ON SOME PROPERTIES OF SOLUTIONS OF QUASILINEAR DEGENERATE PARABOLIC EQUATIONS IN $\mathbb{R}^m \times (0,+\infty)$

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Abstract. We study the asymptotic behaviour near infinity of the weak solutions of the Cauchy-problem.

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1. Introduction

We consider the Cauchy problem

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} a_i(x,t,u,\nabla u) - c_0 u - f(x,t,u,\nabla u) \quad \text{in } Q = \mathbb{R}^m \times (0,+\infty) \tag{1.1}$$

$$u(x,0) = 0 \quad \text{in } \mathbb{R}^m \tag{1.2}$$

assuming degenerate ellipticity condition

$$\lambda(|u|) \sum_{i=1}^{m} a_i(x,t,u,p)p_i \geqslant \nu(x)\psi(t)|p|^2, \tag{1.3}$$

where $\nu(x)$, $\psi(t)$ and $\lambda(s)$ are nonnegative functions verifying additional conditions to be precised later.

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A model representative of (1.1) is as follows

$$\frac{\partial u}{\partial t} = \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(|x|^{\alpha} t^{\beta} \frac{\partial u}{\partial x_i}\right) - \lambda(x,t) u |u|^{p-2},$$

where $0 \leqslant \alpha < 2$, $\beta > 0$, $\lambda(x,t) \in L^1(Q)^+$ and $p \geqslant 2$.

At present time many results have been established concerning linear and quasilinear degenerate parabolic second or high-order equations. Existence and boundedness of weak solutions of equations of the same class as in the present paper have already been studied, for instance, in [2], [3], [4], [10] and [11]. For regularity results such as Hölder continuity we refer the reader to [12]. Our goal is to study the asymptotic behavior near infinity of any weak solution of the problem (1.1)–(1.2). Analogous result for quasilinear degenerate elliptic equation is contained in [5], while results concerning asymptotic properties of the weak solutions to the parabolic equation

$$\frac{\partial u}{\partial t} - \sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(a_{i,j}(x,t) \frac{\partial u}{\partial x_i}\right) + a(x) u |u|^{p-2} = 0,$$

subject to the Neumann boundary condition, are obtained in [6] via comparison principles.

2. Hypotheses and formulation of the main result

Let $\mathbb{R}^m$ denote the Euclidean m-space ($m > 2$) with generic point $x = (x_1, x_2, \dots, x_m)$. We denote by Q_T the cylinder $\mathbb{R}^m \times]0, T[$, $T > 0$.

Hypothesis 2.1. Let $\nu(x)$ be a positive and measurable function defined in Ω such that:

$$\nu(x) \in L^{\infty}_{\mathrm{loc}}(\mathbb{R}^m), \quad \nu^{-1}(x) \in L^{g}_{\mathrm{loc}}(\mathbb{R}^m) \quad \left(g > \frac{m}{2}\right).$$

Hypothesis 2.2. Let $\psi(t)$ be a positive measurable monotone nondecreasing function defined in $]0, +\infty[$.

There exists a positive number $\tilde{g}$ such that $1/\psi \in L^{\tilde{g}}(0,T)$, $\forall T > 0$.

Assumptions (2.1), (2.2) are classical in the theory of weighted parabolic equations (see [10] for more details).

The symbol $W^{1,0}(\nu\psi, Q)$ stands for the set of all real valued functions $u \in L^2(Q)$ such that their derivatives (in the sense of distributions), with respect to x_i, are functions which have the following property

$$\sqrt{\nu\psi}\frac{\partial u}{\partial x_i} \in L^2(Q), \ i = 1, 2, \dots, m.$$

$W^{1,0}(\nu\psi, Q)$ is a Hilbert space with respect to the norm

$$\|u\|_{1,0} = \left(\int_Q \left(|u|^2 + \sum_{i=1}^{m} \nu\psi \left| \frac{\partial u}{\partial x_i} \right|^2 \right) \mathrm{d}x\, \mathrm{d}t \right)^{\frac{1}{2}}.$$

$W^{1,1}(\nu\psi, Q)$ is the subset of $W^{1,0}(\nu\psi, Q)$ of all functions u such that $\partial u/\partial t$ (in the sense of distributions) belongs to $L^2(Q)$. We can suppose that any function of $W^{1,1}(\nu\psi, Q)$ is continuous in $[0, +\infty[$ with respect to values in $L^2(\mathbb{R}^m)$.

