LEXICOGRAPHIC EXTENSIONS OF DUALLY RESIDUATED LATTICE ORDERED MONOIDS

Jiří Rachůnek, Olomouc, Dana Šalounová, Ostrava

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Abstract. Dually residuated lattice ordered monoids ($DR\ell$-monoids) are common generalizations of, e.g., lattice ordered groups, Brouwerian algebras and algebras of logics behind fuzzy reasonings (MV-algebras, BL-algebras) and their non-commutative variants (GMV-algebras, pseudo BL-algebras). In the paper, lex-extensions and lex-ideals of $DR\ell$-monoids (which need not be commutative or bounded) satisfying a certain natural condition are studied.

Keywords: $DR\ell$-monoid, ideal, lex-extension, lex-ideal, algebras of fuzzy logics

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1. Introduction

Recently, the study of algebras which are in connection with logics behind fuzzy reasoning has been intensively developed. Let us recall that e.g. MV-algebras [2] are an algebraic counterpart of the Łukasiewicz infinite valued logic and that BL-algebras are in the same connection with the Hájek basic fuzzy logic [7]. GMV-algebras [17] (equivalently: pseudo MV-algebras [6]) and pseudo BL-algebras [4], respectively, are non-commutative generalizations of the above algebras, and hence one can view them as algebraic semantics of non-commutative reasoning allowing two negations and two implications. At the same time, all the above mentioned algebras are polynomially equivalent to special subclasses of the general class of all dually residuated lattice ordered monoids (briefly: $DR\ell$-monoids) [14], [15], [16], [10],

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which, moreover, contains the classes of lattice ordered groups (briefly: ℓ-groups), positive cones of ℓ-groups and Brouwerian and Boolean algebras.

In the paper we search lexicographic extensions (briefly: lex-extensions) of ideals and lex-ideals of $DR\ell$-monoids. The main results are obtained for $DR\ell$-monoids from the class characterized by condition $(*)$ which contains the above mentioned algebras of logics as well as all ℓ-groups. Special results are then obtained for $DR\ell$-monoids satisfying the cancellation property.

2. Basic notions and notation

Definition. An algebra $M = (M, +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of signature $\langle 2, 0, 2, 2, 2, 2\rangle$ is called a *dually residuated (non-commutative) lattice ordered monoid (a $DR\ell$-monoid)*, if and only if

(M1) $(M, +, 0, \vee, \wedge)$ is a lattice ordered monoid (ℓ-monoid), that is, $(M, +, 0)$ is a (non-commutative) monoid, $(M, \vee, \wedge)$ is a lattice, and for any $x, y, u, v \in M$, the following identities are satisfied:

$$u + (x \vee y) + v = (u + x + v) \vee (u + y + v),$$
$$u + (x \wedge y) + v = (u + x + v) \wedge (u + y + v);$$

(M2) if $\leqslant$ denotes the order on M induced by the lattice $(M, \vee, \wedge)$ then, for any $x, y \in M$, $x \rightharpoonup y$ is the least element $s \in M$ such that $s + y \geqslant x$, $x \leftharpoondown y$ is the least element $t \in M$ such that $y + t \geqslant x$;

(M3) M fulfils the identities

$$((x \rightharpoonup y) \vee 0) + y \leqslant x \vee y,\ y + ((x \leftharpoondown y) \vee 0) \leqslant x \vee y,$$
$$x \rightharpoonup x \geqslant 0,\ x \leftharpoondown x \geqslant 0.$$

An element $x \in M$ is called *positive* if $0 \leqslant x$. Let us denote by $M^+ = \{x \in M;\ 0 \leqslant x\}$ the set of all positive elements in M.

If a $DR\ell$-monoid M is commutative, i.e. $(M, +)$ is a commutative semigroup, then the binary operations $\rightharpoonup$ and $\leftharpoondown$ coincide. Commutative $DR\ell$-monoids were introduced by K. L. N. Swamy in [20] as a common generalization of abelian lattice ordered groups (ℓ-groups) and Brouwerian algebras. General $DR\ell$-monoids (which need not be commutative) were defined by T. Kovář in [9] and their properties were further studied in [10], [11], [12], [13], [18], [19].

Let M be a $DR\ell$-monoid and $x \in M$. Then the *absolute value of an element* x is $|x| = x \vee (0 \rightharpoonup x) = x \vee (0 \leftharpoondown x)$. By [11, Proposition 9], we have $|x| \in M^+$. Further, by the *positive part* of $x \in M$ we mean the element $x^+ = x \vee 0$.

