RADIUS-INVARIANT GRAPHS

V. Bálint, Bratislava, O. Vacek, Zvolen

(Received November 24, 2003)

Abstract. The eccentricity $e(v)$ of a vertex v is defined as the distance to a farthest vertex from v. The radius of a graph G is defined as a $r(G) = \min\limits_{u \in V(G)} \{e(u)\}$. A graph G is radius-edge-invariant if $r(G-e) = r(G)$ for every $e \in E(G)$, radius-vertex-invariant if $r(G-v) = r(G)$ for every $v \in V(G)$ and radius-adding-invariant if $r(G+e) = r(G)$ for every $e \in E(\overline{G})$. Such classes of graphs are studied in this paper.

Keywords: radius of graph, radius-invariant graphs

MSC 2000: 05C12, 05C35, 05C75

1. Introduction

All graphs considered in this paper are undirected, finite connected without loops or multiple edges.

Let G be a graph. Then $V(G)$ denotes the vertex set of G; $E(G)$ the edge set of G; $d_G(u,v)$ (or simply $d(u,v)$) the distance between two vertices u,v in G; $e(u)$ the eccentricity of u. The radius $r(G)$ and the diameter $d(G)$ are the minimum and maximum of the vertex eccentricities, respectively. The center $C(G)$ is the set of vertices with minimum eccentricities and $\Delta(G)$ is the maximum degree of G. The notions and notations not defined here are used accordingly to the book [1].

It is well known (see [1]) that: $r(G-e) \geqslant r(G)$ for all edges e of G; $r(G-v) \lesseqqgtr r(G)$ for all vertices v of G; $r(G+e) \leqslant r(G)$ for all edges of the complement of G. Using these inequalities there have been defined and studied the following graphs (see [1, 6]):

(1) minimal graphs if $r(G-e) > r(G)$ for every $e \in E(G)$;
(2) critical graphs if $r(G-v) \neq r(G)$ for every $v \in V(G)$;

The authors would like to thank Dr. Ferdinand Gliviak for his valuable comments and suggestions. Supported by VEGA grant.

(3) maximal graphs if $r(G+e) < r(G)$ for every $e \in E(\overline{G})$.

The paper [8] defines three changing and three unchanging invariants by i, where i is any graph invariant such as radius, diameter, etc. Three changing invariants are usually used in the same way as minimal, critical and maximal graphs by a given parameter i, see [4], [7]. Three unchanging invariants are new and we shall call them *radius-edge-invariant*, *radius-vertex-invariant*, and *radius-adding-invariant*.

Definition. A graph G is:

(1) radius-edge-invariant (r.e.i.) if $r(G-e) = r(G)$ for every $e \in E(G)$;
(2) radius-vertex-invariant (r.v.i.) if $r(G-v) = r(G)$ for every $v \in V(G)$;
(3) radius-adding-invariant (r.a.i.) if $r(G+e) = r(G)$ for every $e \in E(\overline{G})$.

These classes of graphs were studied up till now in the two papers [2] and [4]. In our paper we first prove some existence results for these graphs, then we show how we can construct radius invariant graphs by some operations on graphs, and finally we give some bounds for the number of vertices as maximum and minimum degree in such graphs.

2. Existence results

Now we give some existence results for radius-edge-invariant, radius-vertex-invariant and radius-adding-invariant graphs. The situation is very easy for $r = 1$.

A graph G with n vertices is r.e.i. if and only if it contains at least three vertices of degree $n-1$, r.v.i. if and only if it contains at least 2 vertices of degree $n-1$. Every graph of radius 1 is r.a.i.

The existence of such graphs is more complicated for radius $r \geqslant 2$. We were unable to find exact characterization of r.e.i. graphs of radius 2. Walikar et. al [2] gives the following observation:

Proposition 2.1. *A graph G with n vertices and diameter 2 is r.e.i. if and only if G contains at least three vertices of degree $n-1$ or G is self-centered.*

The next observation is obvious.

Proposition 2.2. *If a graph G of radius 2 is r.v.i., then*

(1) *the maximum degree $\Delta(G) \leqslant n-3$,*
(2) *$|C(G)| \geqslant 2$,*
(3) *G is vertex 2-connected.*

Proposition 2.3. *Let G be a graph with n vertices which satisfies all properties (1)–(3) of Proposition 2.2 and, moreover let $r(G) = d(G) = 2$. Then G is r.v.i.*

Proof. We will prove this proposition by contradiction.

If $r(G-v) < r(G)$ then there exists a vertex $u \in V(G-v)$ such that $\deg_{G-v}(u) = |V(G-v)| - 1$. It must be $\deg_G(u) \geqslant |V(G)| - 2$, a contradiction.

If $r(G-v) > r(G)$ then for all central vertices c_i of the graph G there holds $d(c_i, v) = 1$. In the other case, for a central vertex c such that $d(c, v) = 2$ there hold $r(G-v) \leqslant e_{G-v}(c) \leqslant 2 = r(G)$. As the graph G is self-centered, we have $(u, v) \in E(G)$ for all $u \in V(G-v)$. Therefore $\deg_G(v) = n-1$ which is a contradiction. $\square$

Proposition 2.4. *A graph G of radius 2 is r.a.i. if and only if the maximum degree $\Delta(G) \leqslant n-3$.*

Proof. ($\Longrightarrow$) Assume that there is a vertex v such that $\deg(v) = n-2$. Then there is a unique vertex u such that $(u, v) \notin E(G)$. If we add the edge (u, v) to the graph G, then the degree of the vertex v increases to $n-1$, a contradiction.