Hypothesis 2.3. The functions $f(x,t,u,p)$, $a_i(x,t,u,p)$ $(i = 1,2,\dots,m)$ are Carathéodory functions in $Q \times \mathbb{R} \times \mathbb{R}^m$, i.e. measurable with respect to (x,t) for any $(u,p) \in \mathbb{R} \times \mathbb{R}^m$, continuous with respect to (u,p) for a.e. (x,t) in Q. $\lambda\colon [0,+\infty[\to [1,+\infty[$ is monotone nondecreasing.

Hypothesis 2.4. There exists a function $f^\star(x,t) \in L^1(Q)$ such that

$$|f(x,t,u,p)| \leqslant \lambda(|u|) \left[f^\star(x,t) + \nu(x)\psi(t)|p|^2 \right] \tag{2.1}$$

holds for almost every $(x,t) \in Q$ and for all real numbers $u, p_1, p_2, \dots, p_m$.

Hypothesis 2.5. There exist a function $f_0(x,t) \in L^1(Q) \cap L^\infty(Q)$ and a non-negative real number $c_1 < c_0$ such that for almost every $(x,t) \in Q$ and for all real numbers $u, p_1, p_2, \dots, p_m$ the inequality

$$uf(x,t,u,p) + c_1^2 + \lambda(|u|)\nu(x)\psi(t)|p|^2 + f_0(x,t) \geqslant 0 \tag{2.2}$$

holds.

Hypothesis 2.6. There exists a function $a^\star(x,t) \in L^2(Q)$ such that, for almost every $(x,t) \in Q$, we have

$$\frac{|a_i(x,t,u,p)|}{\sqrt{\nu\psi}} \leqslant \lambda(|u|)[a^\star(x,t) + \sqrt{\nu\psi}|p|] \tag{2.3}$$

for any real numbers $u, p_1, p_2, \dots, p_m$.

Hypothesis 2.7. The condition (1.3) is satisfied for almost every $(x,t) \in Q$ and for all real numbers $u, p_1, p_2, \dots, p_m$.

Hypothesis 2.8. For almost every $(x,t) \in Q$ we have

$$\sum_{i=1}^{m} [a_i(x,t,u,p) - a_i(x,t,u,q)]\,(p_i - q_i) \geqslant 0 \tag{2.4}$$

for any real numbers $u, p_1, p_2, \dots, p_m, q_1, q_2, \dots, q_m$; the inequality holds if and only if $p \neq q$.

Now, we are in position to give the definition of weak solution of the equation (1.1).

Definition 1. A weak solution of the problem (1.1)–(1.2) in $Q = \mathbb{R}^m \times [0, +\infty[$ is a function $u(x,t) \in W^{1,0}(\nu\psi, Q) \cap L^\infty(Q)$ such that the equality

$$\int_0^{+\infty}\int_{\mathbb{R}^m} \left\{ \sum_{i=1}^m a_i(x,t,u,\nabla u)\frac{\partial w}{\partial x_i} + c_0 uw + f(x,t,u,\nabla u)w - u\frac{\partial w}{\partial t} \right\} \mathrm{d}x\,\mathrm{d}t = 0 \tag{2.5}$$

holds for any $w \in W^{1,1}(\nu\psi, Q) \cap L^\infty(Q)$.

Under Hypotheses 2.1–2.8 the existence of a weak solution u of the equation (1.1) follows from the results of [3], [4], [7], [8] and [9].

The following theorem states the asymptotic behavior of the solutions near infinity.