Definition. a) Let M be a $DR\ell$-monoid and $\emptyset \neq I \subseteq M$. Then I is called an *ideal of* M if the following conditions are satisfied:

(1) $x, y \in I \Longrightarrow x + y \in I$;

(2) $x \in I$, $y \in M$, $|y| \leqslant |x| \Longrightarrow y \in I$.

b) An ideal I is said to be *normal* if for each $x, y \in M$ the following equivalence holds:

(3) $x \rightharpoonup y \in I \Longleftrightarrow x \leftharpoondown y \in I$.

Remark. a) $\emptyset \neq I \subseteq M$ is an ideal of M if and only if I is a convex sub-$DR\ell$-monoid of M. In [11, Theorem 13] (for commutative $DR\ell$-monoids also in [9, Theorem 2.2.4]) it is moreover required that a convex sub$DR\ell$-monoid I satisfies the following condition: for each $x \in M$, $x \in I$ if and only if $|x| \in I$. Let us show that this condition is superfluous and hence it can be omitted. Namely, let $x \in M$. Then $0 \rightharpoonup |x| = 0 \rightharpoonup (x \vee (0 \leftharpoondown x)) \leqslant 0 \rightharpoonup (0 \leftharpoondown x)$. By [9, Lemma 1.1.6], $p \rightharpoonup (q \leftharpoondown r) \leqslant (p \rightharpoonup q) + r$ for any $p, q, r \in M$, therefore in our case we get $0 \rightharpoonup |x| \leqslant (0 \rightharpoonup 0) + x = x$. Hence $0 \rightharpoonup |x| \leqslant x \leqslant |x|$ and so, if $|x| \in I$, then also $x \in I$. The converse implication is obvious.

b) Normal ideals of $DR\ell$-monoids are just the kernels of $DR\ell$-homomorphisms. In particular, for ℓ-groups the normal ideals coincide with kernels of ℓ-homomorphisms and ideals coincide with convex ℓ-subgroups.

c) Let M be a $DR\ell$-monoid and I an ideal of M. If $x \in M$ then by the *left coset* of I containing x we mean the set $x/_l I = \{a \in M; (a \rightharpoonup x) \vee (x \rightharpoonup a) \in I\}$. By [11], left cosets of I form a decomposition $M/_l I$ of M. If we put $x/_l I \leqslant y/_l I$ if and only if $(x \rightharpoonup y)^+ \in I$, then $(M/_l I, \leqslant)$ is a distributive lattice. If, moreover, I is a normal ideal then $M/_l I$ is the underlying set of the factor $DR\ell$-monoid M/I of the $DR\ell$-monoid M by I.

The set $\mathcal{C}(M)$ of all ideals of a $DR\ell$-monoid M was studied in [11] and [12]. Let us recall that $(\mathcal{C}(M), \subseteq)$ is an algebraic Brouwerian lattice, in which infima are formed by set intersections, and that the set $\mathcal{N}(M)$ of all normal ideals of M is a complete sublattice in $\mathcal{C}(M)$. If $A \subseteq M$ then the ideal $C(A)$ generated by A satisfies

$$C(A) = \{x \in M;\ |x| \leqslant |a_1| + \ldots + |a_n|,\ \text{where } a_1, \ldots, a_n \in A,\ n \in \mathbb{N}\}.$$

If $a \in M$ and $C(a)$ is the principal ideal generated by a then

$$C(a) = \{x \in M;\ |x| \leqslant n|a|,\ \text{where } n \in \mathbb{N}\}.$$

If $I, J \in \mathcal{N}(M)$ then

$$I \vee J = \{x \in M;\ |x| \leqslant a + b,\ \text{where } a \in I^+, b \in J^+\}.$$

An ideal I is called a *prime ideal* of M if I is a finitely meet-irreducible element in the lattice $\mathcal{C}(\mathcal{M})$, i.e.

$$I = J \cap K \Longrightarrow I = J \text{ or } J = K$$

for every $J, K \in \mathcal{C}(M)$.

If $I \in \mathcal{C}(M)$ is (infinitely) meet-irreducible in $\mathcal{C}(\mathcal{M})$ then I is called a *regular ideal* of M.

Let $0 \neq a \in M$. Then each ideal of M which is maximal with respect to the property "not containing a" is called a *value of the element* a. Denote by $\operatorname{val}(a)$ the set of all values of a. Then $\operatorname{val}(a) \neq \emptyset$ for any $0 \neq a \in M$. If $I \in \mathcal{C}(M)$ then $I \in \operatorname{val}(a)$ for some $0 \neq a \in M$ if and only if I is regular. An element $0 \neq a \in M$ is said to be *special* if $\operatorname{val}(a)$ is a singleton.

Let $A \subseteq M$. Then the *polar of the set* A is

$$A^{\perp} = \{x \in M;\ |x| \wedge |a| = 0 \text{ for each } a \in A\}.$$

For $A = \{a\}$ we will write $a^{\perp}$ instead of $\{a\}^{\perp}$.

