ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF NONLINEAR DIFFERENCE EQUATIONS

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Abstract. The nonlinear difference equation

$$x_{n+1} - x_n = a_n \varphi_n(x_{\sigma(n)}) + b_n, \tag{E}$$

where $(a_n), (b_n)$ are real sequences, $\varphi_n\colon \mathbb{R} \longrightarrow \mathbb{R}$, $(\sigma(n))$ is a sequence of integers and $\lim\limits_{n\longrightarrow\infty} \sigma(n) = \infty$, is investigated. Sufficient conditions for the existence of solutions of this equation asymptotically equivalent to the solutions of the equation $y_{n+1} - y_n = b_n$ are given. Sufficient conditions under which for every real constant there exists a solution of equation (E) convergent to this constant are also obtained.

Keywords: difference equation, asymptotic behavior

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1. Introduction

In this paper we are concerned with the asymptotic behavior of solutions of nonlinear difference equations of the form

$$x_{n+1} - x_n = a_n \varphi_n(x_{\sigma(n)}) + b_n,\ n = 1, 2, \ldots \tag{E}$$

where (a_n), (b_n) are real sequences, $\varphi_n\colon \mathbb{R} \longrightarrow \mathbb{R}$, $(\sigma(n))$ is a sequence of integers and $\lim\limits_{n\longrightarrow\infty} \sigma(n) = \infty$.

By a solution of equation (E) we mean a real sequence (x_n) defined for $n \geqslant \min\limits_{i\geqslant 1} \sigma(i)$ which satisfies (E) for all sufficiently large n. Equations of the form (E), in particular when $\sigma(n) = n - k$, have been studied by a number of authors, see for example [2–5], [7–12] and the references cited therein. However, in most of these papers the oscillation of equation (E) has been investigated.

Our purpose in this paper is to study asymptotic properties of solutions of equation (E). We will give sufficient conditions for the existence of solutions of equation (E) asymptotically equivalent to the solutions of the equation

$$y_{n+1} - y_n = b_n.$$

We obtain also sufficient conditions under which for every constant $c \in \mathbb{R}$ there exists a solution of equation (E) convergent to c. This result generalizes the main theorem of [7]. A special case of equation (E), namely, the difference equation

$$x_{n+1} - x_n = a_n \varphi_n(x_{n+1}), \quad n = 1, 2, \ldots$$

will also be considered.

Let (X, d), (Y, ϱ) be metric spaces, and let Φ be a family of maps $\varphi\colon X \longrightarrow Y$. Φ is said to be equicontinuous at a point $p \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, p) < \delta$, then $\varrho(\varphi(x), \varphi(p)) < \varepsilon$ for any $\varphi \in \Phi$. We say that Φ is equicontinuous if it is equicontinuous at every point $p \in X$.

If for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\varrho(\varphi(x_1), \varphi(x_2)) < \varepsilon$ for any pair $x_1, x_2 \in X$ such that $d(x_1, x_2) < \delta$ and every $\varphi \in \Phi$, then Φ is said to be uniformly equicontinuous.

Φ is said to be locally bounded if for any point $p \in X$ there exist a neighborhood U of p in X and a constant $M > 0$ such that $|\varphi(t)| \leqslant M$ for every $t \in U$, $\varphi \in \Phi$.

If $|\varphi(t)| \leqslant M$ for all $t \in X$, $\varphi \in \Phi$, then we say that Φ is bounded.

If $\psi\colon X \longrightarrow \mathbb{R}$, $U \subseteq X$, then $\psi|U$ denotes the restriction of ψ defined by $\psi|U\colon U \longrightarrow \mathbb{R}$, $(\psi|U)(x) = \psi(x)$ for $x \in U$.

The space of all sequences $x\colon \mathbb{N} \longrightarrow \mathbb{R}$ is denoted by SQ.

The Banach space of all bounded sequences $x \in SQ$ with the norm $\|x\| = \sup\{|x_n|\colon n \in \mathbb{N}\}$ is denoted by BS.

2. Main results

To prove our results we need some lemmas.

