OPERATORS ON GMV-ALGEBRAS

Filip Švrček, Olomouc

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Abstract. Closure GMV-algebras are introduced as a commutative generalization of closure MV-algebras, which were studied as a natural generalization of topological Boolean algebras.

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1. Introduction

It is well known that Boolean algebras are algebraic counterparts of the classical propositional two-valued logic similarly as MV-algebras (see [1], [2]) are for Łukasiewicz infinite valued logic. Every MV-algebra contains a Boolean algebra, which is formed by the set of its idempotent elements. The same property is possessed also by GMV-algebras, the non-commutative generalization of MV-algebras (see [5] or [9]).

In the paper [11], closure MV-algebras are introduced and studied as a natural generalization of topological Boolean algebras (see [12]). The additive closure operator is here introduced as a natural generalization of the topological closure operator on topological Boolean algebras. The aim of this paper is to generalize the results of [11] to the case of GMV-algebras.

The paper is divided into Introduction and three main sections. In Section 2, the closure GMV-algebras are introduced and the relation between additive closure operators and multiplicative interior operators on GMV-algebras is described. In the case of closure MV-algebras there is a one-to-one correspondence between additive closure operators and multiplicative interior operators. In the paper, it is shown that this correspondence exists also for closure GMV-algebras, but the relation is there a little bit different.

In Section 3 one works with idempotent elemets of a closure GMV-algebra, for example, it is shown that every idempotent element of a closure GMV-algebra induces a new closure GMV-algebra, similarly as is the case for closure MV-algebras.

Finally, in the last section GMV-algebras are factorized via their normal ideals and the connections between congruences and normal c-ideals of closure GMV-algebras are described with help of DRl-monoids, which are studied in [6] or in [13].

2. Closure GMV-algebras

Definition 1. An algebra $\mathscr{A} = (A, \oplus, \neg, \sim, 0, 1)$ of signature $\langle 2, 1, 1, 0, 0\rangle$ is called a *GMV-algebra*, iff the following conditions are satisfied for each $x, y, z \in A$:

(GMV1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$,
(GMV2) $x \oplus 0 = 0 = 0 \oplus x$,
(GMV3) $x \oplus 1 = 1 = 1 \oplus x$,
(GMV4) $\sim 1 = 0$, $\neg 1 = 0$,
(GMV5) $\sim(\neg x \oplus \neg y) = \neg(\sim x \oplus \sim y)$,
(GMV6) $y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y = x \oplus (y \odot \sim x) = (\neg x \odot y) \oplus x$,
(GMV7) $y \odot (x \oplus \sim y) = (\neg y \oplus x) \odot y$,
(GMV8) $\sim(\neg x) = x$,

where $x \odot y := \sim(\neg x \oplus \neg y)$.

R e m a r k 1. We can define the relation of the partial order $\leqslant$ on every GMV-algebra $\mathscr{A}$. We put

$$x \leqslant y \Leftrightarrow \neg x \oplus y = 1 \qquad \forall x, y \in A.$$

Then $(A, \leqslant)$ is a distributive lattice, where each x, y satisfy

- $x \vee y = y \oplus (x \odot \sim y) = (\neg y \odot x) \oplus y$,
- $x \wedge y = y \odot (x \oplus \sim y) = (\neg y \oplus x) \odot y$.

Definition 2. An algebraic structure $G = (G, +, 0, \vee, \wedge)$ of signature $\langle 2, 0, 2, 2\rangle$ is called an *l-group* iff

1. $(G, +, 0)$ is a group,
2. $(G, \vee, \wedge)$ is a lattice,
3. $x + (y \vee z) + w = (x + y + w) \vee (x + z + w) \qquad \forall x, y, z, w \in G$,
   $x + (y \wedge z) + w = (x + y + w) \wedge (x + z + w) \qquad \forall x, y, z, w \in G$.

An element $u \in G$ $(u > 0)$ is said to be a *strong unit* of an l-group G iff

$$(\forall a \in G)(\exists n \in \mathbb{N})(a \leqslant nu),$$

where $nu \overset{\text{def}}{=} \underbrace{u + u + \ldots + u}_{n}$.