A set $B \subseteq M$ is called a *polar in* M if $B = A^{\perp}$ for some $A \subseteq M$. The set $\mathcal{P}(M)$ of all polars in M is a subset of $\mathcal{C}(M)$ and $(\mathcal{P}(M), \subseteq)$ is a complete Boolean lattice.

In the paper we will deal not only with $DR\ell$-monoids in general, but we will very often study properties of $DR\ell$-monoids satisfying the inequalities

$$(*) \qquad \begin{aligned} (x \rightharpoonup y) \wedge (y \rightharpoonup x) &\leqslant 0, \\ (x \leftharpoondown y) \wedge (y \leftharpoondown x) &\leqslant 0. \end{aligned}$$

For instance, any ℓ-group G considered as a $DR\ell$-monoid (i.e. $x \rightharpoonup y = x - y$ and $x \leftharpoondown y = -y + x$ for each $x, y \in G$) satisfies $(*)$.

Further, $DR\ell$-monoids induced by pseudo BL-algebras (that is also by BL-algebras, GMV-algebras and MV-algebras) satisfy condition $(*)$.

3. Lex-extensions of ideals of $DR\ell$-monoids

In this part we introduce the notion of a lexicographic extension of an ideal of a $DR\ell$-monoid. In what follows, we denote by M a $DR\ell$-monoid.

Lemma 1. *If I is an ideal of M and $x \geqslant y$ for each $x \in M^+ \setminus I$ and $y \in I$ then I is comparable with every $K \in \mathcal{C}(M)$.*

Proof. We suppose $K \in \mathcal{C}(M)$, $K \nsubseteq I$ and $x \in K \setminus I$. It is obvious that any element $y \in M$ belongs to the ideal I if and only if $|y|$ belongs to I. Therefore $|x| \in K^+ \setminus I$ and we have $I \subseteq I(|x|) \subseteq K$. $\square$

Lemma 2. *If I is a prime ideal of M which is comparable with every $K \in \mathcal{C}(M)$ then I contains all proper polars in M.*

Proof. Let B be a polar in M such that $B \nsubseteq I$. Then $I \subset B$. We suppose $0 < y \in B \setminus I$. If $z \in y^{\perp}$ then $|z| \wedge y = 0$ and so, as I is a prime ideal of M, by [12, Theorem 2.1] we have $|z| \in I$ and therefore also $z \in I$. That means $y^{\perp} \subseteq I$, hence $B^{\perp} \subseteq I$. So we obtain $B^{\perp} \subseteq I \subset B$, thus $B = M$. □

Lemma 3. *Let $I \in \mathcal{C}(M)$ contain all proper polars in M. Then I contains all minimal prime ideals of M.*

Proof. By [13, Lemma 4], if P is a minimal prime ideal of M then $P = \bigcup\{a^{\perp};\ 0 \leqslant a \notin P\}$. For this reason, if I contains all proper polars in M then it also contains all minimal prime ideals of M. □

Lemma 4. *If $I \in \mathcal{C}(M)$ contains all minimal prime ideals of M then $a^{\perp} = \{0\}$ for each $a \in M \setminus I$.*

Proof. Consider $a \in M \setminus I$. If P is a minimal prime ideal of M then $P \subseteq I$, hence $a \notin P$ and therefore $a^{\perp} \subseteq P$. Each regular ideal of M is prime, the intersection of all regular ideals of M is equal to $\{0\}$ and each prime ideal contains a minimal prime ideal, therefore also the intersection of all minimal prime ideals is equal to $\{0\}$. Consequently, $a^{\perp} = \{0\}$. □

Lemma 5. *If M satisfies condition $(*)$ and if $a^{\perp} = \{0\}$ for any $a \in M \setminus I$, then every element of $M \setminus I$ is special.*

Proof. Assume that $a \in M \setminus I$. Then there exists $H \in \operatorname{val}(a)$ such that $I \subseteq H$. Let $K \in \operatorname{val}(a)$ and $K \neq H$. Let us consider $0 < x \in H \setminus K$ and $0 < y \in K \setminus H$. By [9, Lemma 1.1.14], $p \geqslant q$ implies $(p \rightharpoonup q) + q = p = q + (p \leftharpoondown q)$ for any $p, q \in M$. Thus in our case we obtain

$$x = (x \rightharpoonup (x \wedge y)) + (x \wedge y) \quad \text{and} \quad y = (y \rightharpoonup (x \wedge y)) + (x \wedge y).$$