Lemma 1. *Let X, Y be metric spaces. If X is compact and Φ is a locally bounded family of maps $\varphi\colon X \longrightarrow Y$, then Φ is bounded.*

Proof. Since X is compact, so it follows that there exists a finite covering $\{U_1, \ldots, U_n\}$ of X and constants $M_1, \ldots, M_n$ such that $|\varphi(x)| \leqslant M_i$ for any $x \in U_i$ and arbitrary $\varphi \in \Phi$. If $M = M_1 + \ldots + M_n$, then $|\varphi(x)| \leqslant M$ for any $x \in X$ and $\varphi \in \Phi$. $\square$

Lemma 2. *Let X and Y be metric spaces. If X is compact and Φ is an equicontinuous family of maps $\varphi\colon X \longrightarrow Y$, then Φ is uniformly equicontinuous.*

Proof. Let d, ϱ be metrics of the spaces X, Y, respectively. If $p \in X$, $r > 0$, then $B(p,r)$ denotes the ball $\{x \in X\colon d(p,x) < r\}$. Let $\varepsilon > 0$. For any $p \in X$ there exists such a $\delta_p > 0$ that

$$\varphi(B(p, 2\delta_p)) \subseteq B(\varphi(p), \varepsilon/2)$$

for any $\varphi \in \Phi$. From the covering $\{B(p,\delta_p)\colon p \in X\}$ choose a finite subcovering $\{B(p_1,\delta_{p_1}), \ldots, B(p_n,\delta_{p_n})\}$. Let $\delta = \min(\delta_{p_1}, \ldots, \delta_{p_n})$. If $t, s \in X$, $d(t,s) < \delta$ then $t \in B(p_k, \delta_{p_k})$ for some $k \in \{1, \ldots, n\}$. Then

$$d(s, p_k) \leqslant d(s,t) + d(t, p_k) < \delta + \delta_{p_k} \leqslant 2\delta_{p_k}.$$

Hence, $\varrho(\varphi(t), \varphi(s)) \leqslant \varrho(\varphi(t), \varphi(p_k)) + \varrho(\varphi(p_k), \varphi(s)) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$ □

Lemma 3. *If the sequence of partial sums of a series $\sum b_n$ is bounded, then every solution of the equation $\Delta y_n = b_n$ is bounded.*

Proof. Assume $\Delta y_n = b_n$. Then

$$y_{n+1} - y_1 = \Delta y_1 + \Delta y_2 + \ldots + \Delta y_n = b_1 + \ldots + b_n = s_n.$$

By assumption the sequence (s_n) is bounded, so the sequence (y_n) is bounded, too.

The main results of this paper are the following two theorems. In the proofs of these theorems we use the technique similar to that used in [6]. □

Theorem 1. *Assume the series $\sum a_n$ is absolutely convergent, the sequence of partial sums of the series $\sum b_n$ is bounded, y is a solution of the equation $\Delta y_n = b_n$, Y is the set of values of the sequence y. If there exists a neighbourhood U of the closure $\overline{Y}$ such that the family $\{\varphi_n|U\}$ is locally bounded and equicontinuous, then there exists a solution x of* (E) *such that*

$$x_n = y_n + o(1).$$

Proof. By Lemma 3 the set Y is bounded. Therefore the closure $\overline{Y}$ is compact. Hence, it follows that there exists an open set V such that $\overline{V}$ is compact and $\overline{Y} \subseteq V \subseteq \overline{V} \subseteq U$. Using Lemma 2 and Lemma 1 one can show that the family $\{\varphi_n|V\}$ is bounded and uniformly equicontinuous.

