NEWS AND NOTICES

SIXTY YEARS OF IVAN NETUKA

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Professor Ivan Netuka was born on July 7, 1944 at Hradec Králové. In 1962 he commenced studying mathematics at the Faculty of Mathematics and Physics of Charles University (the Faculty). Among his teachers were Vojtěch Jarník and Jan Mařík. He graduated in 1967 and soon afterwards joined the faculty of his Alma Mater.

Under the supervision of J. Král he completed his thesis [D1] in 1970 and was granted (the equivalent of) a Ph.D. in 1972. At that time he was already deeply interested in potential theory, and this led him to Paris where he spent a year with Marcel Brelot and Gustave Choquet. He defended his thesis [D3] to obtain a DrSc., the highest scientific degree available in Czechoslovakia, and became a professor in 1986. He was active in the Seminar on mathematical analysis (founded by J. Král) which concentrated mainly on potential theory, and he later became one of its leaders.

Netuka was a visiting professor for a semester in Erlangen with Heinz Bauer, and was invited for longer periods to the Netherlands, Sweden, U.S.A. and Great Britain. Five students received doctorates under his guidance. He has lectured abroad on more than fifty occasions in many different countries, frequently as an invited speaker at conferences. He was also a member of organizing committees of international conferences at Praha, Utrecht, Château de Bonas, Kouty, Uppsala, and has served in Editorial boards of several journals. Since 1986 he has been the Editor-in-Chief of Commentationes Mathematicae Universitatis Carolinae. His work in mathematics has received special recognition on several occasions: in particular, he has been made a corresponding member of the Bavarian Academy of Sciences (since 2000) and a honorary member of the Society of Czech Mathematicians and Physicists (since 2002).

Netuka has always been involved with the life of the Faculty. During his academic career, he was a member of the Academic Senate (1989–1991, 1993), the Director of the Mathematical Institute of Charles University (1986–1990), and for three periods he served as a Vice-Dean (for student affairs (1979–1982), for scientific matters, international relations (1993–1996, 1996–1999)). Since 1999, he has been Dean of the Faculty. He has also been involved in various activities within the Czech, as well as the wider European, mathematical community. However, the aim of this article is not to catalogue his numerous activities, but to present an account of his mathematical achievements.

We begin with the Dirichlet problem, a cornerstone of potential theory. Let U be a relatively compact open set in $\mathbb{R}^m$, or, more generally, in a harmonic space. We define $H(U) = \{h \in C(\overline{U});\ h|_U \text{ is harmonic}\}$ and recall that U is said to be *regular* if $H(U)|_{\partial U} = C(\partial U)$, that is, for every continuous boundary condition f there is a uniquely determined $h \in H(U)$, such that $h|_{\partial U} = f$. We call this function h the *solution of the classical Dirichlet problem for f*. For a non-regular U, we try instead to solve the *generalized Dirichlet problem*. This means we seek a reasonable operator T sending $C(\partial U)$ into the space $\mathcal{H}(U)$ of harmonic functions on U such that Tf gives the solution of the classical Dirichlet problem for f when it exists, that is, $T(h|_{\partial U}) = h|_U$ for every $h \in H(U)$. Here *reasonable* means either positive linear or increasing. In the former case T is called a *Keldysh operator*, while in the latter case T is a *K-operator*.

Among methods for producing a Keldysh operator the best known is the *Perron-Wiener-Brelot method* (PWB-solution) based on upper and lower functions. The corresponding operator will be denoted by H_U. Hence there is no problem with the existence of a Keldysh operator. A remarkable result reads as follows: *On every $U \subset \mathbb{R}^m$ there is a unique Keldysh operator.* Keldysh's original proof is difficult. A. F. Monna emphasized the need for an accessible proof. A new and elementary proof is given in [A22].

However, as it was pointed out by J. Lukeš, Keldysh's theorem does not have an analogue for the potential theory associated with the heat equation. Consequently, it is not clear in this case whether the Wiener-type solution introduced by E. M. Landis necessarily coincides with the PWB-solution. An affirmative answer in a much more general context is given in [A17], where interior stability of the PWB-solution is also proved.

Papers [B2], [B4], [B5], [B7] and [B17] are devoted to various aspects of the Keldysh theorem. In [B4], which is a survey article, an interesting new result on the Dirichlet problem on the Choquet boundary is included; the case of discontinuous boundary conditions is also considered. In [A25], a Keldysh-type theorem for the Dirichlet problem on a compact set is proved. Ninomiya operators satisfying weaker requirements than Keldysh operators are studied in [A30].