($\Longleftarrow$) Since by adding an edge to the graph G the degree of two vertices increases by 1, there will be no vertex with degree $n-1$ in the graph $G + (u, v)$ and therefore $r(G + (u, v)) > 1$ for any $(u, v) \in E(\overline{G})$. $\square$

Theorem 2.5. *Let $r \geqslant 2$ be a natural number and let G be a graph with at least one vertex. There exists r.e.i. and r.v.i. graph H such that $r(H) = r$ and G is an induced subgraph of H.*

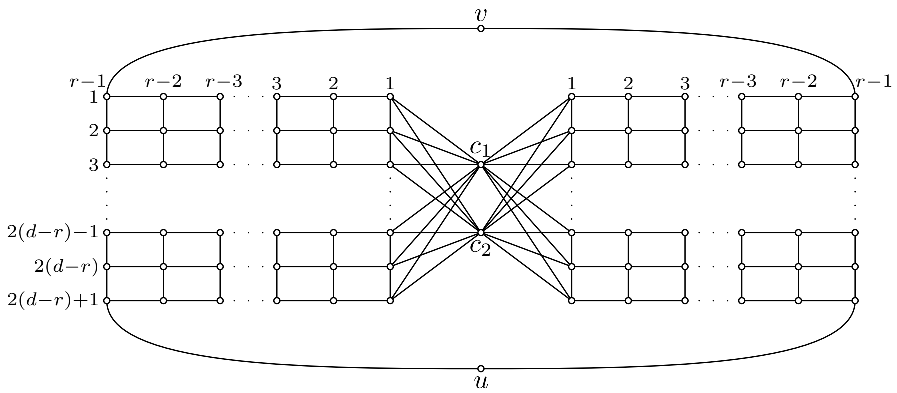
Let G and G' be disjoint graphs and let $u \in V(G')$. We say that a graph H is a substitution of G into G' in place u, if the vertex set $V(H) = (V(G') - \{u\} \cup V(G))$ and the edge set $E(H)$ consists of all edges of the graphs $G' - \{u\}$ and G and, moreover, every vertex of G is joined to every vertex from the neighborhood of u in G'.

Proof. The cycle C_{2r+1}, $r \geqslant 2$, is r.e.i. and r.v.i. graph having radius and diameter equal to r. The demand graph H is obtained from C_{2r+1}, by substituting the graph G instead of any vertex u (i.e. we join any neighbor vertex of u in C_{2r+1} to any vertex in G). $\square$

Theorem 2.6. *Let r, d be natural numbers such that $2 \leqslant r < d \leqslant 2r$. Let G be a graph with at least two vertices. Then there exists an r.e.i. and r.v.i. graph H such that $r(H) = r$, $d(H) = d$, $C(H) = V(G)$ and G is an induced subgraph of H.*

Proof. Consider a graph Q depicted here except the case when $d = 2r-1$. It is clear that $e(c_1) = e(c_2) = r$. Moreover, $d(u, v) = \min\{1 + (d(v, c_i) + d(c_i, u) +$

$1), 2(d-r)+2\}$ where $2(d-r)+2 \leqslant d$ or $2r \leqslant d$ if $d \neq 2r-1$. For any other vertex x we have $d(x,u) \leqslant \min\{2(d-r)+1, 2r-2\} < d$. Let y, z be arbitrary vertices except u, v, c_i. Consider a shortest cycle F, $y \in F$, $z \in F$. The length of this cycle is at most $2 + 2(d-r) + 2(r-1) = 2d$ if it contains vertices c_1 and u (or v) and less otherwise. Therefore $d(y,z) \leqslant d$. There are $2(d-r)+1$ rows and $2(r-1)+1$ columns of vertices in graph Q. To obtain vertices o, p such that $d(o,p) = d$ it is sufficient to take a vertex o in row 1 and column 1 and vertex p in row $2(d-r)+1$ and column $d+1$.



Let $d = 2r-1$. In that case take only $d-1 < 2(d-r)+2$ rows of vertices. It follows directly that $d(u,v) = d$. We can prove the fact that $r(Q) = r$ and $d(Q) = d$ in the same way as above.

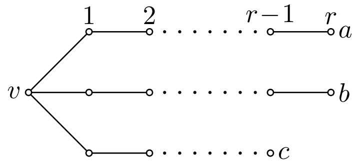
As for each vertex $w \neq c_i$ such that $d(w, c_i) = k$ there are at least two edge and vertex disjoint $c_i - w$ paths of length k we have $r(Q-e) = r(Q)$ for all $e \in E(Q)$ and $r(Q-x) \leqslant r(Q)$ for all $x \in V(Q)$. As there are two vertices u, v such that $d(c_i, u) = d(c_i, v) = r(Q) = r$, we have $r(Q-x) = r$ for all $v \in V(Q)$. Therefore Q is r.e.i. and r.v.i.

The desired graph H is obtained from the graph Q by substituting the graph G instead of the vertices c_1, c_2 (i.e. each neighbor vertex c_1, c_2 in Q will be joined with every vertex of G). No other vertices and edges are added to H.

It is clear that H is an r.e.i. and r.v.i. graph of radius r, diameter d, $C(H) = V(G)$ and G is an induced subgraph of H. □

Theorem 2.7. *Let $r \geqslant 2$ be a natural number and G be a graph with at least one vertex. Then there exists an r.a.i. graph H such that $r(H) = r, C(H) = V(G)$, G is an induced subgraph of H and $C(H) = V(G)$.*

P r o o f. Consider a path P_4 with central vertex v for $r = 2$ and the following tree for $r > 2$:



When we add any other edge to this tree, at least one of the distances $d(a,b)$, $d(a,c)$, $d(b,c)$ remains the same. As $d(G+e) \leqslant 2r(G+e)$, it is impossible to get a graph $G+e$ of radius shorter than r. Thus this graph is r.a.i. A graph H obtained from this tree by substituting the graph G for its central vertex v is r.a.i. and $C(H) = V(G)$. $\square$

Theorem 2.8. *Let r, d be natural numbers such that $r \leqslant d \leqslant 2r$. Then there exists an r.a.i. graph G such that $r(G) = r$ and $d(G) = d$.*

We first give some propositions and then the proof.