If an l-group G contains a strong unit u, then we call it a *unital l-group* and write (G, u).

Let $\leqslant$ be the lattice order on $(G, \vee, \wedge)$. Then for the l-group G we can use notation $G = (G, +, 0, \leqslant)$, which is equivalent to the former notation.

Remark 2.

a) Let $(G, +, 0, \leqslant)$ be an l-group and let u be a strong unit of G. If we put

$$x \oplus y := (x + y) \wedge u, \qquad \neg x := u - x, \qquad \sim x := -x + u,$$

then $\Gamma(G, u) = ([0, u], \oplus, \neg, \sim, 0, u)$ is a GMV-algebra.

b) On the other hand, A. Dvurečenskij has shown that for each GMV-algebra $\mathscr{A}$ there exists a unital l-group (G, u) such that $\mathscr{A} \cong \Gamma(G, u)$—see [4].

We can now define the additive closure and the multiplicative interior operator in the same way as for the MV-algebras. From [12], Theorem 5 and Theorem 6, we know that additive closure operators on an MV-algebra $\mathscr{A}$ generalize topological closure operators on the Boolean algebra $B(\mathscr{A})$ of its idempotent elements.

Definition 3.

a) Let $\mathscr{A} = (A, \oplus, \neg, \sim, 0, 1)$ be a GMV-algebra and $\mathrm{Cl}\colon A \to A$ a mapping. Then Cl is called an *additive closure operator* on $\mathscr{A}$ iff for each $a, b \in A$
   1. $\mathrm{Cl}(a \oplus b) = \mathrm{Cl}(a) \oplus \mathrm{Cl}(b)$;
   2. $a \leqslant \mathrm{Cl}(a)$;
   3. $\mathrm{Cl}(\mathrm{Cl}(a)) = \mathrm{Cl}(a)$;
   4. $\mathrm{Cl}(0) = 0$.

b) If Cl is an additive closure operator on $\mathscr{A}$ then $\mathscr{A} = (A, \oplus, \neg, \sim, 0, 1, \mathrm{Cl})$ is called a *closure GMV-algebra* and $\mathrm{Cl}(a)$ is called the *closure* of an element $a \in A$. An element a is said to be *closed* iff $\mathrm{Cl}(a) = a$.

Definition 4.

a) Let $\mathscr{A} = (A, \oplus, \neg, \sim, 0, 1)$ be a GMV-algebra and $\mathrm{Int}\colon A \to A$ a mapping. Then Int is called a *multiplicative interior operator* on $\mathscr{A}$ if and only if for each $a, b \in A$
   $1'$. $\mathrm{Int}(a \odot b) = \mathrm{Int}(a) \odot \mathrm{Int}(b)$;
   $2'$. $\mathrm{Int}(a) \leqslant a$;
   $3'$. $\mathrm{Int}(\mathrm{Int}(a)) = \mathrm{Int}(a)$;
   $4'$. $\mathrm{Int}(1) = 1$.

b) If Int is a multiplicative interior operator on $\mathscr{A}$, then an algebra $\mathscr{A} = (A, \oplus, \neg, \sim, 0, 1, \mathrm{Int})$ is called an *interior GMV-algebra* and $\mathrm{Int}(a)$ is called the *interior* of an element $a \in A$. An element a is said to be *open* iff $\mathrm{Int}(a) = a$.

Lemma 1. *Let $\mathscr{A} = (A, \oplus, \neg, \sim, 0, 1, \mathrm{Cl})$ be a closure GMV-algebra. We put*

a) $\mathrm{Int}^{\neg}(a) = \neg\mathrm{Cl}(\sim a)$,

b) $\mathrm{Int}^{\sim}(a) = \sim\mathrm{Cl}(\neg a)$

for each $a \in A$. Then these two operators are multiplicative interior operators on $\mathscr{A}$ and for each $a, b \in A$ we have

a) $\mathrm{Cl}(a) = \sim\mathrm{Int}^{\neg}(\neg a)$,

b) $\mathrm{Cl}(a) = \neg\mathrm{Int}^{\sim}(\sim a)$.

Proof. We restrict ourselves to the case a), since b) can be proved analogously.