In [A29] (which is partially based on [D3]) an abstract setting appropriate for the better understanding and study of the Keldysh type operators is presented. To this end, a question of uniqueness of extensions of operators on Riesz spaces is analyzed. The context is then specialized to function spaces and at this point Choquet theory enters quite naturally into the considerations (cf. [B19]). A problem proposed by A. F. Monna is solved in [B4] and [B7] where a uniqueness domain for extensions of Keldysh operators is characterized. Also an interesting connection with Korovkin-type theorems is pointed out.

Recall that a point $z \in \partial U$ is called *regular* provided that $H_U f(x) \to f(z)$ as $x \to z$ for every $f \in C(\partial U)$. The set of all regular points of U is denoted by $\partial_{\mathrm{r}} U$ while $\partial_{\mathrm{irr}} U := \partial U \setminus \partial_{\mathrm{r}} U$. Recall also that the set U is said to be *semiregular* if $H_U f$ is continuously extendible to $\overline{U}$ whenever $f \in C(\partial U)$. J. Král posed the problem of whether, in Brelot harmonic spaces, U is semiregular if and only if $\partial_{\mathrm{r}} U$ is closed. A counterexample may be found in [A11]. In [A10] it is shown that the answer is affirmative under the additional assumption of the axiom of polarity. In 1950 M. Brelot and G. Choquet raised the following question: *for which sets U is it true that*

$$(1) \qquad H_U f = \inf\{h|_U\,;\ h \in H(U),\ h|_{\partial U} \geqslant f\} \quad \textit{for every} \quad f \in C(\partial U)\,?$$

This problem was solved in [A24] by showing that this is true if and only if $\overline{\partial_{\mathrm{r}} U} = \partial U$. The paper also deals with related questions in the context of harmonic spaces. If the pointwise infimum in (1) is replaced by the *specific infimum*, it is proved that (1) holds if and only if the set $\partial_{\mathrm{irr}} U$ is negligible.

Mařík's problem, dating from 1957, concerning solutions of the Dirichlet problem on unbounded open sets, is solved in [A6].

The coarsest topology that makes all hyperharmonic functions continuous is called the *fine topology*. Boundary behaviour of $H_U f$ with respect to the fine topology for resolutive functions f near an irregular point of U is investigated in detail in [A34]. This article extends and completes results previously obtained by H. Bauer. It also includes a new proof of Bauer's result on the coincidence of the Fulks measure known from parabolic potential theory with the balayage measure. Papers [A39], [B10] and [A37] deal with the boundary behaviour of $H_U f$. The survey paper [A37] also contains a new result on the convergence of balayage measures in variation, which solves a problem proposed by T. Gamelin.

Is there a way of recognizing whether a function $f \in C(\partial U)$ admits a solution of the classical Dirichlet problem? Here is an immediate obvious answer: this holds if and only if $H_U f(x) \to f(z)$ as $x \to z$ for every $z \in \partial_{\,\mathrm{irr}} U$. But must one really verify this condition for *all* irregular points? A set $A \subset \partial_{\,\mathrm{irr}} U$ is said to be *regularizing* if the following implication holds: if $f \in C(\partial U)$ and $H_U f(x) \to f(z)$ whenever $z \in A$, then the same is true for every $z \in \partial_{\,\mathrm{irr}} U$. A classical result says that there always exist *countable* regularizing sets. But what do regularizing sets look like? In [A36] a new topology on $\partial_{\,\mathrm{irr}} U$ is introduced, and it is proved that $A \subset \partial_{\,\mathrm{irr}} U$ is regularizing if and only if A is dense in this topology. Special regularizing sets, called *piquetage faible*, were defined in 1969 by G. Choquet. Among other results, the Choquet question of whether every regularizing set is a piquetage faible, is answered in [A36] in the negative.

Other publications related to this subject are [A12], [A13], [A32], [A38], [A42], [A47], [A49], [B3], [B6], [B17], [B19] and [B21].

