; H) + I(R(p^i)G; G))/H]^{p^i} = H[1 + I(R^{p^i}G^{p^i}; H^{p^i})]/H \cong [1 + I(R^{p^i}G^{p^i}; H^{p^i})]/H^{p^i}$ is such. But H being pure in G implies that H^{p^i} is pure in G^{p^i} and so the claim follows automatically from [2, Theorem 6]. The proof is over. $\square$

Now, we come to the first main affirmation motivating the present article.

Theorem 4. *Let G be an abelian group for which G/G_p is p-divisible and let R be a commutative ring with unity with characteristic p for which there exists a nonnegative number i such that R^{p^i} is perfect without nilpotent elements. Then $1 + I(RG; B_p) + I(R(p^i)G; G)$ is a basic subgroup of $S(RG)$, and $[1 + I(RG; B_p) + I(R(p^i)G; G)]G_p/G_p$ is a basic subgroup of $S(RG)/G_p$. Moreover, $1 + I(RG; B_p) + I(R(p^i)G; G)$ is a proper lower basic subgroup of $S(RG)$ if and only if $\inf_{n\in N} \max(|R^{p^n}|, |G^{p^n}|) = \max(|R^{p^i}|, |G/B_p|)$.*

P r o o f. Following [6], we shall inspect that the definition for a basic subgroup given in [6] is satisfied. In fact, the first half holds like this:

1) By virtue of Proposition 3, the desired group is a direct sum of cyclics.

2) Using successively Lemmas 2 and 1, for each natural n we calculate $[1 + I(RG; B_p) + I(R(p^i)G; G)] \cap S^{p^n}(RG) = 1 + I(R^{p^n}G^{p^n}; B_p^{p^n}) + I(R^{p^n}(p^i)G^{p^n}; G^{p^n}) = 1 + I^{p^n}(RG; B_p) + I^{p^n}(R(p^i)G; G) = [1 + I(RG; B_p) + I(R(p^i)G; G)]^{p^n}$, whence the purity is fulfilled.

3) The divisibility is proved in the following manner: Since $R^{p^i} = R^{p^{i+1}}$, for every $r \in R$ we have $r^{p^i} = \alpha^{p^{i+1}}$ for some $\alpha \in R$. Hence $r \in \alpha^p + R(p^i)$. Besides, $G = G_pG^p = B_pG^p$. Let us now choose an arbitrary element $x = r_1g_1 + \ldots + r_sg_s \in S(RG)$. Hence, we can write $r_1 = \alpha_1^p + \beta_1, \ldots, r_s = \alpha_s^p + \beta_s$ and $g_1 = b_1a_1^p, \ldots, g_s = b_sa_s^p$, where $\alpha_j \in R$, $\beta_j \in R(p^i)$; $b_j \in B_p$, $a_j \in G$ whenever $1 \leqslant j \leqslant s \in N$. Consequently, $x = \alpha_1^pb_1a_1^p + \beta_1b_1a_1^p + \ldots + \alpha_s^pb_sa_s^p + \beta_sb_sa_s^p = \alpha_1^pa_1^p + \alpha_1^p(b_1-1)a_1^p + \beta_1b_1a_1^p + \ldots + \alpha_s^pa_s^p + \alpha_s^p(b_s-1)a_s^p + \beta_sb_sa_s^p = (1 - \alpha_1^p - \ldots - \alpha_s^p + \alpha_1^pa_1^p + \ldots + \alpha_s^pa_s^p) + \alpha_1^p(b_1-1)a_1^p + \ldots + \alpha_s^p(b_s-1)a_s^p + \beta_1(b_1a_1^p - 1) + \ldots + \beta_s(b_sa_s^p - 1)$. Denote $u = 1 - \alpha_1^p - \ldots - \alpha_s^p + \alpha_1^pa_1^p + \ldots + \alpha_s^pa_s^p$. Apparently $\alpha_1^p + \beta_1 + \ldots + \alpha_s^p + \beta_s = 1$ yields $\alpha_1^{p^{i+1}} + \ldots + \alpha_s^{p^{i+1}} = 1$, and moreover there exists $t \in N$ such that $g_1^{p^t} = a_1^{p^{t+1}}, \ldots, g_s^{p^t} = a_s^{p^{t+1}}$. Therefore $u^{p^{i+t}} = \alpha_1^{p^{i+t+1}}g_1^{p^{i+t}} + \ldots + \alpha_s^{p^{i+t+1}}g_s^{p^{i+t}} = x^{p^{i+t}}$, thus u is also a normed p-torsion element, i.e. $u \in S^p(RG)$. Now, we may deduce $x = u(1 + \alpha_1^pa_1^pu^{-1}(b_1-1) + \ldots + \alpha_s^pa_s^pu^{-1}(b_s-1) + \beta_1u^{-1}(b_1a_1^p - 1) + \ldots + \beta_su^{-1}(b_sa_s^p - 1)) \in S^p(RG)[1 + I(RG; B_p) + I(R(p^i)G; G)]$.