Since $\overline{Y}$ is compact, so there exists a number $a > 0$ such that if $s \in \overline{Y}$, $t \in \mathbb{R}$ and $|s - t| \leqslant a$ then $t \in V$. There exists $M > 1$ such that $|\varphi_n(t)| \leqslant M$ for any $t \in V$, $n \in \mathbb{N}$. Let us denote $r_n = \sum\limits_{j=n}^{\infty} |a_j|$ for $n \in \mathbb{N}$. There exists $p \in \mathbb{N}$ such that $Mr_n < a$ for any $n \geqslant p$. Let

$$T = \{x \in BS\colon\ x_n = 0 \text{ for } n < p \text{ and } |x_n| \leqslant Mr_n \text{ for } n \geqslant p\}.$$
$$S = \{x \in BS\colon\ x_n = y_n \text{ for } n < p \text{ and } |x_n - y_n| \leqslant Mr_n \text{ for } n \geqslant p\}.$$

Obviously, T is a convex and closed subset of the space BS. Let $\varepsilon > 0$. It is easy to construct a finite ε-net for the set T. Hence, T is compact. Since the mapping $F\colon T \longrightarrow S$ defined by $F(x)\ (n) = x_n + y_n$ is an affine isometry of T onto S, it follows that S is also convex and compact. If $x \in S$ and $k \in \mathbb{N}$, then $y_k \in Y$ and $|x_k - y_k| < a$. Hence, $x_k \in V$. Therefore, $|\varphi_n(x_k)| \leqslant M$ for any $x \in S$, $n, k \in \mathbb{N}$. For $x \in S$, let us define the sequence $A(x)$ by

$$A(x)(n) = \begin{cases} y_n \text{ for } n < p, \\ y_n - \sum\limits_{j=n}^{\infty} a_j \varphi_j(x(\sigma(j))) \text{ for } n \geqslant p. \end{cases}$$

If $x \in S$ and $n \geqslant p$, then

$$|A(x)(n) - y_n| = \left| \sum_{j=n}^{\infty} a_j \varphi_j(x(\sigma(j))) \right| \leqslant \sum_{j=n}^{\infty} |a_j||\varphi_j(x(\sigma(j)))| \leqslant Mr_n.$$

Therefore $A(x) \in S$. Hence, it follows that $A(S) \subseteq S$.

Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that if $t, s \in V$ and $|t - s| < \delta$ then $|\varphi_n(t) - \varphi_n(s)| < \varepsilon$ for any $n \in \mathbb{N}$. Let $x, z \in S$, $\|x - z\| < \delta$. Then $|x_k - z_k| < \delta$ for any $k \in \mathbb{N}$. Hence, $|\varphi_n(x_k) - \varphi_n(z_k)| < \varepsilon$ for any $n, k \in \mathbb{N}$. Therefore

$$\begin{aligned} \|A(x) - A(z)\| &= \sup_{n \geqslant p} \left| \sum_{j=n}^{\infty} a_j \varphi_j(x(\sigma(j))) - \sum_{j=n}^{\infty} a_j \varphi_j(z(\sigma(j))) \right| \\ &\leqslant \sup_{n \geqslant p} \sum_{j=n}^{\infty} |a_j||\varphi_j(x(\sigma(j))) - \varphi_j(z(\sigma(j)))| \\ &= \sum_{j=p}^{\infty} |a_j||\varphi_j(x(\sigma(j))) - \varphi_j(z(\sigma(j)))| \leqslant \varepsilon r_p. \end{aligned}$$

This means that A is a continuous mapping. By Schauder's theorem it follows that there exists $x \in S$ such that $A(x) = x$. Consequently, for $n \geqslant p$ we obtain

$$x_n = y_n - \sum_{j=n}^{\infty} a_j \varphi_j(x(\sigma(j))), \quad x_{n+1} = y_{n+1} - \sum_{j=n+1}^{\infty} a_j \varphi_j(x(\sigma(j))).$$

Hence, if $n \geqslant p$ then $\Delta x_n = \Delta y_n + a_n\varphi_n(x(\sigma(n))) = a_n\varphi_n(x(\sigma(n))) + b_n$. Moreover, the convergence of the series $\sum\limits_{j=1}^{\infty} a_j\varphi_j(x(\sigma(j)))$ implies that $x_n = y_n + o(1)$. This completes the proof. $\square$

Corollary 1. *If the series $\sum a_n$ is absolutely convergent, the sequence of partial sums of the series $\sum b_n$ is bounded, the family $\{\varphi_n\}$ is equicontinuous and locally bounded, then for an arbitrary solution y of the equation $\Delta y_n = b_n$ there exists a solution x of* (E) *such that*

$$x_n = y_n + o(1).$$

Proof. Take $U = \mathbb{R}$ in Theorem 1. $\square$

The next corollary generalizes Theorem 2.1 of [7].