Proposition 2.9. *Let G be an r.a.i. graph with $n \geqslant 3$ vertices, radius r and diameter d. Then there exists an r.a.i. graph G' such that $r(G') = r+1$ and $d(G') = d+2$.*

P r o o f. Let $v_1, \ldots, v_n$ be all vertices of the graph G. We obtain the required graph G' by adding n new vertices $u_1, \ldots, u_n$ to $V(G)$ and n new edges (v_i, u_i) to $E(G)$. The fact that G' is r.a.i. is a special case of Theorem 3.9 given later. $\square$

Proposition 2.10. *For each natural number r there exists an r.a.i. graph G of radius and diameter equal to r and $|V(G)| \geqslant 3$.*

P r o o f. Consider the group $\mathbb{Z}_{2r+1}$ and define a graph $G_{\mathbb{Z}_{2r+1}}$ in the following way:

$$V(G) = \{(i,j);\ i,j \in \mathbb{Z}_{2r+1}\},\ ((i_1,j_1),(i_2,j_2)) \in E(G) \iff |i_1 - i_2| \leqslant 1 \wedge |j_1 - j_2| \leqslant 1.$$

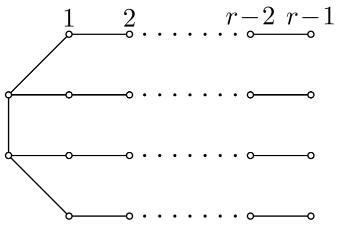
If (i_1, j_1) and (i_2, j_2) are two vertices of $G_{\mathbb{Z}_{2r+1}}$, then $d((i_1,j_1),\ (i_2,j_2)) = \max\{\min\{|i_1 - i_2|, 2r+1-|i_1-i_2|\}, \min\{|j_1-j_2|, 2r+1-|j_1-j_2|\}\} \leqslant r$. As for each vertex $u = (i,j)$, there are $8r$ vertices $u_k = (i_k, j_k)$, $i_k = i + r \bmod (2r+1) \vee i_k = i + r + 1 \bmod (2r+1) \vee j_k = j + r \bmod (2r+1) \vee j_k = j + r + 1 \bmod (2r+1)$ such that $d(u, u_k) = r$, the graph $G_{\mathbb{Z}_{2r+1}}$ is self-centered of radius r.

We will prove the fact that $G_{\mathbb{Z}_{2r+1}}$ is r.a.i. by contradiction. Consider a graph $H = G_{\mathbb{Z}_{2r+1}} + (x,y)$, $x = (i_x, j_x), y = (i_y, j_y)$. Let there be a vertex u such that $e(u)_H < r$. Then the edge (x,y) lies on each shortest $u - u_k$ path, $k = 1, \ldots, 8r$. Without the loss of generality let $d_{G_{\mathbb{Z}_{2r+1}}}(u,x) \leqslant d_{G_{\mathbb{Z}_{2r+1}}}(u,y)$ and $i_y \geqslant i_x$.

Consider a vertex $u_l = (i_y + r - |i_x - i| - 1 \bmod(2r+1), j + r)$. We have $d_{G_{\mathbb{Z}_{2r+1}}}(u, u_l) = r$, $d_{G_{\mathbb{Z}_{2r+1}}}(y, u_l) \geqslant r - |i_x - i| - 1$ and $d_{G_{\mathbb{Z}_{2r+1}}}(x, u_l) \geqslant \min\{|i_x - i_y| + r - |i_x - i| - 1, 2r + 1 - (r - |i_x - i| - 1) - |i_x - i_y|\} \geqslant r - 1 - |i - i_y|$. Therefore the $u - x - y - u_l$ path has length at least $|i_x - i| + 1 + (i_y + r - |i_x - i| - 1) = r$ and the $u - y - x - u_l$ path has length at least $|i - i_y| + 1 + (r - 1 - |i - i_y|) = r$. But then $d_H(u, u_l) = r$, a contradiction.

The graph $G_{\mathbb{Z}_{2r+1}}$ is r.a.i. and $r(G) = d(G) = r$. $\square$

P r o o f o f T h e o r e m 2.8. For $d = 2r$ the desired graph is depicted in the proof of Theorem 2.7. For $d = 2r - 1$ we could consider the following graph:



Finally, consider the case when $d \leqslant 2r - 2$. Let mark $G_0 = G_{\mathbb{Z}_{2k+1}}$ where $k = 2r - d \geqslant 2$. We will construct a graph G_{i+1} from the graph G_i by the concept introduced in the proof of Proposition 2.9. For $i = d - r$ we have an r.a.i. graph G_{d-r} such that $r(G_{d-r}) = i \cdot 1 + d(G_0) = d - r + (2r - d) = r$ and $d(G_i) = i \cdot 2 + d(G_0) = 2(d - r) + 2r - d = d$. $\square$

3. Radius invariant graphs and operations on graphs

3.1 The cartesian product of graphs. Let graphs G and H have vertex sets $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively. Then the Cartesian product $G \times H$ has vertex set w_{ij}, $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, and (w_{hi}, w_{jk}) is an edge of $G \times H$ if and only if either $h = j$ and $(v_i, v_k) \in E(H)$ or $i = k$ and $(u_h, u_j) \in E(G)$.

Directly from the definition it follows that (see [3]):

$$\begin{aligned}
d_{G\times H}(w_{hi}, w_{jk}) &= d_G(u_h, u_j) + d_H(v_i, v_k)\\
e_{G\times H}(w_{ij}) &= e_G(u_i) + e_H(v_j)\\
r(G\times H) &= r(G) + r(H)\\
d(G\times H) &= d(G) + d(H)\\
w_{ij} \in C(G\times H) &\iff u_i \in C(G) \wedge v_j \in C(H)\\
w_{ij} \in \mathrm{Cep}(G\times H) &\iff u_i \in \mathrm{Cep}(G) \wedge v_j \in \mathrm{Cep}(H)
\end{aligned}$$

where $C(G)$ denotes the set of central vertices in G, and $\mathrm{Cep}(G)$ denotes the set of eccentric vertices in $C(G)$.

Now we will prove some facts about stability of Cartesian products.

Theorem 3.1. *Let G be an r.e.i. graph. Then for any connected graph H, $G\times H$ is r.e.i.*

Proof. Consider a graph $G \times H - e$, where e is an arbitrary edge, which could be removed in a following ways:

Case 1: $e = (w_{ig}, w_{jg})$.