Theorem 2.1. *Let Hypotheses* 2.1–2.8 *be satisfied and let* R_0 *be a positive real number such that*

$$\operatorname{supp} a^\star(x,t), \operatorname{supp} f_0(x,t), \operatorname{supp} f^\star(x,t) \subseteq \{x \in \mathbb{R}^m\,;\ |x| \leqslant R_0\} \times [0, +\infty[.$$

Take a function $u(x,t) \in W^{1,0}(\nu\psi, Q) \cap L^\infty(Q)$ *which satisfies* (2.5) *for all* $w \in W^{1,1}(\nu\psi, Q) \cap L^\infty(Q)$*. Then for any* $T > 0$ *there exist two positive constants* β *and* $\tilde{\gamma}$*, depending on* $L = \operatorname{ess\,sup}_Q |u(x,t)|$*, such that*

$$H_R(T) \leqslant \beta \left\{ \|f_0\|_{L^1(Q_T)} + \|f^\star\|_{L^1(Q_T)} \right\} \mathrm{e}^{-\frac{\tilde{\gamma}(R-R_0)^2}{\eta(R)T\psi(T)}} \qquad \forall R > R_0, \tag{2.6}$$

where

$$H_R(T) = \int_{|x|>R} u^2(x,T)\,\mathrm{d}x + \int_0^T\int_{|x|>R} \nu\psi\,|\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t,$$

and

$$\eta(R) = \sup_{R<|x|<2R} \nu(x).$$

3. Proof of Theorem 2.1

Let $R > R_0$, $0 < \varrho \leqslant R$ and

$$\xi(x) = \xi(|x|) = \begin{cases} 0 & \text{if } |x| \leqslant R \\ \dfrac{|x| - R}{\varrho} & \text{if } R < |x| < R + \varrho \\ 1 & \text{if } |x| \geqslant R + \varrho. \end{cases}$$

For fixed $T > 0$, we extend $u(x,t)$ by zero in $\mathbb{R}^m \times]-\infty, +\infty[$ and for any $n, s \in \mathbb{N}$, we define

$$\Theta_n(t) = \begin{cases} 0 & \text{if } t \leqslant 0 \\ nt & \text{if } 0 < t \leqslant \frac{1}{n} \\ 1 & \text{if } \frac{1}{n} < t \leqslant T \\ 1 + n(T - t) & \text{if } T < t \leqslant T + \frac{1}{n} \\ 0 & \text{if } t \geqslant T + \frac{1}{n}, \end{cases}$$

$$v_n^s(x,t) = s\Theta_n(t) \int_t^{t+1/s} u(x,\lambda)|u(x,\lambda)|^\gamma \Theta_n^{\gamma+1}(\lambda) \, \mathrm{d}\lambda$$

where $\gamma > 0$ will be chosen later.

Taking $\xi^2(x) v_n^s(x,t)$ as test function in (2.5) we obtain

$$\begin{aligned} (3.1) \int_0^{+\infty} \int_{\mathbb{R}^m} \Bigg\{ & \sum_{i=1}^m a_i(x,t,u,\nabla u) \frac{\partial}{\partial x_i} (\xi^2(x) v_n^s) + c_0 u \xi^2(x) v_n^s(x,t) \\ & + f(x,t,u,\nabla u) \xi^2(x) v_n^s(x,t) - u \xi^2(x) \Theta'(t) \left(s \int_t^{t+1/s} u(x,\lambda)|u(x,\lambda)|^\gamma \Theta_n^{\gamma+1}(\lambda) \, \mathrm{d}\lambda \right) \\ & - u\xi^2(x) \Theta_n(t) \frac{\partial}{\partial t} \left(s \int_t^{t+1/s} u(x,\lambda)|u(x,\lambda)|^\gamma \Theta_n^{\gamma+1}(\lambda) \, \mathrm{d}\lambda \right) \Bigg\} \, \mathrm{d}x \, \mathrm{d}t = 0. \end{aligned}$$

We note that

$$\frac{\partial v_n^s}{\partial x_i} = s\Theta_n(t) \int_t^{t+1/s} \frac{\partial}{\partial x_i} \left(u(x,\lambda)|u(x,\lambda)|^\gamma \right) \Theta_n^{\gamma+1}(\lambda) \, \mathrm{d}\lambda,$$

so, according to the Hypothesis 2.2, we have

$$\nu(x)\psi(t) \left| \frac{\partial v_n^s}{\partial x_i} \right|^2 \leqslant s\nu(x) \int_t^{t+1/s} \psi(\lambda) \left| \frac{\partial}{\partial x_i} \left(u(x,\lambda)|u(x,\lambda)|^\gamma \right) \right|^2 \mathrm{d}\lambda.$$