1′. $\mathrm{Int}^{\neg}(a\odot b) = \neg\mathrm{Cl}(\sim(a\odot b)) = \neg\mathrm{Cl}(\sim a\oplus\sim b) = \neg(\mathrm{Cl}(\sim a)\oplus\mathrm{Cl}(\sim b)) = \neg\mathrm{Cl}(\sim a)\odot \neg\mathrm{Cl}(\sim b) = \mathrm{Int}^{\neg}(a) \odot \mathrm{Int}^{\neg}(b)$;

2′. $\mathrm{Int}^{\neg}(a) = \neg\mathrm{Cl}(\sim a) \leqslant \neg\sim a = a$;

3′. $\mathrm{Int}^{\neg}(\mathrm{Int}^{\neg}(a)) = \neg\mathrm{Cl}(\sim\neg\mathrm{Cl}(\sim a)) = \neg\mathrm{Cl}(\mathrm{Cl}(\sim a)) = \neg\mathrm{Cl}(\sim a) = \mathrm{Int}^{\neg}(a)$;

4′. $\mathrm{Int}^{\neg}(1) = \neg\mathrm{Cl}(\sim 1) = \neg\mathrm{Cl}(0) = \neg 0 = 1$. □

The next lemma shows that the operator Cl from Definition 3 and the operators $\mathrm{Int}^{\sim}$, $\mathrm{Int}^{\neg}$ from Lemma 1 are all isotone.

Lemma 2. *If $a \leqslant b$ for any $a, b \in A$, then $\mathrm{Cl}(a) \leqslant \mathrm{Cl}(b)$ and $\mathrm{Int}^{\neg}(a) \leqslant \mathrm{Int}^{\neg}(b)$, as well as $\mathrm{Int}^{\sim}(a) \leqslant \mathrm{Int}^{\sim}(b)$.*

Proof. Let $a \leqslant b$. Then $\mathrm{Cl}(b) = \mathrm{Cl}(a \vee b) = \mathrm{Cl}(a \oplus (b \odot \sim a))$. Therefore $\mathrm{Cl}(b) = \mathrm{Cl}(a) \oplus \mathrm{Cl}(b \odot \sim a) \geqslant \mathrm{Cl}(a) \vee \mathrm{Cl}(b \odot \sim a)$, and so $\mathrm{Cl}(a) \leqslant \mathrm{Cl}(b)$.

Similarly from $a \leqslant b$ we have $\mathrm{Int}^{\sim}(a) = \mathrm{Int}^{\sim}(a \wedge b) = \mathrm{Int}^{\sim}(b \odot (a \oplus \sim b)) = \mathrm{Int}^{\sim}(b) \odot \mathrm{Int}^{\sim}(a \oplus \sim b) \leqslant \mathrm{Int}^{\sim}(b) \wedge \mathrm{Int}^{\sim}(a \oplus \sim b)$, hence $\mathrm{Int}^{\sim}(a) \leqslant \mathrm{Int}^{\sim}(b)$ and analogously for $\mathrm{Int}^{\neg}$. □

In the case of closure MV-algebras, here we were able to construct from one closure operator just one interior operator by the rule $\mathrm{Int}(x) = \neg\mathrm{Cl}(\neg x)$ and then get back to the original one. Now, let us try to describe the situation for closure GMV-algebras.

Remark 3. Let us consider a closure GMV-algebra $\mathscr{A}$ and a mapping $f\colon A \to A$. We can define two new operators $\Phi^{\neg}(f)$ and $\Phi^{\sim}(f)$ on A by the reles $\Phi^{\neg}(f)(a) = \neg f(\sim a)$ and $\Phi^{\sim}(f)(a) = \sim f(\neg a)$. Then we clearly have that $\Phi^{\neg}\circ\Phi^{\sim} = \mathrm{id} = \Phi^{\sim}\circ\Phi^{\neg}$ and if we take an additive closure operator Cl on $\mathscr{A}$ instead of the arbitrary mapping f on $\mathscr{A}$, then (by Lemma 1) we see that there exists a one-to-one correspondence between the aditive closure operators and the multiplicative interior operators on the closure GMV-algebras. Compared to closure MV-algebras, the relation is here a little bit different as we are going to show.