Finally, we derive $S(RG) = S^p(RG)[1 + I(RG; B_p) + I(R(p^i)G; G)]$, as claimed.

For the second half, by making use of Proposition 3 we first observe that $[1 + I(RG; B_p) + I(R(p^i)G; G)]G_p/G_p \cong [1 + I(RG; B_p) + I(R(p^i)G; G)]/B_p$ is a direct sum of cyclics. Employing subsequently Lemma 2 and Lemma 1 we compute $[(1 + I(RG; B_p) + I(R(p^i)G; G))G_p] \cap S(R^{p^n}G^{p^n}) = G_p^{p^n}[1 + I^{p^n}(RG; B_p) + I^{p^n}(R(p^i)G; G)] = [G_p(1 + I(RG; B_p) + I(R(p^i)G; G))]^{p^n}$, hence this subgroup is pure in $S(RG)$ or equivalently $[1 + I(RG; B_p) + I(R(p^i)G; G)]G_p/G_p$ is pure in $S(RG)/G_p$ (see [6]). Finally, the divisibility is true because of the fact that $S(RG)/G_p/[1 + I(RG; B_p) + I(R(p^i)G; G)]G_p/G_p \cong S(RG)/[1 + I(RG; B_p) + I(R(p^i)G; G)]G_p$ is an epimorphic image of the divisible group $S(RG)/[1 + I(RG; B_p) + I(R(p^i)G; G)]$, as we have shown.

For the third part concerning the lower basic subgroup, we deduce via the definition stated in [6] along with [15] and [16] that $\inf_{n\in N} \operatorname{rank} (S^{p^n}(RG)) = \operatorname{rank} [S(RG)/[1 + I(RG; B_p) + I(R(p^i)G; G)]]$. But $\operatorname{rank} (S^{p^n}(RG)) = \operatorname{rank} (S^{p^n}(RG)[p]) = \operatorname{rank} (S(R^{p^n}G^{p^n})[p]) = |S(R^{p^n}G^{p^n})[p]| = \max(|R^{p^n}|, |G^{p^n}|)$ since $G_p^{p^n} \neq 1$. On the other hand we calculate the cardinal number $\operatorname{rank} [S(RG)/(1+I(RG; B_p)+I(R(p^i)G; G))]$. Foremost we observe that G/B_p is p-divisible because so are G_p/B_p and $G/G_p \cong G/B_p/G_p/B_p$ (see, for instance, [6]). In the sequel, the natural map $G \to G/B_p$ and the epimorphism $R \to R^{p^i}$ can be naturally extended to a homomorphism $S(RG) \to S(R^{p^i}(G/B_p))$ by the map $\sum_t r_t g_t \to \sum_t r_t^{p^i} g_t B_p$. It is only a routine technical exercise to establish that the kernel of this map is equal to $1 + I(RG; B_p) + I(R(p^i)G; G)$. Furthermore, $S(RG)/(1 + I(RG; B_p) + I(R(p^i)G; G))$ is isomorphic to a subgroup of the divisible group $S(R^{p^i}(G/B_p))$. Moreover, we have seen that the factor-group is divisible. Besides, using [15], [16], we find that rank $(S(R^{p^i}(G/B_p))) = \max(|R^{p^i}|, |G/B_p|)$. Now, since $R/R(p^i) \cong R^{p^i}$ and $S(RG) = 1 + I(RG; B_p) + I(R(p^i)G; G)$ only when $G_p = B_p$, it is elementary to conclude that the rank of the quotient group is the same, as required. The proof is complete. $\square$

Remark. Under the word "a proper basic subgroup" we have in mind that it does not coincide with the reduced part of the group, i.e. the group is not algebraically compact torsion.