Moreover, it follows that

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}^m} u\xi^2(x)\Theta_n(t)\frac{\partial}{\partial t}\bigg(s\int_t^{t+1/s} u(x,\lambda)|u(x,\lambda)|^\gamma\Theta_n^{\gamma+1}(\lambda)\,\mathrm{d}\lambda\bigg)\,\mathrm{d}x\,\mathrm{d}t$$

$$= s\int_{-\infty}^{+\infty}\!\!\int_{\mathbb{R}^m} u(x,t)\xi^2(x)\Theta_n(t)\bigg[u\Big(x,t+\frac{1}{s}\Big)\Big|u\Big(x,t+\frac{1}{s}\Big)\Big|^\gamma$$

$$\times\Theta_n^{\gamma+1}\Big(t+\frac{1}{s}\Big) - u(x,t)|u(x,t)|^\gamma\Theta_n^{\gamma+1}(t)\bigg]\,\mathrm{d}x\,\mathrm{d}t$$

$$\leqslant s\bigg(\int_{-\infty}^{+\infty}\!\!\int_{\mathbb{R}^m} |u(x,t)|^{\gamma+2}\xi^2(x)\Theta_n^{\gamma+2}(t)\,\mathrm{d}x\,\mathrm{d}t\bigg)^{1/(\gamma+2)}\bigg(\int_{-\infty}^{+\infty}\!\!\int_{\mathbb{R}^m}\Big|u\Big(x,t+\frac{1}{s}\Big)\Big|^{\gamma+2}$$

$$\times\xi^2(x)\Theta_n^{\gamma+2}\Big(t+\frac{1}{s}\Big)\,\mathrm{d}x\,\mathrm{d}t\bigg)^{\frac{\gamma+1}{\gamma+2}} - s\int_{-\infty}^{+\infty}\!\!\int_{\mathbb{R}^m}|u(x,t)|^{\gamma+2}\xi^2(x)\Theta_n^{\gamma+2}(t)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

Then, from (3.1), letting $s\to+\infty$, we get

$$\int_0^{+\infty}\!\!\int_{\mathbb{R}^m}\bigg\{\sum_{i=1}^m a_i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\big(u|u|^\gamma\xi^2(x)\big)\Theta_n^{\gamma+2}(t)$$
$$+ c_0|u|^{\gamma+2}\xi^2(x)\Theta_n^{\gamma+2}(t) + f(x,t,u,\nabla u)u|u|^\gamma\xi^2(x)\Theta_n^{\gamma+2}(t)$$
$$- |u|^{\gamma+2}\xi^2(x)\Theta_n'(t)\Theta_n^{\gamma+1}(t)\bigg\}\,\mathrm{d}x\,\mathrm{d}t\leqslant 0. \tag{3.2}$$

On the other hand, for $\sigma\in\,]0,1[$, we have

$$-\int_0^{+\infty}\!\!\int_{\mathbb{R}^m}|u|^{\gamma+2}\xi^2(x)\Theta_n'(t)\Theta_n^{\gamma+1}(t)\,\mathrm{d}x\,\mathrm{d}t$$
$$\geqslant -n\int_0^{1/n}\!\!\int_{\mathbb{R}^m}|u|^{\gamma+2}\xi^2(x)\,\mathrm{d}x\,\mathrm{d}t + n(1-\sigma)^{\gamma+1}\int_T^{T+\sigma/n}\!\!\int_{\mathbb{R}^m}|u|^{\gamma+2}\xi^2(x)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.3}$$