At the same time

$$\begin{aligned}(x \rightharpoonup (x \wedge y)) \wedge (y \rightharpoonup (x \wedge y)) &= ((x \rightharpoonup x) \vee (x \rightharpoonup y)) \wedge ((y \rightharpoonup x) \vee (y \rightharpoonup y)) \\ &= (0 \vee (x \rightharpoonup y)) \wedge ((y \rightharpoonup x) \vee 0) \\ &= (0 \wedge (y \rightharpoonup x)) \vee (0 \wedge 0) \vee ((x \rightharpoonup y) \wedge (y \rightharpoonup x)) \vee ((x \rightharpoonup y) \wedge 0),\end{aligned}$$

and since M satisfies $(*)$, we have

$$(x \rightharpoonup (x \wedge y)) \wedge (y \rightharpoonup (x \wedge y)) = 0.$$

Moreover, $x \rightharpoonup (x \wedge y) \notin K$ and $y \rightharpoonup (x \wedge y) \notin H$, hence $y \rightharpoonup (x \wedge y) \notin I$ and $(y \rightharpoonup (x \wedge y))^{\perp} \neq \{0\}$, a contradiction. Therefore a is a special element. $\square$

Lemma 6. *Let every element from $M \setminus I$ be special. Then $a^{\perp} = \{0\}$ for each $a \in M \setminus I$.*

P r o o f. Suppose $a \in M^{+} \setminus I$ and let H be the only value of the element a. Then $I \subseteq H$. Assume $x \in a^{\perp}$. Then $a \vee |x| \in M \setminus H$ (otherwise, $0 < a \leqslant a \vee |x|$ would imply $a \in H$), and so $H \subseteq K$ where K is the only value of the element $a \vee |x|$. At the same time $a \wedge |x| = 0$ and $a \notin H$, hence $|x| \in H$ and thus also $x \in H$.

If $a \vee |x| \notin C(a)$ held, then it would follow that $C(a) \subseteq K$ and hence $a \vee |x| \in C(a) \vee H \subseteq K$, a contradiction. For this reason $a \vee |x| \in C(a)$, hence also $x \in C(a)$. From this we have $C(a)^{\perp} = a^{\perp} \subseteq C(a)$, i.e. $C(a)^{\perp} = a^{\perp} = \{0\}$. $\square$

Lemma 7. *Let M satisfy $(*)$ and let $a^{\perp} = \{0\}$ for each $a \in M \setminus I$. Then I is a prime ideal of M and $x > y$ for any $x \in M^{+} \setminus I$ and $y \in I$.*

P r o o f. Consider $a \in M^{+} \setminus I$ and $b \in I$. It is true that $a = (a \rightharpoonup (a \wedge b)) + (a \wedge b)$ and $b = (b \rightharpoonup (a \wedge b)) + (a \wedge b)$. Therefore $a \rightharpoonup (a \wedge b) \notin I$, hence $(a \rightharpoonup (a \wedge b))^{\perp} = \{0\}$, and since M satisfies $(*)$, we conclude that $(a \rightharpoonup (a \wedge b)) \wedge (b \rightharpoonup (a \wedge b)) = 0$, which yields $b \rightharpoonup (a \wedge b) = 0$. This implies $b = a \wedge b < a$. $\square$

We have actually proved the following theorem.

Theorem 8. *Let M be a $DR\ell$-monoid satisfying $(*)$ and $I \in \mathcal{C}(\mathcal{M})$. Then the following conditions are equivalent:*

(1) *I is a prime ideal and $x > y$ holds for each $x \in M^{+} \setminus I$ and $y \in I$.*
(2) *I is a prime ideal and I is comparable with every $K \in \mathcal{C}(\mathcal{M})$.*
(3) *I contains all proper polars.*
(4) *I contains all minimal prime ideals.*
(5) *$a^{\perp} = \{0\}$ for any $a \in M \setminus I$.*
(6) *Every element in $M \setminus I$ is special.*

Definition. Let M be a $DR\ell$-monoid and let $I \in \mathcal{C}(M)$. Then M is called a *lexicographic extension of I* (briefly: *lex-extension of I*), if

a) I is a prime ideal of M;
b) $x > y$ for each $x \in M^{+} \setminus I$ and $y \in I$.

R e m a r k. a) If I is not a principal ideal of M generated by an idempotent element (in particular, $I \neq \{0\}$) then condition a) can be omitted from the definition.

Indeed, let $I \in \mathcal{C}(M)$ satisfy condition b) and let $x, y \in M^{+}$ be such that $x \wedge y \in I$. Suppose $x \in M^{+} \setminus I$ and $y \in M^{+} \setminus I$. Then $x > z$, $y > z$ and hence $x \wedge y \geqslant z$ for each

$z \in I$. Therefore if I is not a principal ideal generated by an idempotent element, then $x \wedge y \notin I$. Thus, by [12, Theorem 2.1], I is a prime ideal.

b) In case M satisfies $(*)$ and $I \in \mathcal{C}(M)$, then M is a lex-extension of I if anyone of the conditions from Theorem 8 holds.