Corollary 2. *If the series $\sum a_n$ is absolutely convergent, the series $\sum b_n$ is convergent, the family $\{\varphi_n\}$ is equicontinuous and locally bounded, then for any $c \in \mathbb{R}$ there exists a solution of* (E) *which converges to c.*

Proof. Choose a sequence (z_n) such that $\Delta z_n = b_n$. By the convergence of the series $\sum b_n$, the sequence (z_n) is convergent. Let $\lambda = \lim z_n$ and $y_n = c + z_n - \lambda$. Then $\Delta y_n = b_n$ and $z_n - \lambda = o(1)$. By Corollary 1, there exists a solution x of (E) such that $x_n = y_n + o(1)$. Obviously, x is convergent to c. $\square$

Corollary 3. *If the series $\sum a_n$ is absolutely convergent, the sequence of partial sums of the series $\sum b_n$ is bounded and $\{\varphi_n\}$ is a periodic family (i.e., $\varphi_{n+k} = \varphi_n$ for some $k \in \mathbb{N}$ and every $n \in \mathbb{N}$) of continuous functions, then for an arbitrary solution y of the equation $\Delta y_n = b_n$ there exists a solution x of* (E) *such that*

$$x_n = y_n + o(1).$$

Proof. Obviously, the finite family $\{\varphi_1, \varphi_2, \ldots, \varphi_{k-1}\}$ of continuous functions is equicontinuous and locally bounded. By periodicity of the sequence (φ_n) it follows that the family $\{\varphi_n \colon n \in \mathbb{N}\}$ is equicontinuous and locally bounded. Hence, the assertion follows from Corollary 1. $\square$

Corollary 4. *If the series $\sum a_n$ is absolutely convergent and the series $\sum b_n$ is convergent, and $\{\varphi_n\}$ is a periodic family of continuous functions, then for any $c \in \mathbb{R}$ there exists a solution of* (E) *which converges to c.*

In Theorem 1 (and in Corollary 1) the sequence b is such that all solutions of the equation $\Delta y_n = b_n$ are bounded. In the next theorem b is an arbitrary real sequence.

Theorem 2. *If the series $\sum a_n$ is absolutely convergent and family $\{\varphi_n\}$ is bounded and uniformly equicontinuous, then for an arbitrary solution y of the equation $\Delta y_n = b_n$ there exists a solution x of* (E) *such that $x_n = y_n + o(1)$.*

Proof. Assume y is a solution of the equation $\Delta y_n = b_n$. Let $r_n = \sum\limits_{j=n}^{\infty} |a_j|$ for $n \in \mathbb{N}$. Choose a constant $M > 0$ for which $|\varphi_n(t)| \leqslant M$ for any $t \in \mathbb{R}$ and $n \in \mathbb{N}$. Let

$$T = \{x \in BS\colon\ |x_n| \leqslant Mr_n,\ n \in \mathbb{N}\},$$
$$S = \{x \in SQ\colon\ |x_n - y_n| \leqslant Mr_n,\ n \in \mathbb{N}\},$$

and let $F\colon\ T \longrightarrow S$ be defined by $F(x)\ (n) = y_n + x_n,\ x \in T, n \in \mathbb{N}$. Then the formula $\varrho(x,z) = \sup\{|x_n - z_n|\colon\ n \in \mathbb{N}\},\ x, z \in S$ defines a metric on S such that F is an isometry of T onto S. Since T is a compact and convex subset of the space BS and S is homeomorphic to T it follows by Schauder's theorem that every continuous map $A\colon\ S \longrightarrow S$ has a fixed point.

For $x \in S$ and $n \in \mathbb{N}$, let

$$A(x)(n) = y_n - \sum_{j=n}^{\infty} a_j \varphi_j(x(\sigma(j))).$$

The rest of the proof is similar to the proof of Theorem 1.