Combining (3.2) and (3.3), for $n\to+\infty$, we obtain

$$\int_0^T\!\!\int_{\mathbb{R}^m}\bigg\{\sum_{i=1}^m a_i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\big(u|u|^\gamma\xi^2(x)\big) + c_0|u|^{\gamma+2}\xi^2(x)$$
$$+ f(x,t,u,\nabla u)u|u|^\gamma\xi^2(x)\bigg\}\,\mathrm{d}x\,\mathrm{d}t$$
$$+ (1-\sigma)^{\gamma+1}\sigma\int_{\mathbb{R}^m}|u(x,T)|^{\gamma+2}\xi^2(x)\,\mathrm{d}x\leqslant 0. \tag{3.4}$$

Let us prove, for instance, that

$$\lim_{n\to+\infty}\int_0^{+\infty}\!\!\int_{\mathbb{R}^m}\sum_{i=1}^m a_i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\big(u|u|^\gamma\xi^2(x)\big)\Theta_n^{\gamma+2}(t)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T\!\!\int_{\mathbb{R}^m}\sum_{i=1}^m a_i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\big(u|u|^\gamma\xi^2(x)\big)\,\mathrm{d}x\,\mathrm{d}t.$$

In fact

$$\Theta_n(t) \longrightarrow \begin{cases} 0 & \text{if } t \leqslant 0 \\ 1 & \text{if } 0 < t < T \\ 0 & \text{if } t \geqslant T; \end{cases}$$

moreover, the integration of the first term of the previous relation can be evaluated in $\mathbb{R}^m \times]0, T+1[$ where a.e. we have

$$\begin{aligned} &\Big| \sum_{i=1}^m a_i(x,t,u,\nabla u) \frac{\partial}{\partial x_i} \left(u|u|^\gamma \xi^2(x) \right) \Theta_n^{\gamma+2}(t) \Big| \\ &\qquad \leqslant \sum_{i=1}^m |a_i(x,t,u,\nabla u)| \left| \frac{\partial}{\partial x_i} \left(u|u|^\gamma \xi^2(x) \right) \right| \quad \forall n \in \mathbb{N}. \end{aligned}$$

Then with respect to these facts, our assertion is true due to the Lebesgue theorem.

Next, choosing $\sigma = (\gamma+2)^{-1}$ and using the growth conditions in the right-hand side of the inequality (3.4) we have

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^m} (\gamma+1)|u|^\gamma \frac{\nu\psi}{\lambda(L)} |\nabla u|^2 \xi^2(x) \,\mathrm{d}x \,\mathrm{d}t + (c_0 - c_1) \int_0^T \int_{\mathbb{R}^m} |u|^{\gamma+2} \xi^2(x) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad - \lambda(L) \int_0^T \int_{\mathbb{R}^m} |u|^\gamma \nu\psi |\nabla u|^2 \xi^2(x) \,\mathrm{d}x \,\mathrm{d}t + \frac{1}{\mathrm{e}(\gamma+2)} \int_{\mathbb{R}^m} |u(x,T)|^{\gamma+2} \xi^2(x) \,\mathrm{d}x \\ &\quad \leqslant \int_0^T \int_{\mathbb{R}^m} |f_0||u|^\gamma \xi^2(x) \,\mathrm{d}x \,\mathrm{d}t - 2 \int_0^T \int_{\mathbb{R}^m} \sum_{i=1}^m a_i(x,t,u,\nabla u) \frac{\partial \xi}{\partial x_i} u|u|^\gamma \xi(x) \,\mathrm{d}x \,\mathrm{d}t, \end{aligned}$$

and from this, for γ such that $(\gamma+1)/\lambda(L) - \lambda(L) > 1$ $(\gamma > 1)$,

$$\begin{aligned} \text{(3.5)} \quad &\int_0^T \int_{\mathbb{R}^m} |u|^\gamma \nu\psi |\nabla u|^2 \xi^2(x) \,\mathrm{d}x \,\mathrm{d}t + \frac{1}{\mathrm{e}(\gamma+2)} \int_{\mathbb{R}^m} |u(x,T)|^{\gamma+2} \xi^2(x) \,\mathrm{d}x \\ &\quad \leqslant \int_0^T \int_{\mathbb{R}^m} |f_0||u|^\gamma \xi^2(x) \,\mathrm{d}x \,\mathrm{d}t \\ &\qquad + 2 \int_0^T \int_{\mathbb{R}^m} \sum_{i=1}^m |a_i(x,t,u,\nabla u)| \left| \frac{\partial \xi}{\partial x_i} \right| |u|^{\gamma+1} \xi(x) \,\mathrm{d}x \,\mathrm{d}t. \end{aligned}$$