Let us turn to a b s t r a c t p o t e n t i a l t h e o r y. Recall that the classical theorem of Evans-Vasilesco, also known as the *continuity principle*, states that a Newtonian potential $N\mu$ of a positive measure μ with compact support K is continuous provided that its restriction to K is continuous. In 1973 B.-W. Schulze advanced the following problem: *Does the theorem extend to the case of potentials of signed measures?*

An affirmative answer is given in [E3] and [A15], where a form of the *maximum principle* of Maria-Frostman for signed measures is also proved. In fact, the results are proved within the context of Brelot harmonic spaces; the proof uses balayage and the fine topology. An application to the potential theory of the Helmholz equation is given as well. An important point in [A15] is the construction of a compactly supported signed measure μ with continuous potential in such a way that $N\mu$ cannot be expressed as a difference of two continuous potentials of positive measures. Thus a cancellation of discontinuities of $N\mu^+$ and $N\mu^-$ may occur.

It is known that the *Harnack pseudometric* is a metric if and only if the set of positive harmonic functions separates the points. The paper [A40] presents necessary and sufficient conditions for it. The separation property for other classes of harmonic functions is also characterized in terms of Denjoy domains, Martin compactification and special harmonic morphisms.

Papers [A5], [A20], [A33] and [A39] deal with various problems of abstract potential theory. In [A5], a full characterization of the set of elliptic points for harmonic sheaves on 1-manifolds is given. Properties of balayage defined by neglecting certain small sets are investigated in the framework of standard H-cones in [A33]. Limits of balayage measures in a balayage space are dealt with in [A39].

For the next result denote by F the closure of the Choquet boundary of the closure of a relatively compact open set U with respect to $H(U)$. The following result is proved in [A20]: Every point of $\partial U \setminus F$ is a point of harmonic continuability of any function of $H(U)$, whereas the set of all functions of $H(U)$, for which no point of F is a point of harmonic continuability, is a dense G_δ in $H(U)$. For a more elementary approach applicable in classical potential theory (associated with the Laplace equation in $\mathbb{R}^m$), see [A21]. In [A20], removable singularities in a harmonic space are also studied.

Publications also related to this section are [A11], [A17], [A24], [A25], [A29]–[A31], [A34]–[A39], [A42], [B4], [B5], [B7], [B9], [B10], [B20] and [B21].

It is a well known fact that a continuous function h on an open set $U \subset \mathbb{R}^m$ is harmonic if and only if

$$h(x) = \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} h \,\mathrm{d}\lambda \tag{2}$$

for every closed ball $B(x,r) \subset U$; here λ stands for Lebesgue measure in $\mathbb{R}^m$ and the fact described is called the *mean value property.* If $U = \mathbb{R}^m$, h is continuous and (2) holds for *one* ball centered at each $x \in \mathbb{R}^m$, then h need not be harmonic. This answers a question of J. Mařík from 1956; see [A1].

It is obvious that, for an open ball $A \subset \mathbb{R}^m$ of centre 0 and a harmonic function h integrable on A, the equality

$$h(0) = \frac{1}{\lambda(A)} \int_A h \,\mathrm{d}\lambda \tag{3}$$

holds. The following *inverse mean value property* was proved in 1972 by Ü. Kuran: Let $A \subset \mathbb{R}^m$ be an open set, $0 \in A$ and $\lambda(A) < \infty$. If (3) holds for every integrable harmonic function h on A, then A must be a ball of centre 0. Under various additional assumptions the analogous statement had been proved previously by, for example, W. Brödel, A. Friedman and W. Littman, B. Epstein and M. M. Schiffer, M. Goldstein and W. W. Ow. A series of papers appeared following Kuran's result in which (3) was required to hold for a certain class of harmonic functions only; these results belong to M. Goldstein, W. Hausmann, L. Rogge and D. H. Armitage. The following theorem from [A41] (stated here only for the case $m > 2$) represents a very general form of the inverse mean value property: *Let $A \subset \mathbb{R}^m$ be a Lebesgue measurable set, $0 < \lambda(A) < \infty$ and let B denote the ball of centre 0 such that $\lambda(A) = \lambda(B)$. Then (3) holds for the Newtonian potential h of $\lambda|_C$ for every compact set $C \subset \mathbb{R}^m \setminus A$, if and only if $\lambda(B \setminus A) = 0$.* Other classes of test functions are also investigated, which leads to a description of smallness of the difference between A and B in terms of removable singularities.