Let us denote for each even non-negative integer i and for an operator Cl_0

$$\mathrm{Cl}_i^{\neg} = \underbrace{\Phi^{\neg} \circ \ldots \circ \Phi^{\neg}}_{i}(\mathrm{Cl}_0),$$
$$\mathrm{Cl}_i^{\sim} = \underbrace{\Phi^{\sim} \circ \ldots \circ \Phi^{\sim}}_{i}(\mathrm{Cl}_0)$$

and for each odd non-negative integer i

$$\mathrm{Int}_i^{\neg} = \underbrace{\Phi^{\neg} \circ \ldots \circ \Phi^{\neg}}_{i}(\mathrm{Cl}_0),$$
$$\mathrm{Int}_i^{\sim} = \underbrace{\Phi^{\sim} \circ \ldots \circ \Phi^{\sim}}_{i}(\mathrm{Cl}_0).$$

The following theorem is an easy consequence of the preceding Remark 3 and of Lemma 1.

Theorem 3. *Let Cl_0 be an additive closure operator on a GMV-algebra $\mathscr{A}$. Then we have for each $k \in \mathbb{N} \cup \{0\}$*

a) *$\mathrm{Cl}_{2k}^{\neg}$ and $\mathrm{Cl}_{2k}^{\sim}$ are additive closure operators on $\mathscr{A}$;*

b) *$\mathrm{Int}_{2k+1}^{\neg}$ and $\mathrm{Int}_{2k+1}^{\sim}$ are multiplicative interior operators on $\mathscr{A}$.*

3. Idempotent elements of closure GMV-algebras

Now, we can consider the set $B(\mathscr{A}) = \{a \in A;\ a \oplus a = a\}$ of additively idempotent elements of a GMV-algebra $\mathscr{A}$. One can show that $B(\mathscr{A})$ is just the set of multiplicatively idempotent elements in $\mathscr{A}$. $B(\mathscr{A})$ is a sublattice of the lattice $(A, \vee, \wedge)$, contains 0 a 1 and is also a Boolean algebra. Analogously as for MV-algebras one can show that the operations $\oplus$, $\odot$ coincide on $B(\mathscr{A})$ with the lattice operations $\vee$, $\wedge$—see [10].

Lemma 4. *Let $\mathscr{A}$ be a GMV-algebra and let a be an idempotent element in $\mathscr{A}$. Then*

a) $y \odot a = a \odot y = a \wedge y$,

b) $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$,

c) $(x \oplus y) \odot a = (x \odot a) \oplus (y \odot a)$

for each $x, y \in A$.

P r o o f. a) Let $y \leqslant a$. Then $a \leqslant y \oplus a \leqslant a \oplus a = a$, thus $y \oplus a = a$ and hence, by [9], Theorem 7, $y \odot a = y = y \wedge a$.

Let now $y \in A$ be arbitrary. Clearly $y \odot a \leqslant y, a$. Let $z \in A, z \leqslant y, a$. Then also $z = z \odot a \leqslant y \odot a$, and consequently $y \odot a = y \wedge a$. Similarly $a \odot y = a \wedge y$.

b) Let $a \in B(\mathscr{A})$. Using distributivity of "$\oplus$" over "$\wedge$" we obtain

$$(a \wedge x) \oplus (a \wedge y) = (a \oplus a) \wedge (x \oplus a) \wedge (a \oplus y) \wedge (x \oplus y),$$

hence by a), $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$.

c) Analogously to the case b). □

Similarly as for closure MV-algebras, we can show that every idempotent element a in a closure GMV-algebra $\mathscr{A}$ determines a new closure GMV-algebra on the interval $[0, a]$.