Lex-extensions of ℓ-groups were studied in [3] (see also [1]), of GMV-algebras in [8], of representable commutative $DR\ell$-monoids in [9] and of bounded (non-commutative) $DR\ell$-monoids in [19]. In this paper, we deal with lex-extensions of any $DR\ell$-monoids satisfying condition $(*)$. Under this condition we can recognize the ordering of a $DR\ell$-monoid M which is a lex-extension of its ideal I by means of the orders of I and $M/_l I$. Namely, if M satisfies $(*)$ then by [12] for any $I \in \mathcal{C}(M)$, I is a prime ideal in M if and only if $M/_l I$ is a chain. (Note that the last equivalence is not true for general $DR\ell$-monoids.) Hence let M be a $DR\ell$-monoid satisfying $(*)$. If M is a lex-extension of $I \in \mathcal{C}(M)$ then the lattice M is isomorphic to the lexicographic product of lattices I and $M/_l I$ (since the lexicographic product of lattices A and B is a lattice if and only if B is linearly ordered set). If, moreover, I is a normal ideal then the $DR\ell$-monoid M is isomorphic to the lexicographic product of the $DR\ell$-monoids I and M/I.

4. Values of elements in ideals of $DR\ell$-monoids

Now, we will show connections between values of elements in $DR\ell$-monoids and in their ideals.

Theorem 9. *Let M be a $DR\ell$-monoid and $I \in \mathcal{C}(M)$. Then the assignment $\varphi\colon A \longmapsto A \cap I$ is an isomorphism of the ordered set of all prime ideals of M not containing I onto the ordered set of all proper prime ideals of I. The inverse mapping to φ is the assignment $\psi\colon B \longmapsto \{x \in M;\ |x| \wedge |a| \in B$ for all $a \in I\}$.*

Proof. If A is a prime ideal of M not containing I then $\varphi(A)$ is obviously a prime ideal of I.

Let B be a proper prime ideal of I. Let us denote $\psi(B) = \{x \in M;\ |x| \wedge |a| \in B$ for all $a \in I\}$. Let $y, z \in M^+$ and let $y \notin \psi(B)$ and $z \notin \psi(B)$. Then there exist $a_1, a_2 \in I$ such that $y \wedge |a_1| \notin B$ and $z \wedge |a_2| \notin B$, thus also $y \wedge z \wedge |a_1| \wedge |a_2| \notin B$, and hence $y \wedge z \notin \psi(B)$. Consequently, $\psi(B)$ is a prime ideal of M.

If A is a prime ideal of M then evidently $A \subseteq \psi(A \cap I)$. Let $0 \leqslant x \in \psi(A \cap I)$. Then $x \wedge |a| \in A \cap I$ for each $a \in I$. Let us consider $a \in I \setminus A$. Since A is a prime ideal, $x \in A$. Therefore $\psi(A \cap I) \subseteq A$, and so $\psi(A \cap I) = A$.

If B is a prime ideal of I then $B \subseteq \psi(B) \cap I$. If $0 \leqslant x \in \psi(B) \cap I$ then $|x| \wedge |x| \in B$, hence $x \in B$. Therefore $\psi(B) \cap I = B$.

So we have proved that φ is a bijective mapping of the set of all prime ideals of M not containing I onto the set of all proper prime ideals of I which is an isomorphism of ordered sets, and that $\psi = \varphi^{-1}$. $\square$

Theorem 10. *Let M be a $DR\ell$-monoid, $I \in \mathcal{C}(M)$ and $a \in I$. Then the assignment $P \longmapsto P \cap I$ is a bijection of the set $\operatorname{val}_M(a)$ of all values of a in M onto the set $\operatorname{val}_I(a)$ of all values of a in I.*

P r o o f. Let $I \in \mathcal{C}(M)$ and $a \in I$. Let us consider $P \in \operatorname{val}_M(a)$. Then $P \cap I$ is contained in some value $B \in \operatorname{val}_I(a)$. By Theorem 9, $P \subseteq \varphi^{-1}(B)$ and $a \notin \varphi^{-1}(B)$, thus $\varphi^{-1}(B) = P$. Hence $P \cap I = \varphi^{-1}(B) \cap I = B$, i.e., $P \cap I \in \operatorname{val}_I(a)$.