Now let U be a bounded domain in $\mathbb{R}^m$ and let $0 \in U$. There are many (positive) measures μ on U such that $\mu(U) > 0$ and

$$h(0) = \frac{1}{\mu(U)} \int_U h \,\mathrm{d}\mu \tag{4}$$

holds for every *bounded* $h \in \mathcal{H}(U)$. Such measures were investigated for various purposes by, for example, G. Choquet and J. Deny, L. Flatto, A. Friedman and W. Littman, A. M. Garcia, M. R. Hirschfeld, E. Smyrnélis and L. Zalcman. If desired, the measure μ can be chosen to be absolutely continuous with respect to λ, say $\mu = w\lambda$.

During the International Conference on Potential Theory (Nagoya, 1990), A. Cornea raised the problem whether there always exists a function w such that (4) holds for $\mu = w\lambda$ where w is bounded away from 0 on U. In [A43] it is proved that the answer is negative in general; there always exists a strictly positive $w \in C^\infty(U)$ with the desired property; if U has a smooth enough boundary (for example, of class $C^{1+\alpha}$), then there is a function $w \in C^\infty(U)$ which is bounded away from 0.

Another problem of that kind was proposed in 1994 by G. Choquet. If m_r, $r > 0$, stands for a normalized Lebesgue measure on $B(0,r) \subset \mathbb{R}^m$, it reads as follows: *Let f be a continuous function on $\mathbb{R}^m$ and let $r_1, r_2, \dots$ be strictly positive numbers. Under what conditions on f and $\{r_n\}$ does $\{f * m_{r_1} * m_{r_2} * \dots * m_{r_n}\}$ converge to a harmonic function?* An answer is given in [A44] and the key role is played by the following two facts:

(a) If $\sum r_j^2 = \infty$, then $\{m_{r_1} * m_{r_2} * \dots * m_{r_n}\}$ converges *vaguely* to 0;
(b) If $\sum r_j^2 < \infty$, then the sequence $\{m_{r_1} * m_{r_2} * \dots * m_{r_n}\}$ converges *weakly* to a probability measure on $\mathbb{R}^m$.

In fact, more general measures are investigated.

Publications also related to this section are [A48], [B1] and [B12].

Another group of Netuka's papers is related to h a r m o n i c a p p r o x i m a t i o n. As an answer to a question proposed by J. Lukeš, the following assertion is proved in [A47]: Let $m \geqslant 2$ and let U be the open unit ball in $\mathbb{R}^m$. Then there exists a family $\mathcal{F} \subset H(U)$ such that $u = \inf \mathcal{F}$ is continuous on $\overline{U}$ and there exists a continuous convex function v on $\overline{U}$ such that $u \leqslant v$ and the inequalities $u \leqslant h \leqslant v$ hold for no function $h \in H(U)$. In other words, in contrast to convex analysis, a Hahn-Banach type theorem does not hold for separation by means of elements of $H(U)$. A less sharp result had already been proved in [A28] for the plane case where u, $-v$ are continuous on $\overline{U}$ and superharmonic on U. It gave an answer to a problem proposed by G. A. Edgar who also asked for a comparison of representing measures for harmonic and superharmonic functions.

Now let U be a relatively compact open subset of a harmonic space. The following three subspaces of $\mathcal{H}(U)$ of harmonic functions on U are of interest:

$$\begin{aligned}
\mathcal{H}_1 &= \{h|_U\,;\ h \in H(U)\} \\
&\qquad \text{(solutions of the classical Dirichlet problem)}, \\
\mathcal{H}_2 &= \{H_U f\,;\ f \in C(\partial U)\} \\
&\qquad \text{(solutions of the generalized Dirichlet problem)}, \\
\mathcal{H}_3 &= \{h \in \mathcal{H}(U)\,;\ h \text{ bounded}\}.
\end{aligned}$$

When is $\mathcal{H}_1$ dense in $\mathcal{H}_2$ in the topology of locally uniform convergence? The assumption that the set of irregular points of U is negligible turns out to be sufficient, as proved in [A38]. In [A42] it was shown that this condition is also necessary. On the other hand, [A42] includes an example showing that even in classical potential theory $\mathcal{H}_1$ may not be dense in $\mathcal{H}_3$.