Corollary 5. *If the series $\sum a_n$ is absolutely convergent, $\lim b_n = b \in \mathbb{R}$, the series $\sum (b_n - b)$ is convergent, the family $\{\varphi_n\}$ is bounded and uniformly equicontinuous, then for any $c \in \mathbb{R}$ there exists a solution x of* (E) *such that*

$$x_n = bn + c + o(1).$$

Proof. Choose a sequence (z_n) such that $\Delta z_n = b_n - b$. By the convergence of the series $\sum(b_n - b)$, the sequence (z_n) is corvergent. Let $\lambda = \lim z_n$ and let $y_n = bn + c + z_n - \lambda$. Then $\Delta y_n = b + \Delta z_n = b + b_n - b = b_n$ and $z_n - \lambda = o(1)$. By Theorem 2, there exists a solution x of (E) such that $x_n = y_n + o(1)$. Obviously, $x_n = bn + c + o(1)$. $\square$

Corollary 6. *If the series $\sum a_n$ is absolutely convergent and $\{\varphi_n\}$ is a periodic family of uniformly continuous and bounded functions, then for an arbitrary solution y of the equation $\Delta y_n = b_n$ there exists a solution x of* (E) *such that $x_n = y_n + o(1)$.*

Proof. This corollary is an easy consequence of Theorem 2 and the fact that a finite family of uniformly continuous and bounded functions is uniformly equicontinuous and bounded. $\square$

Theorem 3. *Let x be a solution of* (E). *If the family $\{\varphi_n\}$ is bounded and the series $\sum a_n$ is absolutely convergent, then*

(a) *if the sequence of partial sums of the series $\sum b_n$ is bounded, then x is bounded,*

(b) *if the series $\sum b_n$ is convergent, then x is convergent,*

(c) *if $\sum b_n = \infty$, then x is divergent to ∞,*

(d) *if $\sum b_n = -\infty$, then x is divergent to $-\infty$.*

Proof. For $n \in \mathbb{N}$, let

$$S_n = \sum_{i=1}^{n} a_i \varphi_i(x(\sigma(i))), \quad t_n = \sum_{i=1}^{n} b_i.$$

Since the family $\{\varphi_n\}$ is bounded and the series $\sum a_n$ is absolutely convergent, it follows that the sequence (S_n) is convergent. Moreover,

$$x_n - x_1 = \Delta x_1 + \Delta x_2 + \ldots + \Delta x_{n-1} = S_{n-1} + t_{n-1}.$$

From the convergence of (S_n) it follows now that if the sequence (t_n) is bounded, then (x_n) is also bounded and, moreover, if (t_n) is convergent then (x_n) is convergent, too. Analogously, one can prove (c), (d). $\square$

Now we consider a special case of equation (E):

$$\text{(E1)} \qquad x_{n+1} - x_n = a_n \varphi_n(x_{n+1}), \quad a_n \in \mathbb{R}, \quad \varphi_n \colon \mathbb{R} \longrightarrow \mathbb{R},$$

i.e., we assume that $\sigma(n) = n + 1$ and $b_n = 0$ for every $n \in \mathbb{N}$.

A special case of this equation (when $\varphi_n(t) = t^2$ for $n \in \mathbb{N}$, $t \in \mathbb{R}$) was studied in [7].

We start with some simple lemmas.

Lemma 4. *If $\lambda \in \mathbb{R}$, $\varphi_n(\lambda) = 0$ for any $n \in \mathbb{N}$, then the constant sequence $x_n = \lambda$ is a solution of* (E1).