It is clear, that for any vertices w_{hk}, w_{hl} we have $d_{G\times H-e}(w_{hk}, w_{hl}) = d_H(v_k, v_l)$. Because of this, as G is r.e.i., we have that for any vertex $w_{st} \in G \times H$ and central vertex $w \in G\times H-e$ it is $d_{G\times H-e}(w, w_{rs}) \leqslant r(G-(u_i, u_j))+r(H) = r(G)+r(H) = r(G \times H)$. Because of this $e_{G\times H-e}(w) = r$ and, therefore $r(G \times H - e) = r(G + H)$.

Case 2: $e = (w_{hi}, w_{hj})$ and G has at least two central vertices.

As G has at least two central vertices, there exists a central vertex w_{xy} of the graph $G \times H$ such that $x \neq h$. It is clear that v_y is a central vertex of H. Because of this, for any vertex w_{rs} we have $d_{G\times H-e}(w_{xy}, w_{rs}) \leqslant r(G) + r(H) = r(G \times H)$.

Case 3: $e = (w_{hi}, w_{hj})$ and G has only one central vertex.

It is clear that $r(G) > 1$. Let w_{xy} be a central vertex of the graph $G \times H$. For any other vertex w_{rs} such that $r \neq x$, $s \neq y$ there are at least two edge disjoint paths of length at most $r(G \times H)$. Then $d_{G\times H-e}(w_{x,y}, w_{rs}) \leqslant r(G \times H)$. If $r \neq x$, $s = y$ then, as G is r.e.i. we have that $d_{G\times H-e}(w_{x,y}, w_{rs}) \leqslant r(G) < r(G \times H)$. If $r = x, s \neq y$ then if v_z is some neighbour vertex of v_y in the graph G, thus $d_{G\times H-e}(w_{x,y}, w_{rs}) \leqslant \min\{d_H(u_y, u_s), 2 + d_H(u_y, u_s)\} \leqslant 2 + r(H) \leqslant r(G \times H)$.

This completes proof. $\square$

Theorem 3.2. *Let G and H be an r.v.i. graphs. Then $G \times H$ is r.v.i.*

Proof. It is clear that for each vertex $u_a \in G$ ($v_k \in H$) there are at least two vertices u_b, u_c (v_l, v_m) such that $d(u_a, u_b) \geqslant r(G)$, $d(u_a, u_c) \geqslant r(G)$ and $d(v_k, v_l) \geqslant r(H)$, $d(v_k, v_m) \geqslant r(H)$. Because of this for each vertex w_{ak} of the graph $G \times H$ there are at least two vertices at distance at least $r(G) + r(H)$ and for any other $w_{xy} \in G \times H$ there is $e_{G-w_{xy}}(w_{ak}) \geqslant r(G) + r(H) = r(G \times H)$.

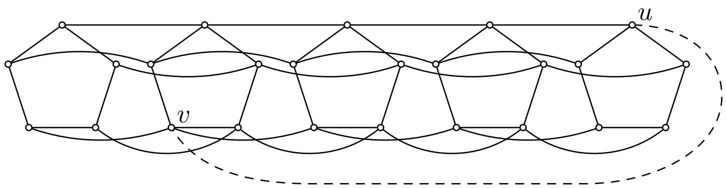
Let $w_{hi} \in V(G \times H)$. Then $u_h \in V(G)$ and $v_i \in V(H)$. Since G and H are r.v.i., there exist vertices $u_r \in C(G - u_h)$ of eccentricity $r(G)$ and $v_s \in C(H - v_i)$ of eccentricity $r(H)$.

Then the vertex w_{rs} of graph $G \times H - w_{hi}$ will have eccentricity equal to $r(G) + r(H)$. So the graph $G \times H$ is r.v.i. The theorem holds. $\square$

Theorem 3.3. *Let $G \times H$ be an r.a.i. graph. Then G is r.a.i. and H is r.a.i.*

P r o o f. We will prove the theorem by contradiction. Assume that for example G is not r.a.i. Consider a central vertex u_s of the graph $G + (u_l, u_k)$ of radius less than or equal to $r(G) - 1$. Let v_t be a central vertex of the graph H. Now let $G \times H$ be the cartesian product of the graphs G and H and let $F = G \times H + (w_{lt}, w_{kt})$. Now check the eccentricity of the vertex w_{st}. Let w_{mn} be an arbitrary vertex of F. Then $d_F(w_{st}, w_{mn}) \leqslant d_{G+(u_l,u_k)}(u_s, u_m) + d_H(v_t, v_n) \leqslant r(G) - 1 + r(H)$. Then $r(G \times H + (w_{lt}, w_{kt})) < r(G \times H)$. $\square$

The converse is false. For example $G = C_5 \times P_4$ is not r.a.i.



We have $r(G) = 4$, but $r(G + uv) = e_{G+(uv)}(v) = 3$.

3.2 The join of graphs. Recall that the join of graphs G and H is denoted $G + H$ and consists of $G \cup H$ and all edges of the form (u_i, v_j) where are $u_i \in G$, $v_j \in H$. It is clear that $r(G + H) = 2$, or $r(G + H) = 1$ if $\Delta(G) = |V(G)| - 1$ or $\Delta(H) = |V(H)| - 1$. Also $\deg_{G \cup H}(v) = \deg_G(v) + |V(H)|$ for all $v \in V(G)$ and $\deg_{G \cup H}(u) = \deg_H(u) + |V(G)|$ for all $u \in V(H)$. If $G + H$ has radius 2, it is self-centered. Now we show how we can construct r.v.i., r.e.i. and r.a.i. graphs by the operation of join.

Theorem 3.4. *Let graphs G and H have both at least two vertices. Then:*

(1) *$G + H$ is r.e.i. of radius 1 if and only if the graphs G and H contain together at least three vertices such that $\deg_G(u_i) = |V(G)| - 1$, $u_i \in G$ or $\deg_H(v_j) = |V(H)| - 1$, $v_j \in H$,*

(2) *$G + H$ is r.e.i. of radius 2 if and only if the graphs G and H have no vertex with $\deg_G(u_i) = |V(G)| - 1$, $u_i \in G$ or $\deg_H(v_j) = |V(H)| - 1$, $v_j \in H$.*

P r o o f. (1) The graph $G + H$ is r.e.i. of radius 1 if and only if it contains at least 3 vertices of degree $|V(G)| + |V(H)| - 1$ which is the same as the requirement (1).