By Hypothesis 2.6 and the Young inequality it results

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^m} \sum_{i=1}^m |a_i(x,t,u,\nabla u)| \left| \frac{\partial \xi}{\partial x_i} \right| |u|^{\gamma+1} \xi(x) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad \leqslant \beta_1 \Big\{ \int_0^T \int_{\mathbb{R}^m} a^\star \sqrt{\nu\psi} |u|^\gamma \xi \, |\nabla \xi| \,\mathrm{d}x \,\mathrm{d}t + \frac{\varepsilon}{2} \int_0^T \int_{\mathbb{R}^m} |u|^\gamma \nu\psi |\nabla u|^2 \xi^2(x) \,\mathrm{d}x \,\mathrm{d}t \\ &\qquad + \frac{1}{2\varepsilon} \int_0^T \int_{\mathbb{R}^m} \nu\psi |u|^{\gamma+2} |\nabla \xi|^2 \,\mathrm{d}x \,\mathrm{d}t \Big\}. \end{aligned}$$

Hence, taking into account that $\operatorname{supp} a^\star(x,t)$ and $\operatorname{supp} f_0(x,t)$ are subsets of $\{x \in \mathbb{R}^m\,;\ |x| \leqslant R_0\} \times [0,+\infty[$, from (3.5) for $\varepsilon > 0$ sufficiently small, we obtain

$$\int_0^T\!\!\int_{\mathbb{R}^m} |u|^\gamma \nu\psi \left|\nabla u\right|^2 \xi^2(x)\,\mathrm{d}x\,\mathrm{d}t \leqslant \beta_2 \int_0^T\!\!\int_{\mathbb{R}^m} \nu\psi|u|^2 \left|\nabla\xi\right|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{3.6}$$

On the other hand, from (3.4) for $\gamma = 0$ and $\sigma = \frac{1}{2}$, we get

$$\int_0^T\!\!\int_{\mathbb{R}^m} \left\{ \sum_{i=1}^m a_i(x,t,u,\nabla u)\frac{\partial}{\partial x_i}\left(u\xi^2(x)\right) + c_0u^2\xi^2(x) + f(x,t,u,\nabla u)u\xi^2(x)\right\}\mathrm{d}x\,\mathrm{d}t \\ + \frac{1}{2\mathrm{e}}\int_{\mathbb{R}^m} |u(x,T)|^2\xi^2(x)\,\mathrm{d}x \leqslant 0.$$

From this, according to Hypotheses 2.4, 2.6 and 2.7, we have

$$\begin{aligned}
&\frac{1}{\lambda(L)}\int_0^T\!\!\int_{\mathbb{R}^m} \nu\psi\left|\nabla u\right|^2\xi^2(x)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2\mathrm{e}}\int_{\mathbb{R}^m}|u(x,T)|^2\xi^2(x)\,\mathrm{d}x \\
&\quad\leqslant \lambda(L)\Big(\int_0^T\!\!\int_{\mathbb{R}^m}|u|^\gamma\nu\psi\left|\nabla u\right|^2\xi^2(x)\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{1}{\gamma}}\Big(\int_0^T\!\!\int_{\mathbb{R}^m}\nu\psi\left|\nabla u\right|^2\xi^2(x)\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{\gamma-1}{\gamma}} \\
&\qquad + 2\lambda(L)\int_0^T\!\!\int_{\mathbb{R}^m}\nu\psi\left|\nabla u\right||u|\xi(x)\left|\nabla\xi\right|\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