Theorem 5. *Let $\mathscr{A} = (A, \oplus, \neg, \sim, 0, 1, \mathrm{Cl})$ be a closure GMV-algebra and let a be an idempotent element in $\mathscr{A}$. We put*

- $x \oplus_a y = x \oplus y$,
- $\neg_a x = \neg(x \oplus \sim a)$,
- $\sim_a x = \sim(\neg a \oplus x)$,
- $0_a = 0$,
- $1_a = a$,
- $\mathrm{Cl}_a(x) = a \odot \mathrm{Cl}(x)$

for each $x, y \in A$. Then $\mathscr{A}_a = ([0, a], \oplus_a, \neg_a, \sim_a, 0_a, 1_a, \mathrm{Cl}_a)$ is a closure GMV-algebra and we have

- $x \odot_a y = x \odot y$,
- $\mathrm{Int}_a^{\neg}(x) = a \odot \mathrm{Int}^{\neg}(\neg a \oplus x)$,
- $\mathrm{Int}_a^{\sim}(x) = a \odot \mathrm{Int}^{\sim}(x \oplus \sim a)$.

P r o o f. Availability of axioms (GMV1)–(GMV8) from Definition 1 for the algebra $([0, a], \oplus_a, \neg_a, \sim_a, 0, a)$ are proved in [9], so $\mathscr{A}_a$ is a GMV-algebra. In the second part of the proof we need to show that Cl_a is an additive closure operator on $\mathscr{A}_a$.

1. $\mathrm{Cl}_a(x \oplus y) = a \odot \mathrm{Cl}(x \oplus y) = a \odot (\mathrm{Cl}(x) \oplus \mathrm{Cl}(y)) = (a \odot \mathrm{Cl}(x)) \oplus (a \odot \mathrm{Cl}(y)) = \mathrm{Cl}_a(x) \oplus \mathrm{Cl}_a(y)$;
2. $\mathrm{Cl}_a(x) = a \odot \mathrm{Cl}(x) \geqslant a \odot x = a \wedge x = x$;
3. $\mathrm{Cl}_a(\mathrm{Cl}_a(x)) = a \odot \mathrm{Cl}(a \odot \mathrm{Cl}(x)) \leqslant a \odot \mathrm{Cl}(\mathrm{Cl}(x)) = a \odot \mathrm{Cl}(x) = \mathrm{Cl}_a(x)$; on the other hand, according to 2 we get $\mathrm{Cl}_a(x) = a \odot \mathrm{Cl}(x) \leqslant \mathrm{Cl}_a(a \odot \mathrm{Cl}(x)) = \mathrm{Cl}_a(\mathrm{Cl}_a(x))$, so, together we have $\mathrm{Cl}_a(\mathrm{Cl}_a(x)) = \mathrm{Cl}_a(x)$;
4. $\mathrm{Cl}_a(0) = a \odot \mathrm{Cl}(0) = a \odot 0 = a \wedge 0 = 0$.

Further, $\mathrm{Int}_a^{\neg}(x) = \neg_a \mathrm{Cl}_a(\sim_a x) = \neg((a \odot \mathrm{Cl}(\sim(\neg a \oplus x))) \oplus \sim a) = (\neg a \oplus \neg \mathrm{Cl}(\sim(\neg a \oplus x))) \odot a = (\neg a \oplus \mathrm{Int}^{\neg}(\neg a \oplus x)) \odot a = \mathrm{Int}^{\neg}(\neg a \oplus x) \wedge a = a \odot \mathrm{Int}^{\neg}(\neg a \oplus x)$. Analogously for $\mathrm{Int}_a^{\sim}$. □

Corollary 6. *Let $\mathscr{A}$ be a GMV-algebra and $a \in A$ an idempotent element. Then a mapping h given by the formula $h(x) = a \odot x$ for each $x \in A$ is a homomorphism from $\mathscr{A}$ onto $\mathscr{A}_a$.*

Proof. Let $x, y \in A$. Then

$$h(x \odot y) = a \odot (x \odot y) = a \odot a \odot (x \odot y) = a \odot (a \odot x) \odot y.$$

By Lemma 4a) we have

$$a \odot (a \odot x) \odot y = a \odot (x \odot a) \odot y = (a \odot x) \odot (a \odot y) = h(x) \odot_a h(y).$$