(2) The graph $G + H$ has radius 2 if and only if the graphs G and H have no vertex with $\deg_G(u_i) = |V(G)| - 1$, $u_i \in G$ or $\deg_H(v_j) = |V(H)| - 1$, $v_j \in H$. As every graph $G + H$ of radius 2 is self-centered, it is also r.e.i. $\square$

Theorem 3.5. *Let graphs G and H have both at least two vertices. Then:*

(1) *$G+H$ is r.v.i. of radius 1 if and only if the graphs G and H contain together at least two vertices such that $\deg_G(u_i)=|V(G)|-1$, $u_i\in G$ or $\deg_H(v_j)=|V(H)|-1$, $v_j\in H$,*

(2) *$G+H$ is r.v.i. of radius 2 if and only if the graphs G and H have no vertex with $\deg_G(u_i)=|V(G)|-1$, $\deg_G(u_i)=|V(G)|-2$, $u_i\in G$ or $\deg_H(v_j)=|V(H)|-1$, $\deg_H(v_j)=|V(H)|-2$, $v_j\in H$.*

P r o o f. (1) The graph $G+H$ of radius 1 is r.v.i. if and only if it contains at least 2 vertices of degree $|V(G)|+|V(H)|-1$ which is the same as the requirement (1).

(2) The graph $G+H$ has radius 2 if and only if the graphs G and H have no vertex with $\deg_G(u_i)=|V(G)|-1$, $u_i\in G$ or $\deg_H(v_j)=|V(H)|-1$, $v_j\in H$. Moreover, if it is r.v.i. then $\Delta(G)\leqslant|V(G)|+|V(H)|-3$ by Proposition 2.2 which is the same as the requirement (2). Conversely, as a consequence of Proposition 2.3, the graph $G+H$ is r.v.i. □

Theorem 3.6. *For any graphs G and H*

(1) *$G+H$ is r.a.i. of radius 1 if and only if the graphs G and H contain together at least one vertex $u_i(v_j)$ such that $\deg_G(u_i)=|V(G)|-1$, $u_i\in G$ or $\deg_H(v_j)=|V(H)|-1$, $v_j\in H$,*

(2) *$G+H$ is r.a.i. of radius 2 if and only if the graphs G and H have no vertex $u_i(v_j)$ with $\deg_G(u_i)=|V(G)|-1$, $\deg_G(u_i)=|V(G)|-2$, $u_i\in G$ or $\deg_H(v_j)=|V(H)|-1$, $\deg_H(v_j)=|V(H)|-2$, $v_j\in H$.*

P r o o f. (1) Every graph of radius 1 is r.a.i. Therefore this case is obvious.

(2) We can use Proposition 2.4. The graph $G+H$ has $\Delta(G+H)\leqslant n-3$ if and only if it satisfies the condition (2). □

3.3 The corona of graphs. The corona $G\circ H$ of graphs G and H was defined by Harary and Frucht ([5], see also [1]) as the graph obtained by taking one copy of G of order p_G and p_G copies of H, and then joining the i'th vertex of G to every vertex in the i'th copy of H.

It is clear that if $p_G>1, r(G)=r_G$, then $r(G\circ H)=r_G+1$ and v is a central vertex of $G\circ H$ if and only if v is a central vertex of G. Removal of any vertex of G will change the radius of $G\circ H$ to infinity, so for any graphs G, H the graph $G\circ H$ is not r.v.i. and we will not discuss r.v.i. graphs later in this section.

One can show that if $G\circ H$ is r.e.i., then G must be r.e.i. and H must be a graph with at least two vertices and no isolated vertex.

Theorem 3.7. *Let G be a self-centered graph with at least three vertices and let H have no vertex with* $\deg(v) = 0$. *Then $G \circ H$ is r.e.i.*

P r o o f. As was shown in [4], every self centered graph with at least three vertices is r.e.i. Consider three different forms in which an edge could be removed.

C a s e 1: We remove an edge which joins vertices from G. As G is r.e.i., radius of G remains unchanged and radius of $G \circ H$ is still $r_G + 1$.

C a s e 2: We remove an edge which joins vertices from H. As the shortest path from central vertex of $G \circ H$ (which is a vertex of G) does not contain an edge from any copy of H the radius remains unchanged.

C a s e 3: We remove an edge which joins the vertex v from G with any vertex u in some copy of H. Then $r(G \circ H) \geqslant 2$. As G is self-centered, v is a central vertex of $G \circ H$. Distances to all vertices of the copy of H except u are 1. As H has no isolated vertices, $d(u, v) = 2$. The distance from the vertex v to any other vertex of $G \circ H$ remains unchanged, so $G \circ H$ is r.e.i. □

Theorem 3.8. *The corona $G \circ H$ of graphs G, H is r.e.i. graph if and only if*

(1) *G is r.e.i. graph,*
(2) *for each vertex $v \in G$ there exists a central vertex $c \in G$ such that $d(u, c) \leqslant r - 1$,*
(3) *H has no vertex $v \in V(G)$ such that* $\deg(v) = 0$.

P r o o f. ($\Longleftarrow$) We will prove this similarly as in the previous theorem. The cases (1), (2) can be proved by the same way as above. Now consider the third case.

C a s e 3: We remove an edge which joins a vertex v from G with any vertex u in some copy of H. Then $r(G \circ H) \geqslant 2$. For the vertex v there exists a vertex $c \in G$ such that $d(u, c) \leqslant r - 1$. As u is not isolated in H, we have $d(u, v) = 2$. Because of this, the distances from the vertex c are still not longer than $r + 1$. Then $G \circ H$ is r.e.i.