and, after a simple calculation,

$$\begin{aligned}
&\int_0^T\!\!\int_{\mathbb{R}^m}\nu\psi\left|\nabla u\right|^2\xi^2(x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^m}|u(x,T)|^2\xi^2(x)\,\mathrm{d}x \\
&\quad\leqslant \beta_3\Big(\int_0^T\!\!\int_{\mathbb{R}^m}|u|^\gamma\nu\psi\left|\nabla u\right|^2\xi^2(x)\,\mathrm{d}x\,\mathrm{d}t\Big) + \beta_4\int_0^T\!\!\int_{\mathbb{R}^m}\nu\psi|u|^2\left|\nabla\xi\right|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The above inequality and (3.6) give

$$\int_0^T\!\!\int_{\mathbb{R}^m}\nu\psi\left|\nabla u\right|^2\xi^2(x)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}^m}|u(x,T)|^2\xi^2(x)\,\mathrm{d}x \\ \leqslant \beta_5\int_0^T\!\!\int_{\mathbb{R}^m}\nu\psi|u|^2\left|\nabla\xi\right|^2\,\mathrm{d}x\,\mathrm{d}t;$$

in this way, by the definition of $\xi(x)$, we get

$$H_{R+\varrho}(T) \leqslant \frac{\beta_5}{\varrho^2}\bar{\nu}(R,R+\varrho)\int_0^T\psi(\tau)H_R(\tau)\,\mathrm{d}\tau, \tag{3.7}$$

where if $R_1 < R_2$, $\bar{\nu}(R_1,R_2) = \sup\limits_{R_1<|x|<R_2}\nu(x)$.

Let us prove, by induction, the following inequality

$$H_{R_0+k\varrho}(T) \leqslant \beta^k \left\{\|f_0\|_{L^1(Q_T)} + \|f^\star\|_{L^1(Q_T)}\right\} \frac{(T\psi(T))^k}{\varrho^{2k}k!} \overline{\nu}\left[(R+\varrho, R_0+k\varrho)\right]^k . \tag{3.8}$$

Our next claim is that to prove (3.8) where $k=0$.

Choosing $v_n^s(x,t)$ as test function in (2.5) and proceeding analogously to the proof of (3.4), we get

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^m}(\gamma+1)\sum_{i=1}^m a_i(x,t,u,\nabla u)\frac{\partial u}{\partial x_i}|u|^\gamma \,\mathrm{d}x\,\mathrm{d}t + c_0\int_0^T\int_{\mathbb{R}^m}|u|^{\gamma+2}\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T\int_{\mathbb{R}^m} f(x,t,u,\nabla u)u|u|^\gamma\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\mathrm{e}(\gamma+2)}\int_{\mathbb{R}^m}|u(x,T)|^{\gamma+2}\,\mathrm{d}x \leqslant 0\end{aligned} \tag{3.9}$$

and from this for $\gamma > \lambda^2(L)+\lambda(L)-1$

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^m}|u|^\gamma\nu\psi\left|\nabla u\right|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\mathrm{e}(\gamma+2)}\int_{\mathbb{R}^m}|u(x,T)|^{\gamma+2}\,\mathrm{d}x \\ \leqslant \Big(\operatorname*{ess\,sup}_{\mathbb{R}^m\times]0,+\infty[}|u|\Big)^\gamma \int_0^T\int_{\mathbb{R}^m}|f_0(x,t)|\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{3.10}$$

On the other hand if we write (3.9) for $\gamma=0$ we obtain

$$\begin{aligned}\frac{1}{\lambda(L)}\int_0^T\int_{\mathbb{R}^m}\nu\psi\left|\nabla u\right|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2\mathrm{e}}\int_{\mathbb{R}^m}|u(x,T)|^2\,\mathrm{d}x \\ \leqslant \lambda(L)\int_0^T\int_{\mathbb{R}^m}[|f^\star(x,t)| + \nu\psi\left|\nabla u\right|^2]|u|\,\mathrm{d}x\,\mathrm{d}t \leqslant L\lambda(L)\int_0^T\int_{\mathbb{R}^m}|f^\star(x,t)|\,\mathrm{d}x\,\mathrm{d}t \\ + \lambda(L)\Big(\int_0^T\int_{\mathbb{R}^m}|u|^\gamma\nu\psi\left|\nabla u\right|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{1}{\gamma}}\Big(\int_0^T\int_{\mathbb{R}^m}\nu\psi\left|\nabla u\right|^2\,\mathrm{d}x\,\mathrm{d}t\Big)^{\frac{\gamma-1}{\gamma}}.\end{aligned}$$