Lemma 5. *Let x be a solution of* (E1). *If $\lambda \in \mathbb{R}$, $p \in \mathbb{N}$, $\varphi_n(\lambda) = 0$ for any $n \in \mathbb{N}$ and $x_p \neq \lambda$, then $x_n \neq \lambda$ for any $n \geqslant p$.*

Proof. Assume $x_{p+1} = \lambda$. Then $x_p = x_{p+1} - a_p \varphi(x_{p+1}) = \lambda$, a contradiction. Hence, $x_{p+1} \neq \lambda$ and so on. $\square$

Corollary 7. *Let x be a solution of* (E1). *If $\lambda \in \mathbb{R}$, $\varphi_n(\lambda) = 0$ for any $n \in \mathbb{N}$, then either $x_n = \lambda$ for all n or there exists $p \in \mathbb{N}$ such that $x_n = \lambda$ for any $n < p$ and $x_n \neq \lambda$ for $n \geqslant p$.*

Theorem 4. *Assume $\varepsilon \in (0,1)$, $\varphi_n(0) = 0$ for any $n \in \mathbb{N}$, $A, B \subseteq \mathbb{R}$, $a_n \in A$ for any $n \in \mathbb{N}$, $\varphi_n(t)/t \in B$ for $n \in \mathbb{N}$ and $t \neq 0$. Let x be a nontrivial solution of* (E1)*. Then*

(a) *if $AB \subseteq (-\infty, 1]$, then x has a constant sign for large n,*
(b) *if $AB \subseteq [1, \infty)$, then x is alternating for large n,*
(c) *if $AB \subseteq [0, 2]$, then $|x|$ is nondecreasing,*
(d) *if $AB \subseteq (0, 2)$, then $|x|$ is increasing for large n,*
(e) *if $AB \subseteq [\varepsilon, 2-\varepsilon]$, then x is unbounded,*
(f) *if $AB \subseteq (-\infty, 0] \cup [2, \infty)$, then $|x|$ is nonincreasing for large n,*
(g) *if $AB \subseteq (-\infty, 0) \cup (2, \infty)$, then $|x|$ is decreasing for large n,*
(h) *if $AB \subseteq (-\infty, -\varepsilon] \cup [2+\varepsilon, \infty)$, then x is convergent to zero.*

Proof. By Lemma 5 it follows that there exists $p \in \mathbb{N}$ such that $x_n \neq 0$ for any $n \geqslant p$. Let $n \geqslant p$. Since $x_{n+1} - x_n = a_n \varphi_n(x_{n+1})$ and $x_{n+1} \neq 0$, so

$$x_n/x_{n+1} = 1 - a_n \varphi_n(x_{n+1})/x_{n+1}.$$

Let $\alpha_n = x_n/x_{n+1}$. Then $\alpha_n \in 1 - AB$. If $AB \subseteq (-\infty, 1]$, then $\alpha_n \geqslant 0$ and $x_n \neq 0 \neq x_{n+1}$. Hence, $x_n/x_{n+1} > 0$. This proves (a). Analogously, one can prove (b), (c), (d), (f), (g). If $AB \subseteq [\varepsilon, 2-\varepsilon]$ then $\alpha_n \in [-1+\varepsilon, 1-\varepsilon]$. Hence, $|x_n/x_{n+1}| \leqslant (1-\varepsilon)$, therefore $|x_n| \leqslant (1-\varepsilon)|x_{n+1}|$. By induction one can get $|x_n| \leqslant (1-\varepsilon)^k |x_{n+k}|$ for any $k \in \mathbb{N}$. Hence, (e) holds. Similarly one can show (h). □

Corollary 8. *Assume $\lambda \in \mathbb{R}$, $c > 0$, $\varphi_n(\lambda) = 0$, $a_n \in [0, c]$, $c\varphi_n(t+\lambda) \leqslant t$ for any $n \in \mathbb{N}$ and any $t \in \mathbb{R}$. If x is a solution of* (E1)*, then the sequence $(x_n - \lambda)$ has a constant sign for large n.*

Proof. Let $\psi_n(t) = \varphi_n(t+\lambda)$, $A = [0, c]$, $B = (-\infty, 1/c]$. Then $\psi_n(0) = 0$, $a_n \in A$, $\psi_n(t)/t \in B$ for any $n \in \mathbb{N}$ and $t \neq 0$. Since $AB = (-\infty, 1]$, it follows from Theorem 4(a) that an arbitrary solution y of the equation $\Delta y_n = a_n \psi_n(y_{n+1})$ has a constant sign for large n. Let $y_n = x_n - \lambda$. Then

$$\Delta y_n = \Delta x_n = a_n \varphi_n(x_{n+1}) = a_n \psi_n(x_{n+1} - \lambda) = a_n \psi_n(y_{n+1}).$$