($\Longrightarrow$) It is clear that G is r.e.i. and H has no isolated vertices. Let there exist a vertex v such that the nearest central vertex lies at distance at least r. Assume that we remove some edge (v, u) when u is in the copy of H which belongs to v. Then $d_{G \circ H - (vu)}(v, u) = 2$, $d_{G \circ H - (vu)}(u, c) > r + 2$ for all vertices c which were central in $G \circ H$. It follows that $e(w) > r$ for all vertices $w \in G \circ H - (vu)$. Thus $G \circ H$ is not r.e.i., which is a contradiction. □

Theorem 3.9. *For any graphs $G, H, |V(G)| \geqslant 3$ the corona $G \circ H$ is r.a.i. if and only if G is r.a.i.*

P r o o f. ($\Longrightarrow$) Assume that $G, r(G) = r_G$ is not r.a.i. Then $r(G + (u, v)) < r_G$. As $r((G + (u, v)) \circ H) = r(G + (u, v)) + 1 < r(G) + 1 = r(G \circ H)$ we have that $G \circ H$ is not r.a.i., a contradiction.

($\Longleftarrow$) Assume that $G \circ H$ is not r.a.i. As G is r.a.i., adding an edge which joins vertices of G does not change $r(G \circ H)$. The same is true for any edge between vertices of the unique copy of H.

Let u, v be two vertices of G and let an edge be added between two copies H_u, H_v of the graph H where each vertex of H_u is connected to u and each vertex of H_v is connected to v.

Let an edge (h_v, h_u), $h_v \in H_v$, $h_u \in H_u$ be added and assume that the radius has changed. It is enough to consider the case when the new central vertex c lies in G. If a vertex $h_d \in H_d$ is a central vertex of the graph $G \circ H + (h_u, h_v)$, then d must be a central vertex of the graph $G \circ H + (u, v)$ and $e_{G \circ H+(u,v)}(d) \leqslant e_{G \circ H+(h_u,h_v)}(h_d)$ which leads to the previous case.

So let $c \in G$ be a new central vertex and let $r(G \circ H + (h_u, h_v)) < r(G \circ H)$. Then every $c - x$ path which has length at least r in $G \circ H$ was shortened. But then it contains $u - h_u - h_v$ or $v - h_v - h_u$ path. If we add the edge (v, u) to the graph $G \circ H$, then these paths could be shortened by same distance replacing $u - h_u - h_v$ by $u - v - h_v$ or $v - h_v - h_u$ by $v - u - h_u$. But this case was discussed above.

Because of this, if the radius was changed, then an edge between some vertex $v \in G$ and some vertex $w_u \in H_u$ of the copy of the graph H must have been added. By the same arguments as above we should consider only the case when the new central vertex c of the graph $G \circ H + (v, w_u)$ is in G. G is r.a.i., so there are at least two vertices y, z at distance r or longer from c. If not, then adding the edge (c, y) (if y is the only such vertex) will decrease the radius of G.

It follows that distances from the coronas H_y, H_z to c are at least $r + 1$. Then the shortest $c - h_y$, $c - h_z$ paths in the graph $G \circ H + (u, w_u)$ for some $h_y \in H_y$, $h_z \in Z$ must contain the edge (v, w_u). Moreover, $y \neq u$ or $z \neq u$. Without the loss of generality let us consider the first case. Then

$$r \geqslant d(h_y, c) > d(y, c) > d(u, c).$$

All shortest paths satisfying this inequality contain the $v - w_u - u$ path. If we substitute this path by an edge (v, u) then $d(c, b) \leqslant r$ for all $b \in G \circ H + (v, u)$. But this case was discussed above. The proof is now complete. $\square$

4. Some bounds

Now we will prove some estimations for the number of vertices and degree of a vertex for radius invariant graphs.

A *k-depth spanning tree* (k-DST) of a graph G is a spanning tree of G of height k. It must be true that $k \geqslant r$, and if $k = r$, such trees must be rooted at a central vertex. A breadth first search algorithm beginning with any vertex v such that $e(v) = k$ will always produce an k-DST. Moreover, if $d(u, v) = i$ then the vertex u belongs to the level i. Vertices of the spanning tree with degree 1 will be called leaves. We would consider only breadth first search distance spanning trees later in this section.

Theorem 4.1. *Let G be an r.e.i. graph of radius $r \geqslant 1$. Then $|V(G)| \geqslant 2r$.*

P r o o f. A graph of radius r has at least $2r$ vertices. $\square$

The cycle C_{2r} is r.e.i., so we can't find any better estimation.

Theorem 4.2. *Let G be an r.v.i. graph of radius $r \geqslant 1$. Then $|V(G)| \geqslant 2r + 1$.*

P r o o f. Consider an r-DST of the graph G. If it has only one vertex on level r, its removal will decrease radius to $r - 1$. Therefore there are at least two vertices on level r. Assume that there is a level i, $1 \leqslant i < r$ on which there appears only one vertex a. Then the graph $G - a$ is disconnected, which is a contradiction. Finally we have the central vertex v and at least two vertices at every level $1, \ldots, r$. $\square$

It's clear (for example as a consequence of Theorem 2.5) that there is no upper bound for $|V(G)|$ both for r.e.i. and r.v.i. graphs.

Theorem 4.3. *Let G be an r.a.i. graph of radius $r \geqslant 3$. Then $|V(G)| \geqslant 3r$.*

At first we give four lemmas and then the proof.

Lemma 4.4. *Let G be an r.a.i. graph of radius $r \geqslant 2$. Then every r-DST has at least two vertices on level r.*

P r o o f. Assume that r-DST is rooted at central vertex v and there is only one vertex u on level r. By adding an edge (u, v) to G we obtain a graph of radius no greater than $r - 1$. Thus G is not r.a.i., a contradiction. $\square$

Lemma 4.5. *Let G be an r.a.i. graph of radius $r \geqslant 3$ and let v be a central vertex of G. Consider an r-DST rooted at a central vertex v and the set $\{u_1, \ldots, u_s\}$ of all vertices on level r. Now if w is some joint vertex of all $v - u_1, \ldots, v - u_n$ paths then $d(v, w) = 1$ or $v = w$.*

Proof. Assume that $d(v, w) \geqslant 2$. Adding an edge (v, w) to G decreases all distances $d(v, u_i)$ by $(d(v, w) - 1)$. The radius of $G + (v, w)$ is smaller than the radius of G, a contradiction. $\square$