In [A49], for classical harmonic functions, uniform approximation of functions from $\mathcal{H}_3$ by functions in $\mathcal{H}_2$ is studied; similarly for $\mathcal{H}_2$ and $\mathcal{H}_1$ and also for $\mathcal{H}_3$ and $\mathcal{H}_1$. The results obtained involve the oscillation of functions from $\mathcal{H}_3$ or $\mathcal{H}_2$ at the boundary as a measure of how close the approximation can be. It is shown that the results cannot be improved. As a consequence of the approximation investigations, the following Sarason-type theorem is proved: *The space $\mathcal{H}_3 + C(\overline{U})|_U$ is uniformly closed.* For regular U, the result had recently been proved by D. Khavinson and H. S. Shapiro.

If U is not regular, then one may try, for a given $f \in C(\partial U)$, to find amongst the functions of $H(U)|_{\partial U}$ the best uniform approximant to f. Such an approximation problem is investigated in [A32]. It turns out that this is intimately related to the following property of $H(U)$: If $U \subset \mathbb{R}^m$ is a bounded domain satisfying $\partial U = \partial\overline{U}$, then the space $H(U)|_{\partial U}$ is *pervasive*, in the sense that $H(U)|_F$ is uniformly dense in $C(\partial U)$ whenever F is a nonempty proper closed subset of ∂U. We note that the assumption $\partial U = \partial\overline{U}$ cannot be omitted. In [A32], approximation properties of general pervasive function spaces are established, which made it possible to clear up the question of best harmonic approximation stated above.

Publications also related to this section include [A30], [A46] and [A50].

Let us return to the f i n e t o p o l o g y which is the coarsest topology making all hyperharmonic functions continuous. It is known that functions continuous in the fine topology for classical potential theory are approximately continuous and thus Baire-one functions with respect to original topology. Such an approach is not available for the parabolic potential theory associated with the heat equation. In [A14] it is proved that, also in this situation, finely continuous functions are Baire-one with respect to the Euclidean topology; this implies, for example, that the fine topology is not normal. In a way it is not surprising that the fine topology is not "nice", for example, general topological considerations from [A35] show that, in interesting cases, the fine topology fails to be Čech complete. This is also the case for density topologies investigated in real analysis.

In [A31] and [B9], for a Borel measurable function $f\colon \mathbb{R}^m \to \mathbb{R}$, the set of *fine strict maxima* (that is, strict maxima with respect to the fine topology) is shown to be polar, and thus small in the potential theoretic sense. In fact, polarity characterizes the size of the set of strict fine maxima.

Recall that a set A is said to be *thin* at a point $x \notin A$ provided that the complement of A is a fine neighbourhood of x. For parabolic potential theory, a geometric condition for thinness is established in [A13]. The result obtained generalizes that of W. Hansen as well as the "tusk condition" of E. G. Effros and J. L. Kazdan. Since a boundary point z of an open set U is regular if and only if the complement of U is thin at z, the result in [A13] provides a geometric regularity criterion.

Publications also related to this section are [A10], [A15], [A34], [A37], [A41], [A50], and [B10].

Now let us turn to work related to integral equation method for boundary value problems. Netuka's Ph.D. thesis [D1] was written under the supervision of J. Král and was published in papers [A7], [A8] and [A9]. The *classical* formulation of the *third boundary value problem* for the Laplace equation requires smoothness of the boundary of the domain. For the case of non-smooth boundaries, it is thus appropriate to choose the weak (distributional) formulation. In the integral equation method, a solution is sought in the form of a single layer potential of a signed measure. The starting point of the investigation is to identify when the corresponding distribution is representable by means of a signed measure. A necessary and sufficient condition is proved in [A7] in terms of the so-called *cyclic variation* studied by J. Král in the sixties. Under this condition, the distribution can be identified with a bounded operator on the Banach space of signed measures on the boundary, and thus the third boundary value problem is transformed into the problem of solving the corresponding operator equation. Properties of this operator are investigated in detail in [A7] and [A8]. The dual operator connected with the *double layer potential* plays an important role here.

For non-smooth domains, the operators studied are not compact and so, in view of the applicability of the Riesz-Schauder theory, it is useful to calculate the essential norm, that is, the distance from the space of compact operators. This is done in [A8], and in [A9] the solvability of the corresponding formulation of the third boundary value problem is proved. The results obtained generalize those of V. D. Sapozhnikova and complete Král's investigations of the Neumann problem.

The applicability of the integral equation method depends on the geometrical nature of the boundary of the domain in question. In general, C^1-domains do not enjoy the geometric conditions involving the boundedness of the cyclic variation, whereas $C^{1+\alpha}$-domains do. In [A3] it is shown that most (in the sense of Baire category) smooth surfaces even have the cyclic variation infinite everywhere.