□

Analogously, using Theorem 4 (b) we obtain

Corollary 9. *If $\lambda \in \mathbb{R}$, $c > 0$, $\varphi_n(\lambda) = 0$, $a_n \geqslant c$, $c\varphi_n(t+\lambda) \geqslant t$ for $n \in \mathbb{N}$, $t \in \mathbb{R}$ and x is a solution of* (E1)*, then the sequence $(x_n - \lambda)$ is alternating for large n.*

Theorem 5. *Assume* $\varepsilon \in (0,1)$, $c > 0$, $c\varphi_n(t)/t \in [\varepsilon, 2]$ *for any* $n \in \mathbb{N}$ *and* $t \neq 0$, $a_n \in [0,c]$, $\varphi_n(0) = 0$ *for all* $n \in \mathbb{N}$, *and let the series* $\sum \min(a_n, c - a_n)$ *be divergent. Then every nontrivial solution of* (E1) *is unbounded.*

P r o o f. Let x be a nontrivial solution of (E1). By Corollary 7, there is an index $p \in \mathbb{N}$ such that $x_n \neq 0$ for any $n \geqslant p$. For $n \geqslant p$ let $\alpha_n = a_n\varphi_n(x_{n+1})/x_{n+1}$. Then $\alpha_n \in [\varepsilon a_n/c, 2a_n/c]$. Since

$$2a_n = 2(c + a_n - c)/c = 2 - 2(c - a_n)/c$$

we have

$$1 - \alpha_n \in [-1 + 2(c - a_n)/c, 1 - \varepsilon a_n/c].$$

Hence,

$$\begin{aligned}
|x_n/x_{n+1}| &= |1 - \alpha_n| \leqslant \max(|-1 + 2(c - a_n)/c|, |1 - \varepsilon a_n/c|) \\
&= \max(1 - 2(c - a_n)/c, 1 - \varepsilon a_n/c) = 1 - \min(2(c - a_n)/c, \varepsilon a_n/c) \\
&= 1 - \min(2(c - a_n), \varepsilon a_n)/c.
\end{aligned}$$

Let $\beta_n = \min(2(c - a_n), \varepsilon a_n)/c$. Then $\beta_n \in [0,1)$ and $|x_n| \leqslant (1 - \beta_n)|x_{n+1}|$. If $k \geqslant 1$ then, by induction, we obtain

$$|x_n| \leqslant (1 - \beta_n)(1 - \beta_{n+1}) \ldots (1 - \beta_{n+k-1})|x_{n+k}|.$$

Since the series $\sum \min(a_n, c - a_n)$ is divergent, the series $\sum \beta_n$ is also divergent. Hence, the infinite product $(1 - \beta_1)(1 - \beta_2)(1 - \beta_3) \ldots$ is divergent to zero (i.e., $\lim p_n = 0$ where $p_n = (1 - \beta_1)(1 - \beta_2) \ldots (1 - \beta_n)$). Therefore, the sequence (x_n) is unbounded. $\square$

R e m a r k. If x is a solution of (E1), $A = [0,c]$, $B = [\varepsilon/c, 2/c]$ and the assumptions of Theorem 5 are satisfied, then $a_n \in A$, $\varphi_n(t)/t \in B$ for $n \in \mathbb{N}$, $t \neq 0$ and $AB = [0,2]$. Hence, by Theorem 4(c), the sequence $|x|$ is nondecreasing.