Lemma 4.6. *Let G be an r.a.i. graph of radius $r \geqslant 3$ and let v be a central vertex of G. Consider an r-DST rooted at central vertex v and the set $\{u_1, \ldots, u_s\}$ of all vertices on level r. Let w be a joint vertex of $v - u_1, \ldots, v - u_s$ paths such that $v \neq w$. Then there exist at lest two vertices y, z on level $r - 1$ for which w does not lie on $v - y$, $v - z$ paths.*

Proof. Assume that there is only one such vertex (for example y). Now consider distances from the vertex w in graph $G + (w, y)$. We have $d_G(w, u_i) = r - 1 \geqslant d_{G+(w,y)}(w, u_i)$, $d_{(G+(w,y))}(w, y) = 1$ and the distance from the vertex w to all other vertices in the graph $G + (w, y)$ are no longer than in the graph G where it was less than r. Adding the edge (w, y) decreases the radius of G, a contradiction. $\square$

Lemma 4.7. *Let G be an r.a.i. graph of radius r, v be a central vertex of graph G and let for vertices u_1, u_2 on level r of r-DST the paths $v - u_1$, $v - u_2$ have a unique joint vertex v. Then at least one of the following properties holds:*

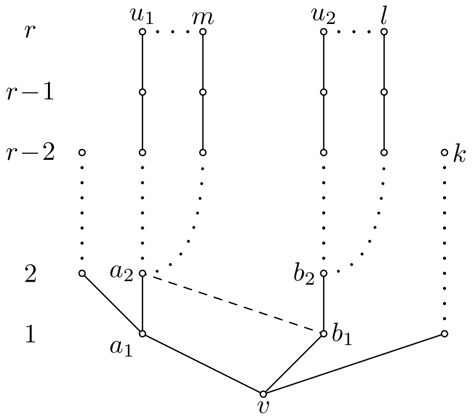
(1) *there exists a vertex y on level at least $(r - 1)$ such that each pair of paths $v - u_1$, $v - u_2$, $v - y$ has only one joint vertex v,*

(2) *there exist vertices p, q on level at least $(r - 1)$ such that the paths $v - p$, $v - q$ and $v - u_1$, $v - u_2$ have just two joint vertices.*

Proof. Let a_1, a_2 be the second and the third vertices on the path $v - u_1$, and let b_1, b_2 be the second and the third vertices on $v - u_2$ path. Assume that none of the conditions (1), (2) holds. Without loss of generality assume that there does not exist a vertex p on level $r - 1$ such that the paths $v - p$, $v - u_1$ have just the two vertices v, a_1 in common. Consider the graph $G + (b_1, a_2)$ and distances from the vertex b_1.

Case 1: Let k be a vertex such that the path $v - k$ has no joint vertex with $v - u_1$, $v - u_2$ paths except v. The property (1) is false, so $d(v, k) \leqslant r - 2$ which implies $d(b_1, k) \leqslant r - 1$.

Case 2: Let l be a vertex such that the paths $v - l$, $v - u_2$ have joint vertices v and b_1. Then $d(v, l) \leqslant r$, so $d(b_1, l) \leqslant r - 1$.



C a s e 3: Let m be a vertex such that the paths $v - m$, $v - u_1$ have joint vertices v, a_1, a_2. Then $d(m, v) \leqslant r$, so $d(m, a_2) \leqslant r - 2$ and $d(m, b_1) \leqslant r - 1$.

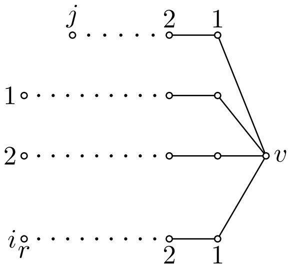
C a s e 4: At last let o be a vertex such that the paths $v - o$, $v - u_1$ have joint vertices v and a_1. The property (2) is false, so $d(v, o) \leqslant r-2 \Longrightarrow d(a_1, o) \leqslant r-3 \Longrightarrow d(a_2, o) \leqslant r - 2 \Longrightarrow d(b_1, o) \leqslant r - 1$.

Hence in the graph $G + (b_1, a_2)$ we have $e_{G+(b_1,a_2)}(b_1) = r - 1$, so the graph G is not r.a.i., a contradiction. □

P r o o f o f T h e o r e m 4.3. An r.a.i. graph G with $r \geqslant 3$ must satisfy the property from Lemma 4.6, or at least one property (1), (2) from Lemma 4.7. In every case it has at least $3r$ vertices. □

This estimation is the best possible (see for example the tree from proof of Theorem 2.7).

There is no upper bound for the number of vertices of r.a.i. graph of radius r. For example if $p = r \cdot i + j + 1 \geqslant 3r$, $i, j \in \mathbb{N}$, the tree with central vertex v depicted here has radius r and is r.a.i.



Theorem 4.8. *Let G be an r.e.i. graph with n vertices and radius r. Then for each vertex $v \in G$*

$$2 \leqslant \deg(v) \leqslant n - 2r + 2.$$

Proof. As G is r.e.i., it has no bridges and because of this no vertex of degree 1. Thus the first inequality is obvious. The second inequality holds for all graphs (for example see [9]). □

Theorem 4.9. *Let G be an r.v.i. graph with n vertices and radius r. Then for each vertex $v \in G$*

$$2 \leqslant \deg(v) \leqslant n - 2r + 1.$$

Moreover, if $e(v) = k > r$, then

$$\deg(v) \leqslant n - 2e(v).$$

Proof. As G is r.v.i., it has no cut-vertices and because of this no vertex of degree 1. Then the first inequality is obvious. Let $e(v) = k$. Consider a k-DST rooted at the vertex v. If v is a central vertex of G, then there are at least two vertices at every level $1, 2, \ldots, r$. If v is not a central vertex, then there are at least two vertices at every level $1, 2, \ldots, k-1$ and at least one vertex at level k. □