Theorem 5. *Suppose that G is an abelian group whose G/G_p is p-divisible and R is a commutative unitary ring whose R^{p^i} is perfect with no nilradical for some nonnegative number i. Then $B[1+I(RG; B_p)+I(R(p^i)G; G)]$ is a p-basic subgroup of $V(RG)$, and $G[1+I(RG; B_p)+I(R(p^i)G; G)]/G$ is a p-basic p-subgroup of $V(RG)/G$.*

Proof. First, we see that $B[1+I(RG; B_p)+I(R(p^i)G; G)]/B \cong [1+I(RG; B_p)+ I(R(p^i)G; G)]/B_p$. Therefore in view of Proposition 3 the last factor-group is a direct sum of cyclic groups. On the other hand, B is pure in $B[1 + I(RG; B_p) + I(R(p^i)G; G)]$. Really, for all naturals n and primes $q \neq p$, owing to the modular law we derive $B \cap [B(1 + I(RG; B_p) + I(R(p^i)G; G))]^{q^n} = B \cap [B^{q^n}(1 + I(RG; B_p) + I(R(p^i)G; G))] = B^{q^n}[B \cap (1 + I(RG; B_p) + I(R(p^i)G; G))] = B^{q^n} B_p = B^{q^n}$. Moreover, by definition B is p-pure in G, and as is well-known G is p-pure in $V(RG)$, hence by [6] we obtain that B is p-pure in $V(RG)$ and thus also in every subgroup of $V(RG)$ that contains B. This substantiates our claim.

So, applying a classical Kulikov's theorem (see [6], p. 143, Theorem 28.2), B is a direct factor of $B[1 + I(RG; B_p) + I(R(p^i)G; G)]$ with a direct sum of p-cyclics as the complementary factor, hence the last mentioned group is a direct sum of p-cyclics and infinite cyclics, as well.

Further, combining Lemmas 2 and 1 together with folklore technical matters, the p-purity follows: $[B[1+I(RG;B_p)+I(R(p^i)G;G)]]\cap V^{p^n}(RG)=[B[1+I(RG;B_p)+I(R(p^i)G;G)]]\cap V(R^{p^n}G^{p^n}) = B^{p^n}[1+I(R^{p^n}G^{p^n};B_p^{p^n})+I(R^{p^n}(p^i)G^{p^n};G^{p^n})] = B^{p^n}[1+I^{p^n}(RG;B_p)+I^{p^n}(R(p^i)G;G)] = B^{p^n}[1+I(RG;B_p)+I(R(p^i)G;G)]^{p^n} = [B(1+I(RG;B_p)+I(R(p^i)G;G))]^{p^n}$, as required.

For the p-divisibility, we obtain: Because for some $t\in N$ and each normed element $v\in RG$ we have $v\in V(RG)$ if and only if $v^{p^t}\in V(RG)$, by the same token as in Theorem 4 we get that $V(RG)=V^p(RG)[1+I(RG;B_p)+I(R(p^i)G;G)] = V^p(RG)B[1+I(RG;B_p)+I(R(p^i)G;G)]$, as claimed.

Because of the isomorphism $[1+I(RG;B_p)+I(R(p^i)G;G)]G/G\cong[1+I(RG;B_p)+I(R(p^i)G;G)]/B_p\cong G_p[1+I(RG;B_p)+I(R(p^i)G;G)]/G_p$, the final part holds as in the above formulated Theorem 4. The proof is finished. □

Theorem 6. *Suppose G is an abelian group whose G/G_p is p-divisible and R is a perfect commutative ring with identity of prime characteristic p. Then $BV(RG;B_p)$ is a p-basic subgroup of $V(RG)$, and $GV(RG;B_p)/G$ is a p-basic p-subgroup of $V(RG)/G$. Moreover, B is a direct factor of $BV(RG;B_p)$.*

P r o o f. We shall show that the three conditions from the definition of a p-basic subgroup are satisfied (see [6]):

Really, $BV(RG;B_p)/B\cong V(RG;B_p)/B_p$ is a direct sum of cyclics according to [2]. But B is pure in $BV(RG;B_p)$. In fact, $B\cap[BV(RG;B_p)]^{q^n}=B\cap[B^{q^n}V(RG;B_p)] = B^{q^n}(B\cap V(RG;B_p))=B^{q^n}B_p=B^{q^n}$ for all primes $q\neq p$ and naturals n. Continuing in this direction, we compute $B\cap[BV(RG;B_p)]^{p^n}=B\cap[B^{p^n}V(R^{p^n}G^{p^n};B_p^{p^n})] = B^{p^n}(B\cap V(R^{p^n}G^{p^n};B_p^{p^n}))=B^{p^n}B_p^{p^n}=B^{p^n}$. Combining the two equalities, we conclude that the claim on the purity is valid [6]. Further, referring to ([6], p. 143, Theorem 28.2 of L. Kulikov) B is a direct factor of $BV(RG;B_p)$ with a direct sum of p-cyclics complement, hence $BV(RG;B_p)$ is also a direct sum of p-cyclics and infinite cyclics.