Corollary 10. *If* $\varepsilon \in (0,1)$, $c > 0$, $a_n \in [0, c/2]$, $c\varphi_n(t)/t \in [\varepsilon, 2]$, $\varphi_n(0) = 0$ *for* $n \in \mathbb{N}$ *and* $t \neq 0$, *and the series* $\sum a_n$ *is divergent, then every nontrivial solution of* (E1) *is unbounded.*

P r o o f. Let $n \in \mathbb{N}$. Since $a_n \leqslant c/2$, so $a_n \leqslant c - a_n$. Hence, $\min(a_n, c - a_n) = a_n$ and the assertion follows from Theorem 5. $\square$

Corollary 11. *If $\varepsilon \in (0,1)$, $c > 0$, $a_n \in [c/2, c]$, $c\varphi_n(t)/t \in [\varepsilon, 2]$, $\varphi_n(0) = 0$ for $n \in \mathbb{N}$ and $t \neq 0$, and the series $\sum(c - a_n)$ is divergent, then every nontrivial solution of* (E1) *is unbounded.*

P r o o f. If $n \in \mathbb{N}$ then $c \leqslant 2a_n$. Hence, $c - a_n \leqslant a_n$ and $\min(a_n, c - a_n) = c - a_n$ and the assertion follows from Theorem 5. □

Theorem 6. *Let x be a solution of* (E1)*. If $c > 0$, $|a_n| \geqslant c$, $c|\varphi_n(t)/t| \geqslant 2$ for any $n \in \mathbb{N}$ and $t \neq 0$, then $|x|$ is nonincreasing. Moreover, if the series $\sum(|a_n| - c)$ is divergent, then x converges to zero.*

P r o o f. For $n \in \mathbb{N}$, let $\alpha_n = 2(|a_n| - c)/c$. If $x_{n+1} \neq 0$, then

$$|x_n/x_{n+1}| = |1 - a_n\varphi(x_{n+1})/x_{n+1}|.$$

Since

$$|a_n\varphi(x_{n+1})/x_{n+1}| \geqslant 2|a_n|/c = 2(|a_n| - c + c)/c = 2 + \alpha_n$$

we obtain $|x_n| \geqslant (1 + \alpha_n)|x_{n+1}|$. Obviously, the last inequality is also valid in the case $x_{n+1} = 0$. Hence, the sequence $|x_n|$ is nonincreasing. Moreover, by induction we obtain

$$|x_n| \geqslant (1 + \alpha_n)(1 + \alpha_{n+1}) \ldots (1 + \alpha_{n+k-1})|x_{n+k}|.$$

If the series $\sum(|a_n| - c)$ is divergent, then the series $\sum \alpha_n$ is divergent, too. Hence, the infinite product $(1 + \alpha_1)(1 + \alpha_2)(1 + \alpha_3) \ldots$ is also divergent (i.e., $\lim p_n = \infty$ where $p_n = (1 - \alpha_1)(1 - \alpha_2) \ldots (1 - \alpha_n)$). It follows that the sequence (x_n) is convergent to zero. □

R e m a r k. If x is a solution of (E1), $A = (-\infty, -c] \cup [c, \infty)$, $B = (-\infty, -2/c] \cup [2/c, \infty)$ and the assumptions of Theorem 6 are satisfied, then $a_n \in A$, $\varphi_n(t)/t \in B$ for $n \in \mathbb{N}$, $t \neq 0$ and $AB = (-\infty, -2] \cup [2, \infty)]$. Hence, the fact that the sequence $|x|$ is nonincreasing for large n follows also from Theorem 4(f).

Theorem 7. *Assume $c > 0$ and $a_n \geqslant c$ for any $n \in \mathbb{N}$. If all functions φ_n are positive and there exists $L > 0$ such that $c\varphi_n(t) \geqslant t$ for any $t > L$ and $n \in \mathbb{N}$, then all solutions of* (E1) *are increasing and convergent.*

P r o o f. Since the sequence (a_n) and the functions φ_n are positive, so all solutions of (E1) are increasing. Assume a solution (x_n) is unbounded. There exists an index p such that $x_n > L$ for any $n \geqslant p$.

If $n \geqslant p$, then

$$1 - x_n/x_{n+1} = a_n\varphi_n(x_{n+1})/x_{n+1}.$$

But $0 < (x_n/x_{n+1}) < 1$ and $(a_n\varphi_n(x_{n+1})/x_{n+1}) < 1$. Hence,

$$(\varphi_n(x_{n+1})/x_{n+1}) < (1/a_n) \leqslant (1/c),$$

a contradiction. Thus (x_n) is bounded and therefore it is convergent. $\square$

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