Hence using the Young inequality we conclude that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^m}\nu\psi\left|\nabla u\right|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2\mathrm{e}}\int_{\mathbb{R}^m}|u(x,T)|^2\,\mathrm{d}x \\ \leqslant \beta_6\int_0^T\int_{\mathbb{R}^m}|f^\star(x,t)|\,\mathrm{d}x\,\mathrm{d}t + \beta_7\Big(\int_0^T\int_{\mathbb{R}^m}|u|^\gamma\nu\psi\left|\nabla u\right|^2\,\mathrm{d}x\,\mathrm{d}t\Big)\end{aligned}$$

and finally, according to (3.10), that

$$H_{R_0}(T) \leqslant \beta\left\{\|f_0\|_{L^1(Q_T)} + \|f^\star\|_{L^1(Q_T)}\right\}. \tag{3.11}$$

Let us assume that the inequality (3.8) holds for some integer $k > 0$. Due to (3.7) and (3.8), we obtain

$$
\begin{aligned}
H_{R_0+(k+1)\varrho}(T) &\leqslant \frac{\beta}{\varrho^2}\overline{\nu}\left(R_0+k\varrho, R_0+(k+1)\varrho\right)\int_0^T \psi(\tau)H_{R_0+k\varrho}(\tau) \\
&\leqslant \frac{\beta}{\varrho^2}\overline{\nu}\left(R_0+k\varrho, R_0+(k+1)\varrho\right)\int_0^T \psi(\tau)\beta^k\left\{\|f_0\|_{L^1(Q_T)}+\|f^\star\|_{L^1(Q_T)}\right\} \\
&\quad\times\frac{\left(\tau\psi(\tau)\right)^k}{\varrho^{2k}k!}\left[\overline{\nu}\left(R_0+k\varrho, R_0+(k+1)\varrho\right)\right]^k\,\mathrm{d}\tau \\
&= \frac{\beta^{k+1}}{\varrho^{2(k+1)}k!}\left[\overline{\nu}\left(R_0+k\varrho, R_0+(k+1)\varrho\right)\right]^{k+1}\left\{\|f_0\|_{L^1(Q_T)}+\|f^\star\|_{L^1(Q_T)}\right\} \\
&\quad\times\int_0^T \tau^k\left[\psi(\tau)\right]^{k+1}\,\mathrm{d}\tau.
\end{aligned}
$$

According Hypothesis 2.1, taking into account that

$$
\overline{\nu}\left(R_0+k\varrho, R_0+(k+1)\varrho\right)\leqslant \sup_{R+\varrho<|x|<R_0+(k+1)\varrho}\nu(x),
$$

the last inequality implies (3.8) for $k+1$. Let $k \geqslant 1$. Choosing in (3.8) $\varrho = (R-R_0)/k$ we obtain

$$
H_R(T)\leqslant \beta^k\left\{\|f_0\|_{L^1(Q_T)}+\|f^\star\|_{L^1(Q_T)}\right\}\frac{\left(T\psi(T)\right)^k k^{2k}}{\left(R-R_0\right)^{2k}k!}\left[\eta(R)\right]^k.
$$

From this inequality, using also Stirling's formula, it follows that

$$
H_R(T)\leqslant\left\{\|f_0\|_{L^1(Q_T)}+\|f^\star\|_{L^1(Q_T)}\right\}\mathrm{e}^{-k\log\frac{\beta(R-R_0)^2}{\mathrm{e}k\eta(R)T\psi(T)}}. \tag{3.12}
$$

Now, if

$$
\frac{\beta\left(R-R_0\right)^2}{\mathrm{e}\eta(R)T\psi(T)}\leqslant \mathrm{e}
$$

the estimate (2.6) easily follows from (3.11). Otherwise, we can obtain (2.6) from (3.12) taking as k the integer part of

$$
\left\{\frac{\beta\left(R-R_0\right)^2}{\mathrm{e}^2\eta(R)T\psi(T)}\right\}.
$$

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