In [A12] and [E2], the representability of solutions of the Dirichlet problem (with possibly discontinuous boundary data) by means of a generalized double layer potentials is studied. Š. Schwabik's and W. Wendland's modification of the Riesz-Schauder theory turned out to be useful in this context. For a class of non-smooth domains, the harmonic measure is shown to be absolutely continuous with respect to surface measure and non-tangential boundary behaviour of solutions is analysed.

In [A16] the essential radius of a potential theoretic operator for convex sets in $\mathbb{R}^m$ is evaluated in terms of metric density at boundary points. The formula obtained is a higher-dimensional analogue of J. Radon's result established in 1919 for plane domains bounded by *curves of bounded rotation*.

Definitive results concerning the contractivity of C. Neumann's operator considered in full generality are proved in [A18]: non-expansiveness is shown to be equivalent to convexity, and the contractivity of the second iterate of C. Neumann's operator holds for all convex sets. The paper [A18] was inspired by the investigation of R. Kleinman and W. Wendland on the Helmholz equation.

The applicability of the method of integral equations to the *mixed boundary value problem for the heat equation* is investigated in [D2] and [E4]. No a priori smoothness restrictions on the boundary are imposed. A weak characterization of the boundary condition is introduced and, under suitable geometric assumptions involving cyclic variation, the existence and uniqueness result is proved.

Publications also related to this section are [B6] and [E1].

Now we will describe works on real and complex analysis and on measure theory. P. M. Gruber proved in 1977 that *most convex bodies are smooth but not too smooth*. More specifically, considering the Hausdorff metric on convex bodies, the set of convex bodies with C^1-boundary is residual whereas that with C^2-boundary is of the first Baire category. The paper [A23], where convex functions are treated instead of convex bodies, gives a more precise information on the *gap* between C^1 and C^2 smoothness. A special case of the result of [A23] says that a typical convex function is of the class $C^{1+\alpha}$ on no (non-empty) open subset of the domain. In fact a much richer scale of moduli than t^α is considered.

The note [A2] solves a problem proposed by J. Mařík in 1953 concerning uniform continuity of functions with bounded gradient on some (non-convex) open sets possessing a certain geometrical property.

The paper [A19] deals with arbitrary finite sums of vectors in $\mathbb{R}^m$. For a finite set $F = \{x_1, \dots, x_n\} \subset \mathbb{R}^m$ put

$$\sum F = \sum_{j=1}^{n} x_j, \quad \sum |F| = \sum_{j=1}^{n} |x_j|.$$

Denote by $T(u, \delta)$ the cone $\{x \in \mathbb{R}^m\,;\ x \cdot u \geqslant \delta|x|\}$, where $\delta > 0$ and $u \in \mathbb{R}^m$, $|u| = 1$. The result: *There exists $C > 0$ such that for any finite set $F \subset \mathbb{R}^m$ with $\sum |F| > 0$ there is a unit vector u such that*

$$\left| \sum (F \cap T(u, \delta)) \right| > C \sum |F|.$$

The exact (maximal) value of C depending only on m and δ is determined. The result generalizes inequalities previously obtained by W. W. Bledsoe, D. E. Dynkin and A. Wilansky.

In [A45], a general construction of *regularly open subsets* of $\mathbb{R}^m$ (that is , those coinciding with the interior of their closure) *having a boundary of positive Lebesgue measure* is given. This is related to an article of R. Börger published in 1999, where a special construction for $\mathbb{R}$ is presented.

Given a probability measure μ on $\mathbb{R}^m$, write $c(\mu)$ for the *barycentre* of μ and put

$$\|\mu\|_2 = \left(\int_{\mathbb{R}^m} |x - c(\mu)|^2 \, \mathrm{d}\mu(x) \right)^{1/2}.$$

For sequences of probability measures $\mu_1, \mu_2, \dots$ the limit behaviour (with respect to vague and weak convergence) of successive convolutions $\mu_1 * \dots * \mu_n$ is investigated in [A48]. It turns out that the character of convergence is closely related to the convergence or divergence of $\sum \|\mu_k\|_2^2$, respectively. A detailed analysis of the divergence case has to do with the *central limit theorem* and the *Lindeberg condition* from probability theory. Special cases have already been studied in [A44].