Further,

- $h(\sim_a x) = a \odot \sim x = a \wedge \sim x = \sim x \wedge a = a \odot (\sim x \oplus \sim a) = a \odot \sim(x \odot a) = a \odot \sim(a \odot x) = a \odot \sim h(x) = \sim(\neg a \oplus h(x)) = \sim_a h(x)$,
- $h(\neg_a x) = a \odot \neg x = a \wedge \neg x = \neg x \wedge a = (\neg a \oplus \neg x) \odot a = \neg(a \odot x) \odot a = \neg h(x) \odot a = \neg(h(x) \oplus \sim a) = \neg_a h(x)$,
- $h(0) = 0 = 0_a$

and finally

- $h(x \oplus y) = h(\sim(\neg x \oplus \neg y)) = \sim_a h(\neg x \odot \neg y) = \sim_a (h(\neg x) \odot_a h(\neg y)) = \sim_a (\neg_a h(x) \odot_a \neg_a h(y)) = h(x) \oplus_a h(y)$.

So h is a homomorphism from the GMV-algebra $\mathscr{A}$ into the GMV-algebra $\mathscr{A}_a$ and since $x = a \odot x = h(x)$ for each $x \in [0, a]$, h is surjective. $\square$

Definition 5. Let $\mathscr{A}_1 = (A_1, \oplus_1, \neg_1, \sim_1, 0_1, 1_1, \mathrm{Cl}_1)$ and $\mathscr{A}_2 = (A_2, \oplus_2, \neg_2, \sim_2, 0_2, 1_2, \mathrm{Cl}_2)$ be closure GMV-algebras and let $h\colon A_1 \to A_2$ be a homomorphism from $\mathscr{A}_1$ into $\mathscr{A}_2$. Then h is said to be a *c-homomorphism* from $\mathscr{A}_1$ into $\mathscr{A}_2$ iff

(C1) $h(\mathrm{Cl}_1(x)) = \mathrm{Cl}_2(h(x))$

for each $x \in A_1$.

Lemma 7. *Let us consider closure GMV-algebras $\mathscr{A}_1$ and $\mathscr{A}_2$. A homomorphism h from the GMV-algebra $\mathscr{A}_1$ into the GMV-algebra $\mathscr{A}_2$ is a c-homomorphism from $\mathscr{A}_1$ into $\mathscr{A}_2$ if and only if one of the following two equivalent conditions is satisfied:*

(C2) $h(\mathrm{Int}_1^{\neg}(x)) = \mathrm{Int}_2^{\neg}(h(x))$,

(C3) $h(\mathrm{Int}_1^{\sim}(x)) = \mathrm{Int}_2^{\sim}(h(x))$

for each $x \in A_1$.

Proof. A homomorphism h from $\mathscr{A}_1$ into $\mathscr{A}_2$ is a c-homomorphism iff

$$h(\mathrm{Cl}_1(x)) = \mathrm{Cl}_2(h(x))$$

for each $x \in A_1$, so for $\neg_1 x$, too. From the last equation we get

$$\sim_2 h(\mathrm{Cl}_1(\neg_1 x)) = \sim_2 \mathrm{Cl}_2(h(\neg_1 x)).$$

Since h is a homomorphism from $\mathscr{A}_1$ into $\mathscr{A}_2$, we have got $h(\neg_1 x) = \neg_2 h(x)$ and also $h(\sim_1 x) = \sim_2 h(x)$ for each $x \in A_1$. Therefore we can write instead of the last equation

$$h(\sim_1 \mathrm{Cl}_1(\neg_1 x)) = \sim_2 \mathrm{Cl}_2(\neg_2 h(x)),$$

which is equivalent to the axiom (C3), thus

$$h(\mathrm{Int}_1^{\sim}(x)) = \mathrm{Int}_2^{\sim}(h(x)).$$

The equivalence of the conditions (C1), (C2) we can be proved analogously. $\square$

The following theorem refers to Theorem 5 and Corollary 6 and completes our description of the relation of closure GMV-algebras $\mathscr{A} = (A, \oplus, \neg, \sim, 0, 1, \mathrm{Cl})$ and $\mathscr{A}_a = ([0,a], \oplus_a, \neg_a, \sim_a, 0_a, 1_a, \mathrm{Cl}_a)$.