Conversely, let $C \in \operatorname{val}_I(a)$. Then $a \notin \varphi^{-1}(C)$, hence $\varphi^{-1}(C) \subseteq Q$ for some $Q \in \operatorname{val}_M(a)$. We have $C = \varphi^{-1}(C) \cap I \subseteq Q \cap I$ and $a \notin Q \cap I$, thus $\varphi^{-1}(C) = Q$, that means $\varphi^{-1}(C) \in \operatorname{val}_M(a)$. $\square$

Corollary. *Let M be a $DR\ell$-monoid, $a \in M$, and let $I \in \mathcal{C}(M)$ contain a. Then a is special in M if and only if a is special in I. In particular, a is special in M if and only if $C(a)$ has a greatest proper ideal.*

5. Lex-ideals of $DR\ell$-monoids

Proposition 12. *Let a $DR\ell$-monoid M satisfy condition $(*)$, let $I, J \in \mathcal{C}(M)$ and let $I \subset J$. Then M is a lex-extension of I if and only if M is a lex-extension of J and J is a lex-extension of I.*

P r o o f. The assertion follows from condition (6) of Theorem 8, because by Corollary 11, an element belonging to J is special in J if and only if it is special in M. $\square$

Throughout this section, we will suppose that a $DR\ell$-monoid M satisfies $(*)$.

Definition. a) The join of all proper polars (i.e. different from M) in the lattice $\mathcal{C}(M)$ is called the *lex-kernel* of a $DR\ell$-monoid M. (Notation: $\operatorname{lex} M$.)

b) If $\operatorname{lex} M = M$ then M is called a *lex-simple $DR\ell$-monoid.*

Proposition 13. *If $I \in \mathcal{C}(M)$ then M is a lex-extension of I if and only if $\operatorname{lex} M \subseteq I$.*

P r o o f. It follows from condition (3) of Theorem 8. $\square$

Proposition 14. *The ideal* $\operatorname{lex} M$ *is the greatest ideal in* $\mathcal{C}(M)$ *which is lex-simple.*

P r o o f. Let $I \in \mathcal{C}(M)$, $\operatorname{lex} I = I$. If $\operatorname{lex} M \subset I$, then I is a lex-extension of $\operatorname{lex} M$. Hence $\operatorname{lex} M \subset \operatorname{lex} I$, a contradiction. Therefore by condition (2) of Theorem 8, we have $I \subseteq \operatorname{lex} M$.

Let $J \in \mathcal{C}(M)$, $J \subset \operatorname{lex} M$. If $\operatorname{lex} J \neq J$ then $\operatorname{lex} M$ is a lex-extension of J, and thus M is also a lex-extension of J. Then $J = \operatorname{lex} M$, a contradiction. Hence $\operatorname{lex} J = J$. (From the proof it is also obvious that $\operatorname{lex} M$ is lex-simple.) $\square$

Definition. If $I \in \mathcal{C}(M)$ then I is called a *lex-ideal* of M provided $\operatorname{lex} I \neq I$.

Proposition 15. *If* $a \in M$ *then* a *is a special element in* M *if and only if* $C(a)$ *is a lex-ideal.*

P r o o f. Let an element $a \in M$ be special and let P be its unique value in M. Then by Corollary 11, $P \cap C(a)$ is the unique value of a in $C(a)$, and thus it is the greatest proper ideal in $C(a)$. Hence by Theorem 8 we get that $C(a)$ is a lex-extension of $P \cap C(a)$.

Conversely, let $a \notin \operatorname{lex} C(a)$. Then by Theorem 8, a is special in $C(a)$ and hence, by Corollary 11, a is special in M. $\square$

Proposition 16. *Any two lex-ideals in* M *are either comparable or their intersection is a principal ideal generated by an idempotent element (in a special case they are orthogonal).*

P r o o f. Let I and J be non-comparable ideals of M. Then there exist $0 < a \in I$ and $0 < b \in J$ such that $a \notin J \cup \operatorname{lex} I$ and $b \notin I \cup \operatorname{lex} J$. Since $I \cap J \in \mathcal{C}(I)$, by Theorem 8 we conclude $I \cap J$ is comparable with $\operatorname{lex} I$. If $\operatorname{lex} I \subset I \cap J$, then I is a lex-extension of $I \cap J$, and hence also $I \cap J < a$. Analogously we obtain $I \cap J < b$. Therefore $a \wedge b$ is the greatest element in $I \cap J$. Since $0 \leqslant a \wedge b, a \wedge b \leqslant 2(a \wedge b) \in I \cap J$, thus $a \wedge b = 2(a \wedge b)$, consequently $a \wedge b$ is idempotent in $(M, +)$ and $I \cap J = C(a \wedge b) = \{x \in M;\ |x| \leqslant a \wedge b\}$. If in particular $a \wedge b = 0$ then I and J are orthogonal. $\square$

6. Lex-ideals and lex-extensions of $DR\ell$-monoids with the cancellation property

In this part we will study further properties of lex-ideals and lex-extensions in the class of $DR\ell$-monoids with the cancellation property that satisfy condition $(*)$.

Definition. We say that a $DR\ell$-monoid M possesses *the cancellation property* if

$$x+y=x+z \Longrightarrow y=z \quad \text{and} \quad y+x=z+x \Longrightarrow y=z$$

for each $x,y,z \in M$.