Second, $BV(RG;B_p)$ is p-pure in $V(RG)$ since in accordance with [2], [3], we calculate $[BV(RG;B_p)]\cap V^{p^n}(RG)=[BV(RG;B_p)]\cap V(R^{p^n}G^{p^n})=B^{p^n}V(R^{p^n}G^{p^n};B_p^{p^n})$ $=B^{p^n}V^{p^n}(RG;B_p)=[BV(RG;B_p)]^{p^n}$.

Third, because of the isomorphism $V(RG)/V(RG;B_p)\cong V(R(G/B_p))$ and the fact that G/B_p is p-divisible [1], we easily derive that $V(RG)/BV(RG;B_p)$ is p-divisible as an epimorphic image of the p-divisible $V(RG)/V(RG;B_p)$.

Combining these conclusions we obtain the first half.

Further, $GV(RG;B_p)/G\cong V(RG;B_p)/B_p$ and we may copy our method in [2] to complete the proof in all generality. □

The following questions are actual.

P r o b l e m. Since when R possesses zero divisors or G is not p-mixed, $V(RG)/G$ may not be a p-group, then of some interest and importance is the question what is the mixed p-basic subgroup of $V(RG)/G$? Moreover, whether the condition G/G_p to be p-divisible or R^{p^i} to be without nilradical when $i \in N$ can be omitted? We observe that it is needed (but no directly) only for the divisibility of the quotient group.

Finally, we note that the basic and p-basic subgroups established by us enlarge the main result in [17] and also are supplement to those in [1] and [2]. Generalizations of the last cited papers in another direction are given in [5], too.

In the next section, we shall examine some crucial applications of the above proved facts concerning the group structure and the invariant properties in RG.

3. Applicable results

Let us start with the computation of the cardinality of the basic subgroups and with the power estimation of the set of all distinguished basic subgroups in commutative modular group rings. This will be made by the following useful

Proposition 7. *Let G be abelian and such that G/G_p is p-divisible and either R is perfect or there is a natural number i such that R^{p^i} is perfect with trivial nilradical. Then the basic subgroup of $S(RG)$ and the p-basic subgroup of $V(RG)$ have power* $\max(|R|, |G|)$, *and the set of all different basic subgroups of $S(RG)$ and the set of all different p-basic subgroups of $V(RG)$ have cardinality* $\max(|R|, |G|)^{\max(|R|,|G|)}$. *In particular, no every basic subgroup is of the established above kind.*

P r o o f. Follows directly by Theorems 4, 5 and 6 plus the methodology used by us in [2]. □

We continue with a study of

Commutative group algebras of starred abelian groups. The definition for a starred abelian group is given in [10] or [8, p. 446]. This class of groups is large and contains as subclasses certain types of abelian groups. Here we shall use the following two simple equivalent conditions.

C r i t e r i o n (Khabbaz [10]). An infinite abelian p-group A is starred if and only if one of the following identities holds:

$(*)$ $|A| = |A/A^p|$

$(**)$ $|A| = |B|$, where B is a basic subgroup of A.

We shall provide the proof of the equivalence. In fact, because $A = BA^p$ we conclude $A/A^p \cong B/B^p$ (cf. [6]) and thus $|A| = |A/A^p| = |B/B^p| = |B|$, where the last equality is valid since B as a direct sum of cyclics is starred. This verifies the proof.

R e m a r k . The class of starred abelian groups is quite general and it contains for example all reduced direct sums of countable abelian groups. Moreover, it is evident that nontrivial divisible groups are not starred. A major property argued by Khabbaz [10] is that any abelian p-group is a direct factor of some starred abelian p-group.

A key consequence is the following

Corollary 8. *Let $A = M \times C$ be infinite.*

(i) *If $|A| = |M| > |C| \geqslant \aleph_0$, then A is starred if and only if M is starred.*
(ii) *If M and C are starred, then A is starred.*
(iii) *If A is starred, then M or C is starred.*

P r o o f . Indeed, in virtue of the above criterion and of the fact that $|A/A^p| = |M/M^p \times C/C^p| = \max(|M/M^p|, |C/C^p|)$ whenever $|A/A^p| \geqslant \aleph_0$, we can conclude:

If A is starred, then $|A| = |A/A^p| = |M/M^p| = |M|$; otherwise $|A| = |C/C^p| \leqslant |C|$, a contradiction. Thus M is starred as well.