Let F map conformally the open unit disc in $\mathbb{C}$ onto the interior of a polygon. The article [A4] deals with a very detailed investigation of the (multivalued) analytic function determined by the analytic element $\{0, F\}$.

In [A32], as we have already mentioned, the space $H(U)|_{\partial U}$ was shown to be *pervasive*, provided U satisfies a mild topological condition. This result suggests the question of whether, substituting $\mathbb{C}$ for $\mathbb{R}^m$, the space of harmonic functions can be replaced by the space $(\operatorname{Re} A(U))|_{\partial U}$; here $A(U)$ is the *disc algebra*, that is, the algebra of functions continuous on $\overline{U}$ and holomorphic on U. A complete characterization of the (real) pervasiveness of $(\operatorname{Re} A(U))|_{\partial U}$ and the complex pervasiveness of $A(U)|_{\partial U}$ is given in [A46].

Publications also related to this section are [A3], [A27], [A35], [A39], [A50], [B8] and [B13]–[B19].

Some articles deal with problems of functional analysis, partial differential equations and statistics. In [A50], two important function spaces are studied from the point of view of Choquet's theory: the space of continuous affine functions on a compact convex set in a locally convex space and the space $H(U)$ introduced above. It turns out that Baire-one functions generated by each of these spaces behave quite differently. Unlike the affine case, the space of bounded $H(U)$-Baire-one functions is not uniformly closed and the *barycentric formula* fails for functions of this space. On the other hand, every Baire-one $H(U)$-affine function (in particular a fine extension of a solution of the generalized Dirichlet problem for continuous boundary data) is a pointwise limit of a bounded sequence of functions from $H(U)$. It is shown that such a situation always occurs for simplicial spaces, but not for general function spaces. Baire-one functions which can be pointwise approximated by bounded sequences of elements of a given function space are characterized.

R. R. Phelps in his monograph on Choquet's theorem asks for an elementary proof of the fact that every extreme point of the convex set of normalized harmonic functions on a ball coincides with a Poisson kernel. The note [A51] brings a contribution in this direction.

For a nonlinear second order *very strongly elliptic system*, every solution with a bounded gradient has affine components (the *Liouville condition*). This result is proved in [A26] and, as a consequence, $C^{1,\mu}$ regularity for a wide class of elliptic systems is obtained.

A threshold autoregressive process of the first order with Gaussian innovations is investigated in [A27]. Several methods of finding its stationary distribution are used; one of them is based on solving a *special integral equation*. Its solution is found for some values of parameters which makes it possible to compare the exact values with results obtained by Markov approximation, numerical solutions and simulations.

Publications also related to this section are [A7]–[A9], [A12], [A15], [A29], [A32], [A46], [A49], [B4], [B5] and [B7].

A series of Netuka's papers is devoted to history of mathematics including some biographies. A long series of texts describes the evolution of mathematical analysis; see [B1]–[B3], [B6], [B12], [B16], [B20], [B21], [C1]–[C7], [C9], [C15], [C16] and [C21]. Some of these papers include biographies of I. Fredholm, E. Helly, H. Lebesgue, K. Löwner, G. Mittag-Leffler, G. Pick, J. Radon, B. Riemann and F. Riesz. Publications [C8], [C10]–[C14], [C17]–[C20], [C22], [F11] written on various occasions are devoted to the life and work of Netuka's teachers and/or colleagues: H. Bauer, M. Brelot, G. Choquet, I. Černý, V. Jarník, J. Král, J. Mařík and J. Veselý.

A contribution to the history of potential theory is contained in [A18]. C. Neumann's original proof of the *contractivity lemma* for plane convex domains from 1877 contained a gap. Neumann's error was sharply criticized by H. Lebesgue in his work of 1937. However, as documented in [A18], C. Neumann corrected his proof in his treatise in 1887, a fact of which H. Lebesgue was apparently unaware.

List of works of I. Netuka

Publications containing new results with complete proofs

[A1] Solution of the problem N$^{\mathrm{o}}$ 10 (author Jan Mařík) from *81* (1956), p. 470. Čas. Pěst. Mat. *94* (1969), 223–225. (In Czech.)

[A2] Solution of the problem N$^{\mathrm{o}}$ 3 (author Jan Mařík) from *81* (1956), p. 247. Čas. Pěst. Mat. *94* (1969), 362–364. (In Czech.)

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