Theorem 8. *Let $\mathscr{A}$ be a closure GMV-algebra and let a be its idempotent element, which is open to at least one of multiplicative interior operators* $\mathrm{Int}^{\neg}$ *and* $\mathrm{Int}^{\sim}$ *on $\mathscr{A}$. Finally, let $h\colon A \to [0,a]$ be a mapping such that $h(x) = a \odot x$ for each $x \in A$. Then h is a surjective c-homomorphism $\mathscr{A}$ onto $\mathscr{A}_a$.*

P r o o f. Let us consider a mapping $h\colon A \to [0,a]$ such that $h(x) = a \odot x$ for each $x \in A$. We know from Lemma 6 that h is a surjective homomorphism of GMV-algebras $\mathscr{A}$ and $\mathscr{A}_a$.

We need to show now that h is a c-homomorphism. Let a be open for example with respect to $\mathrm{Int}^{\sim}$. Then it is enough to check availability of the condition (C3) from Lemma 7. For each $x \in A$ we have

$$h(\mathrm{Int}^{\sim}(x)) = a \odot \mathrm{Int}^{\sim}(x) = \mathrm{Int}^{\sim}(a) \odot \mathrm{Int}^{\sim}(x) = \mathrm{Int}^{\sim}(a \odot x) = \mathrm{Int}^{\sim}(h(x)).$$

Let $y \leqslant a$. Then

$$\mathrm{Int}^{\sim}(y) = \mathrm{Int}^{\sim}(a \wedge y) = \mathrm{Int}^{\sim}(a \odot (y \oplus \sim a)) = a \odot \mathrm{Int}^{\sim}(y \oplus \sim a) = \mathrm{Int}_a^{\sim}(y).$$

Altogether we have

$$h(\mathrm{Int}^{\sim}(x)) = \mathrm{Int}^{\sim}(h(x)) = \mathrm{Int}_a^{\sim}(h(x))$$

for each $x \in A$. $\square$

N o t e. If a is open with respect to $\mathrm{Int}^{\neg}$, then we check availability of the condition (C2) from Lemma 7.

4. Factorization on closure GMV-algebras

Definition 6. Let us consider a GMV-algebra $\mathscr{A}$. Then a set $I \subset A$, $\emptyset \neq I$ is called an *ideal* of the GMV-algebra $\mathscr{A}$ iff

(I1) $0 \in I$;

(I2) if $x, y \in I$, then $x \oplus y \in I$;

(I3) if $x \in I, y \in A$ a $y \leqslant x$, then $y \in I$.

An ideal I of a GMV-algebra $\mathscr{A}$ is called a *normal ideal* iff for each $x, y \in A$

(I4) $\neg x \odot y \in I \Leftrightarrow y \odot \sim x \in I$.

Definition 7. A normal ideal I of a closure GMV-algebra $\mathscr{A}$ is called a *normal c-ideal* iff $\mathrm{Cl}(a) \in I$ for each $a \in I$.

Remark 4. Normal ideals of GMV-algebra $\mathscr{A}$ are in a one-to-one correspondence with congruences on $\mathscr{A}$.

a) If $\equiv$ is a congruence on $\mathscr{A}$, then $0/{\equiv} = \{x \in A;\ x \equiv 0\}$ is a normal ideal of $\mathscr{A}$.

b) Let H be a normal ideal of $\mathscr{A}$. The relation $\equiv_H$, where

$$x \equiv_H y \Longleftrightarrow (\neg y \odot x) \oplus (\neg x \odot y) \in H,$$

or equivalently

$$x \equiv_H y \Longleftrightarrow (y \odot \sim x) \oplus (x \odot \sim y) \in H,$$

is a congruence on $\mathscr{A}$ and $H = \{x \in A;\ x \equiv_H 0\} = 0/{\equiv_H}$ holds.

More detail is found in [5].

Note.

a) We denote by $\mathscr{A}/I = \mathscr{A}/{\equiv_I}$ the factor GMV-algebra of a GMV-algebra $\mathscr{A}$ according to a congruence $\equiv_I$ on $\mathscr{A}$ and by $\overline{x}$ the class of A/I which contains the element x.

b) Let $\mathscr{A}$ be a closure GMV-algebra and let I be its normal c-ideal. Let us put $\mathrm{Cl}_I(\overline{x}) := \overline{\mathrm{Cl}(x)}$ for each $x \in A$. This definition of the operator Cl_I is correct as we will show in the proof of Theorem 9.