Conversely, if M is starred, then $|A/A^p| = |M/M^p| = |M| = |A|$ since $|M/M^p| = |M| > |C| \geqslant |C/C^p|$. So A is starred.

Let now M and C be both infinite starred. Hence $|A/A^p| = \max(|M|, |C|) = |A|$, i.e. A is starred. If M is finite, $|A/A^p| = |C/C^p| = |C| = |A|$.

By hypothesis $\max(|M|, |C|) = |A| = |A/A^p| = \max(|M/M^p|, |C/C^p|)$. It is no harm in assuming that $|M| \geqslant |C|$. Therefore if $|M/M^p| \geqslant |C/C^p|$ we have $|M| = |M/M^p|$ and we are done. In the remaining case $|M| = |C/C^p| \geqslant |C|$ and thus $|C| = |C/C^p|$, i.e. in other words C is starred. □

Now we are ready to formulate

Theorem 9. *Suppose R is perfect and G/G_p is p-divisible. Then $S(RG)$ and $S(RG)/G_p$ are starred or divisible groups. In particular so is $V(FG)/G$ provided G is p-mixed and F is a perfect field of* $\operatorname{char} F = p > 0$.

P r o o f . Employing [2], we obtain that $V(RG; B_p)$ is basic in $S(RG)$ and $G_pV(RG; B_p)/G_p$ is basic in $S(RG)/G_p$. Moreover, it is clear that $|V(RG; B_p)| = \max(|R|, |G|) = |S(RG)|$ when R or G are infinite and $B_p \neq 1$ (the other case is elementary if we observe that $S(RG)$ and $S(RG)/G_p$ are finite whence starred, or are divisible since G_p must be divisible, hence the same p-divisibility holds for G- cf. [6]) whence $(**)$ is applicable. Besides, as we have just seen $G_pV(RG;B_p)/G_p \cong V(RG;B_p)/B_p$. That is why, constructing the elements $[1 + rg(1 - b_p)]B_p$ and $[1 + rg(1 - g_p)]G_p$ where $r \in R$, $g \in G$, $b_p \in B_p$ and $g_p \in G_p$ are special selected, (we omit the details) we deduce that $|V(RG;B_p)/B_p| = \max(|R|, |B_p|, |G/B_p|) = \max(|R|, |G|) = (|R|, |G_p|, |G/G_p|) = |S(RG)/G_p|$, whence we obtain the equality of the powers needed for the application of $(**)$.

Now, when G has a p-torsion part it is well-known that ([12], [13] plus [3]) $V(FG) = GS(FG)$ and thus $V(FG)/G \cong S(FG)/G_p$. □

As an immediate consequence we obtain

Corollary 10. *Let G be p-primary and R perfect. Then $V(RG)$ and $V(RG)/G$ are starred or divisible groups.*

Next we treat the case when R is not obviously perfect.

Proposition 11. *Suppose G is a starred abelian p-group. Then $V(RG)$ is starred.*

Proof. We shall consider only the infinite case since the other is routine; G bounded implies $V(RG)$ is bounded whence starred. As $|V(RG)| = \max(|R|, |G|) \geqslant \aleph_0$, $|V(R(G/G^p))| = \max(|R|, |G/G^p|)$ and $V(RG)/V(RG;G^p) \cong V(R(G/G^p))$, we have $|V(RG)/V^p(RG)| \geqslant |V(RG)/V(RG^p)| \geqslant |V(RG)/V(RG;G^p)| = |V(R(G/G^p))| = |V(RG)|$. Thus, $|V(RG)| = |V(RG)/V^p(RG)|$ and the Khabbaz criterion leads to $V(RG)$ starred, as claimed. □

The next example shows that the converse implication is not always true.

Example. Assume $S(RG)/G_p$ is an infinite direct sum of (reduced) countables and G_p is not starred; for instance G is a p-group with $\mathrm{length}(G) = \omega_1, |G| = \aleph_1$ and $|B| = \aleph_0$ (see [13]). Then $S(RG)$ can be possibly starred. In fact, if $|R| > |G|$ then because $|S(RG)| = |R|$ when $G_p \neq 1$ (otherwise we are done) and $S(RG) \cong G_p \times S(RG)/G_p$, we derive $|S(RG)| = |S(RG)/G_p|$ and so we need only to apply the group-theoretic corollary to get the claim.