R e m a r k. a) It is obvious that every ℓ-group viewed as a $DR\ell$-monoid possesses the cancellation property.

b) Let G be an ℓ-group and $G^+ = \{x \in G;\ 0 \leqslant x\}$ its positive cone. Put

$$x \rightharpoonup y = (x-y) \vee 0 \quad \text{and} \quad x \leftharpoondown y = (-y+x) \vee 0$$

for any $x,y \in G^+$. Then $G^+ = (G^+, +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ is a $DR\ell$-monoid which also possesses the cancellation property.

Definition. We will say that a $DR\ell$-monoid M is the *(inner) direct product* of its ideals I and J if there is an isomorphism φ of M onto the (external) direct product $I \times J$ of $DR\ell$-monoids I and J such that for each $x \in I$ and each $y \in J$ the relations $\varphi(x) = (x,0)$ and $\varphi(y) = (0,y)$ are valid.

By [18, Theorem 6 and Corollary 8], M is the direct product of $I, J \in \mathcal{C}(M)$ if and only if

1. $I + J = M$, $I \cap J = \{0\}$;
2. $\forall x, x' \in I,\ y, y' \in J;\ x+y = x'+y' \Rightarrow x = x',\ y = y'$.

Definition. If for $I \in \mathcal{C}(M)$ there is $J \in \mathcal{C}(M)$ such that I and J satisfy conditions 1 and 2 then I is called a *direct factor* of M.

By [18, Propositions 2 and 3 and Theorem 6] we have that if M is the direct product of I and J then $I = J^\perp$ and $J = I^\perp$. Therefore now we denote by $(**)$ the following condition for an ideal I of a $DR\ell$-monoids M:

$(**)$ $\quad x+y = x'+y'$ implies $x = x'$ and $y = y'$ for every $x, x' \in I$ and $y, y' \in I^\perp$.

Lemma 17. *Let M be a $DR\ell$-monoid and let $I \in \mathcal{C}(M)$ satisfy condition $(**)$. If there exists an element $0 \leqslant a \in M$ such that $I < a$ and $a \in I + I^\perp$, then M does not possess the cancellation property and $I = C(x_1)$, where x_1 is an idempotent element.*

P r o o f. Let $I \in \mathcal{C}(M)$ satisfy $(**)$ and let $I < a$. Let us suppose that $a \in I + I^\perp$, $a = x_1 + x_2$, where $x_1 \in I$ and $x_2 \in I^\perp$. By [18, Lemma 4], $x_1 \in I^+$ and $x_2 \in \left(I^\perp\right)^+$.

If $x \in I$ then $x + 0 = x < x_1 + x_2$. Since I satisfies condition $(**)$, by [18, Corollary 8] we get that $I + I^\perp$ is isomorphic to the direct product $I \times I^\perp$. Hence $x \leqslant x_1$, and that means x_1 is the greatest element in I.

In the case that M possesses the cancellation property, we have a contradiction. In the other case, x_1 is an idempotent element in M and $I = C(x_1)$. $\square$

Theorem 18. *Let M be a $DR\ell$-monoid with the cancellation property satisfying $(*)$. If I is a lex-ideal in M which satisfies $(**)$ and $0 \leqslant a \in M$, then $a \notin I + I^{\perp}$ if and only if $I < a$.*

Proof. a) If $I < a$ then $a \notin I + I^{\perp}$ by the preceding lemma.

b) Let $0 \leqslant a$ and $a \notin I + I^{\perp}$. Let us suppose that there exists $b \in I$ such that $b \nleqslant a$. Without loss of generality we can suppose that $b \notin \operatorname{lex} I$ and $0 < b$.

By [9, Lemma 1.1.14], in any $DR\ell$-monoid, $x \geqslant y$ implies $x = y + (x \leftharpoondown y)$. Hence in our case we have

$$2b = (2b \wedge a) + (2b \leftharpoondown (2b \wedge a)), \;\; a = (2b \wedge a) + (a \leftharpoondown (2b \wedge a)),$$

and since M satisfies $(*)$, analogously to the proof of Lemma 5 we obtain

$$(2b \leftharpoondown (2b \wedge a)) \wedge (a \leftharpoondown (2b \wedge a)) = 0.$$

Now $0 \leqslant 2b \leftharpoondown (2b \wedge a) \leqslant 2b$ implies $2b \leftharpoondown (2b \wedge a) \in I$. Let us suppose that $2b \leftharpoondown (2b \wedge a) \in \operatorname{lex} I$. Then (because $b \notin \operatorname{lex} I$) $2b \leftharpoondown (2b \wedge a) < b$, therefore

$$b + (2b \leftharpoondown (2b \wedge a)) \leqslant 2b = (2b \wedge a) + (2b \leftharpoondown (2b \wedge a)) \leqslant a + (2b \leftharpoondown (2b \wedge a)).$$

By assumption, M possesses the cancellation property, thus, by [9, Theorem 1.4.7], $b \leqslant a$, a contradiction. Therefore $2b \leftharpoondown (2b \wedge a) \notin \operatorname{lex} I$.