Remark 5. A *DRl*-monoid is an algebraic structure $\mathscr{A} = (A, +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of signature $\langle 2, 0, 2, 2, 2, 2\rangle$, where $(A, +, 0)$ is a monoid, $(A, \vee, \wedge)$ is a lattice, $(A, +, \vee, \wedge, 0)$ is a lattice ordered monoid and the operations $\rightharpoonup$ and $\leftharpoondown$ are left and right dual residuations—see e.g. [6].

There are mutual relations between GMV-algebras and *DRl*-monoids which are described in [9], Theorems 12, 13.

Theorem 9. *Let $\mathscr{A}$ be a closure GMV-algebra and let I be its normal c-ideal. Then the factor GMV-algebra $\mathscr{A}/I$ endowed with the operator Cl_I from the preceding Note* b) *is a closure GMV-algebra.*

P r o o f. Let us consider $x \equiv_I y$. Then $(\neg x \odot y) \oplus (\neg y \odot x) \in I$, therefore $\neg x \odot y, \neg y \odot x \in I$ and $\mathrm{Cl}(\neg x \odot y), \mathrm{Cl}(\neg y \odot x) \in I$. Further we have

$$\mathrm{Cl}(\neg y \odot x) \oplus \mathrm{Cl}(y) = \mathrm{Cl}((\neg y \odot x) \oplus y) = \mathrm{Cl}(x \vee y) \geqslant \mathrm{Cl}(x).$$

Since $\mathscr{A}$ is actually a DRl-monoid, we get

$$\mathrm{Cl}(\neg y \odot x) \geqslant \mathrm{Cl}(x) \rightharpoonup \mathrm{Cl}(y) = \neg\mathrm{Cl}(y) \odot \mathrm{Cl}(x).$$

So we have $\neg\mathrm{Cl}(y) \odot \mathrm{Cl}(x) \in I$, since $\mathrm{Cl}(\neg y \odot x) \in I$. We can show analogously that $\neg\mathrm{Cl}(x)\odot\mathrm{Cl}(y) \in I$. Therefore we can see that $(\neg\mathrm{Cl}(x)\odot\mathrm{Cl}(y))\oplus(\neg\mathrm{Cl}(y)\odot\mathrm{Cl}(x)) \in I$, so $\mathrm{Cl}(x) \equiv_I \mathrm{Cl}(y)$, and the operation Cl_I is therefore correctly defined on A/I. Moreover, $\mathrm{Cl}_I\colon A/I \to A/I$ satisfies axioms 1–4 from Definition 3, because

1. $\mathrm{Cl}_I(\bar a \oplus \bar b) = \mathrm{Cl}_I(\overline{a \oplus b}) = \overline{\mathrm{Cl}(a \oplus b)} = \overline{\mathrm{Cl}(a) \oplus \mathrm{Cl}(b)} = \overline{\mathrm{Cl}(a)} \oplus \overline{\mathrm{Cl}(b)} = \mathrm{Cl}_I(\bar a) \oplus \mathrm{Cl}_I(\bar b)$,
2. $\mathrm{Cl}_I(\bar a) = \overline{\mathrm{Cl}(a)} \geqslant \bar a$,
3. $\mathrm{Cl}_I(\mathrm{Cl}_I(\bar a)) = \mathrm{Cl}_I(\overline{\mathrm{Cl}(a)}) = \overline{\mathrm{Cl}(\mathrm{Cl}(a))} = \overline{\mathrm{Cl}(a)} = \mathrm{Cl}_I(\bar a)$,
4. $\mathrm{Cl}_I(\bar 0) = \overline{\mathrm{Cl}(0)} = \bar 0$. □

Corollary 10. *There is a one-to-one correspondence between the normal c-ideals and the congruences of the closure GMV-algebras.*

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Author's address: *Filip Švrček*, Department of Algebra and Geometry, Faculty of Science, Palacký University, Tomkova 40, 779 00 Olomouc, Czech Republic, e-mail: `svrcekf@seznam.cz`.