In that aspect, the readers can see directly Proposition 7.

Theorem 12. *Suppose R is weakly perfect whose maximal perfect subring is without nilpotents and G/G_p is p-divisible. Then $S(RG)$ and $S(RG)/G_p$ are starred groups or algebraically compact groups.*

Proof. Invoking to Theorem 4 and to arguments similar to the Theorem 9, we obtain that the basic subgroups of $S(RG)$ and $S(RG)/G_p$ have powers equal to

$\max(|R|, |G|)$ when $B_p \neq 1$ or in other words, G_p is not divisible. Therefore it is obvious that $S(RG)$ and $S(RG)/G_p$ are indeed starred groups. Further, if G_p is divisible, i.e. G is p-divisible (see for example [6]), we easily deduce that both $S(RG)$ and $S(RG)/G_p$ are weakly divisible, whence algebraically compact (cf. [6]). This completes the proof. □

Recall that as usual F denotes a field with $\operatorname{char} F = p > 0$.

Proposition 13. *Suppose G is an abelian group and $FH \cong FG$ as F-algebras for some group H. If G_p is starred, then H_p is starred.*

Proof. The case for finite G_p is routine. We will give two different type of arguments to confirm our claim, namely:

1. Applying the Main Proposition in [3] we derive $F(G/G_p^p) \cong F(H/H_p^p)$. On the other hand using an assertion of Karpilovsky [9] we obtain the equalities $|G_p| = |H_p|$ and $|G_p/G_p^p| = |H_p/H_p^p|$. Therefore the above criterion $(*)$ is applicable to get the first part.
2. As above we deduce $|G_p| = |H_p|$. Besides, it follows from an excellent result of May [11] along with [6] that $|B_p| = |B'_p|$, where B'_p is a basic subgroup of H_p. Consequently, applying $(**)$ we obtain the second part. □

Corollary 14. *Let G be an abelian group whose G_p is a direct sum of countables. Then $FH \cong FG$ as F-algebras for any group H implies that H_p is a direct sum of a divisible and a starred group.*

Proof. Write $G_p = (G_p)_d \times G_p/(G_p)_d$, where $(G_p)_d$ is the maximal divisible subgroup of G_p. Apparently $G_p/(G_p)_d$ is starred. But the Main Proposition in [3] guarantees that $F(H/(H_p)_d) \cong F(G/(G_p)_d)$. Furthermore, by what we have just proved, $(H/(H_p)_d)_p = H_p/(H_p)_d$ is starred, thus completing the proof. □

Remark. The last claim gives a hint that H_p may be a direct sum of countables, thus confirming in the affirmative a May's question (see [12], [9], [3]).

We conclude the investigation with one new and interesting group-theoretical fact which depends on the continuum hypothesis.

Group Theorem. Suppose A is an abelian p-group of cardinality $\kappa < \kappa^{\aleph_0}$. Then A is a starred torsion complete group if and only if it is bounded.

Proof. The sufficiency is apparent. For the necessity, if A is bounded torsion complete, we are done. Otherwise, it follows from [6, p. 29, Exercise 7] that $|A| = |B|^{\aleph_0}$, where B is the basic subgroup of A. On the other hand, by the above stated Khabbaz criterion, $|A| = |B|$. Therefore we elementarily derive $|A| = |A|^{\aleph_0}$, a contradiction. This completes the proof. □

R e m a r k. The assumption concerning the power can not be omitted (see [6]).

4. Concluding discussion

In [7], P. Hill and W. Ullery have stated that $V(FG)/G$ is almost totally projective, provided F is a perfect field of $\operatorname{char} F = p > 0$ and G is an abelian p-group. Here we have established that under these restrictions $V(FG)/G$ is starred. This is a good supplement to the Hill-Ullery's result. Probably many other facts of this type are needed to prove in general that $V(FG)/G$ is totally projective, an old and very difficult Direct Factor Conjecture (see, for example, [7] and [9]). Referring to the above mentioned two descriptions for $V(FG)/G$ and to the easy fact that each factor-group of a primary torsion complete group modulo a balanced subgroup is torsion complete too, then because of the balancedness of the p-group G in $V(FG)$, combining it in one of the cases with the last proved Group Theorem, we may successfully attack the problem of torsion completeness of $V(FG)$ to confirm once again the main results in [14] and [4].

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