If $0 \leqslant c \in I$ then $(2b \leftharpoondown (2b \wedge a)) \wedge (a \leftharpoondown (2b \wedge a)) \wedge c = 0$, hence (by Theorem 8) $(a \leftharpoondown (2b \wedge a)) \wedge c = 0$. That means $a \leftharpoondown (2b \wedge a) \in I^{\perp}$, and so $a \in I + I^{\perp}$. $\square$

The following theorem is now a consequence of the previous theorem.

Theorem 19. *If M is a $DR\ell$-monoid with the cancellation property satisfying $(*)$ then every $I \in \mathcal{C}(M)$ which is non-bounded and satisfies condition $(**)$ is a direct factor in M.*

Theorem 20. *Let M be a $DR\ell$-monoid with the cancellation property satisfying $(*)$, let $I, J \in \mathcal{C}(M)$, $I \subset J$, and let J satisfy $(**)$. Then the following conditions are equivalent:*

(1) *J is a lex-extension of I.*
(2) *$J^{\perp\perp}$ is a lex extension of I.*
(3) *$a^{\perp} = J^{\perp}$ for each $a \in J \setminus I$.*

Proof. (1) $\Longrightarrow$ (2): Let J be a lex-extension of I. Let us consider $0 < a \in J^{\perp\perp} \setminus J$. Since in $\mathcal{C}(M)$

$$(J \vee J^{\perp}) \cap J^{\perp\perp} = (J \cap J^{\perp\perp}) \vee (J^{\perp} \cap J^{\perp\perp}) = J,$$

we get $a \notin J \vee J^{\perp}$, and thus $a \notin J + J^{\perp}$. Hence by Theorem 18, $J < a$. This means $J^{\perp\perp}$ is a lex-extension of J, and therefore, by Theorem 8, $J^{\perp\perp}$ is also a lex-extension of I.

(2) $\Longrightarrow$ (1): Follows from Proposition 12.

(1) $\Longrightarrow$ (3): Let J be a lex-extension of I. If $0 < a \in J \setminus I$ then $J^{\perp} \subseteq a^{\perp}$. Let $0 \leqslant x \in a^{\perp}$. Then for each $0 \leqslant y \in J$ we have $a \wedge x \wedge y = 0$, thus $x \wedge y = 0$ and hence $a^{\perp} \subseteq J^{\perp}$.

(3) $\Longrightarrow$ (1): Let $b^{\perp} = J^{\perp}$ for every $b \in J \setminus I$. Let $0 < a \in J \setminus I$. Let us consider $x \in J$ such that $x \wedge a = 0$. Then $x \in J \cap a^{\perp} = J \cap J^{\perp} = \{0\}$, hence $x = 0$. □

As a consequence we obtain the following theorem.

Theorem 21. *If M is a $DR\ell$-monoid with the cancellation property satisfying $(*)$ and if $I \in \mathcal{C}(M)$ then every lex-extension J of I which satisfies $(**)$ is contained in the unique maximal lex-extension $J^{\perp\perp}$.*

Proposition 22. *Let M be a $DR\ell$-monoid with the cancellation property satisfying $(*)$, let $I, J \in \mathcal{C}(M)$, and let J be a lex-extension of I. Then $I^{\perp} = J^{\perp}$.*

Proof. Let $0 \leqslant x \in I$, $0 \leqslant y \in I^{\perp}$ and $0 \leqslant z \in J \setminus I$. If $y \wedge z \notin I$ then $y \wedge z \geqslant x$, hence $x = x \wedge y \wedge z = x \wedge y = 0$, a contradiction. Thus $y \wedge z \in I \cap I^{\perp} = \{0\}$. That means $I^{\perp} \subseteq J^{\perp}$, and so $I^{\perp} = J^{\perp}$. □

The following theorem is now an immediate consequence.

Theorem 23. *Let M be a $DR\ell$-monoid with the cancellation property satisfying $(*)$. If $\{0\} \neq I \in \mathcal{C}(M)$ satisfies $(**)$ and has a proper lex-extension then the lex-extensions of I form a chain with the greatest element $I^{\perp\perp}$.*

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Authors' addresses: *J. Rachůnek*, Department of Algebra and Geometry, Faculty of Sciences, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: `rachunek@inf.upol.cz`; *D. Šalounová*, Department of Mathematical Methods in Economy, Faculty of Economics, VŠB-Technical University Ostrava, Sokolská 33, 701 21 Ostrava, Czech Republic, e-mail: `dana.salounova@vsb.cz`.