Theorem 4.10. *Let G be an r.a.i. graph with n vertices and radius r. Then*

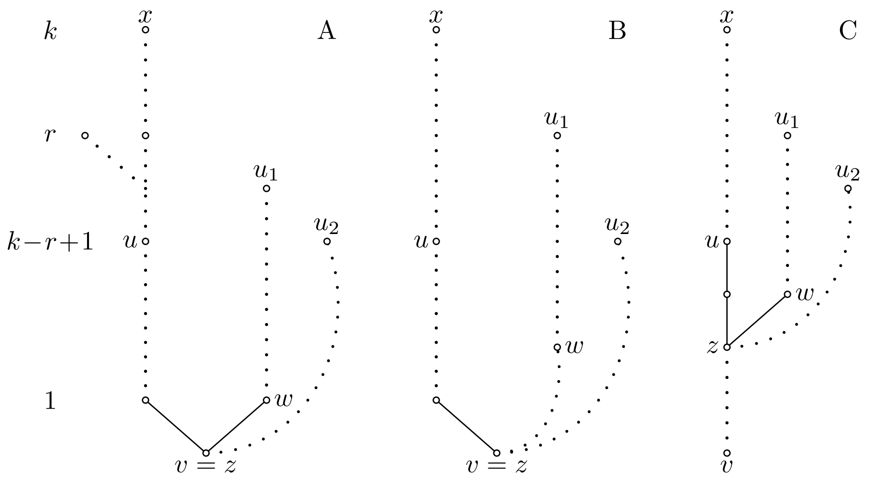
$$\Delta(G) \leqslant n - 3r + 3.$$

Proof. We prove this similarly as Theorem (4.3). Consider an arbitrary vertex $v \in V(G)$. If $v \in C(G)$, then we should use Lemmas (4.6), (4.7) and the proof is obvious. Let $v \notin C(G)$. Consider an k-DST rooted at the vertex v. Then $k = e(v) > r$. Let x be a vertex on level k and u be a vertex such that $d(u, x) = r - 1$.

Then there are at least two or more leaves $u_1, u_2, \ldots, u_n$ such that $d(u, u_i) \geqslant r$. Otherwise if $d(u, v) < r$ and u_1 is a leaf such that $d(u, u_1) \geqslant r$, then $r(G + (u, u_1)) \leqslant e_{G+(u,u_1)}(u) = r - 1$. If $d(u, v) \geqslant r$ and there is only one such leaf, it must be $\deg(v) \leqslant 2$. But as $|V(G)| \geqslant 3r$ it is immediate that $\deg(v) \leqslant n - 3r + 3$.

Let u_1 be a leaf such that $d(u, u_1) = \max\limits_{u_i \in G}(d(u, u_i))$ and w be a vertex such that $d(w, u_1) = r - 2$. It follows that there is at least one leaf u_j with $d(u_j, w) = r - 1$. Otherwise $r(G + (u, w)) \leqslant r - 1$ as $e_{G+(u,w)}(u) = r - 1$.

Consider two cases:



C a s e 1: w lies on the path $u - u_j$ for all u_j. But then $d(w, u_1) = r - 2 < r - 1 \leqslant d(w, u_j)$ which is not possible as u_1 is a vertex at maximal distance from u, and then must be a vertex (of $u_1, u_2, \ldots$) at maximum distance from w.

C a s e 2: w is not on the path $u - u_2$. Let z be the last joint vertex of the paths $u - u_1$, $u - u_2$.

A) $z = v$, w is joined to z.

In this case all vertices $u_1, u_2, \ldots$ are at level at most $r - 1$. But as G is r.a.i., there are at least 2 vertices at level r. Finally, we have r vertices on the path $x - u$, r vertices on the $u - u_2$ path without the vertex u, $r - 1$ vertices on the $z - u_1$ path without the vertex z and at least one additional vertex on level r. At most 3 of these $3r$ vertices are joined to v.

B) $z = v$, w is not joined to z.

We have r vertices on the $x - u$ path, r vertices on the $u - u_2$ path without the vertex u. As w is at level at least 2 and $d(w, u_1) = r - 2$, we have at least r vertices on the $z - u_1$ path without the vertex z. At most 3 of these vertices are joined to v.

C) $z \neq v$.

We have already $3r - 1$ vertices of which only 2 are joined to v. This completes the proof. $\square$

References

[1] *Buckley, F., Harary, F.*: Distance in Graphs. Addison-Wesley, Redwood City, 1990.

[2] *Buckley, F., Itagi K. M., Walikar, H. B.*: Radius-edge-invariant and diameter-edge-invariant graphs. Discrete Math. *272* (2003), 119–126.

[3] *Buckley, F., Lewinter, M.*: Graphs with all diametral paths through distant central nodes. Math. Comput. Modelling *17* (1990), 35–41.

[4] *Dutton, R. D., Medidi, S. R., Brigham, R. C.*: Changing and unchanging of the radius of graph. Linear Algebra Appl. *217* (1995), 67–82.
[5] *Frucht, R., Harary, F.*: On the corona of two graphs. Aequationes Math. *4* (1970), 322–325.
[6] *Gliviak, F.*: On radially extremal graphs and digraphs, a survey. Math. Bohem. *125* (2000), 215–225.
[7] *Graham, N., Harary, F.*: Changing and unchanging the diameter of a hypercube. Discrete Appl. Math. *37/38* (1992), 265–274.
[8] *Harary, F.*: Changing and unchanging invariants for graphs. Bull. Malaysian Math. Soc. *5* (1982), 73–78.
[9] *Vizing, V. G.*: The number of edges in a graph of given radius. Dokl. Akad. Nauk *173* (1967), 1245–1246. (In Russian.)

Authors' addresses: *V. Bálint*, Department of Numerical and Optimization Methods, Faculty of Mathematics, Physics and Informatics, Comenius University, 842 48 Bratislava, Slovak Republic, e-mail `vojtob@centrum.sk`; *O. Vacek*, Department of Mathematics and Descriptive Geometry, Faculty of Wooden Science, Technical University Zvolen, T. G. Masaryka 24, 960 53 Zvolen, Slovak Republic, e-mail `o.vacek@vsld.tuzvo.sk`.