SPECTRAL TOPOLOGIES OF DUALLY RESIDUATED LATTICE-ORDERED MONOIDS

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Abstract. Dually residuated lattice-ordered monoids ($DR\ell$-monoids for short) generalize lattice-ordered groups and include for instance also GMV-algebras (pseudo MV-algebras), a non-commutative extension of MV-algebras. In the present paper, the spectral topology of proper prime ideals is introduced and studied.

Keywords: $DR\ell$-monoid, prime ideal, spectrum

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1. Introduction

K. L. N. Swamy [19] introduced commutative dually residuated lattice-ordered monoids ($DR\ell$-semigroups) as a common abstraction of Abelian lattice-ordered groups and Brouwerian algebras (by a Brouwerian algebra we mean a dually relatively pseudo-complemented lattice). J. Rachůnek [14], [15] proved that well-known MV-algebras [2], an algebraic counterpart of Łukasiewicz's logic, and BL-algebras [9], structures for Hájek's basic logic [8] that captures the three most significant fuzzy logics (Łukasiewicz logic, Gödel logic and product logic), can be viewed as particular kinds of bounded commutative $DR\ell$-monoids.

In the paper we deal with (non-commutative) $DR\ell$-monoids which include lattice-ordered groups and likewise non-commutative generalizations of the above mentioned MV-algebras and BL-algebras, i.e., pseudo MV-algebras [6] called also GMV-algebras [16], and pseudo BL-algebras [3], [4], respectively. Analogously to the case of ℓ-groups we define the (prime) spectrum of a $DR\ell$-monoid to be the set of all proper prime ideals endowed with the spectral topology, that is, the topology whose open sets are exactly the sets consisting of all primes not containing a certain ideal. Prime spectra of commutative representable $DR\ell$-monoids were studied by J. Rachůnek [13] who also examined spectra of GMV-algebras [17].

The present definition of a (non-commutative) dually residuated lattice-ordered monoid is due to T. Kovář [10]:

An algebra $(A; +, 0, \vee, \wedge, \rightharpoonup, \leftharpoondown)$ of signature $\langle 2, 0, 2, 2, 2, 2\rangle$ is called a *dually residuated lattice-ordered monoid ($DR\ell$-monoid)* if

(1) $(A; +, 0, \vee, \wedge)$ is a lattice-ordered monoid (ℓ-monoid), i.e., $(A; +, 0)$ is a monoid, $(A; \vee, \wedge)$ is a lattice and the semigroup operation distributes over the lattice operations,

(2) for any $a, b \in A$, $a \rightharpoonup b$ is the least $x \in A$ such that $x + b \geqslant a$, and $a \leftharpoondown b$ is the least $y \in A$ such that $b + y \geqslant a$,

(3) A fulfils the identities

$$((x \rightharpoonup y) \vee 0) + y \leqslant x \vee y, \quad y + ((x \leftharpoondown y) \vee 0) \leqslant x \vee y,$$
$$x \rightharpoonup x \geqslant 0, \quad x \leftharpoondown x \geqslant 0.$$

As observed in [10], the identities in (3) hold even with "$\leqslant$" replaced by "$=$" and it can be easily shown that the condition (2) is equivalent to the system of identities

$$(x \rightharpoonup y) + y \geqslant x, \quad y + (x \leftharpoondown y) \geqslant x,$$
$$x \rightharpoonup y \leqslant (x \vee z) \rightharpoonup y, \quad x \leftharpoondown y \leqslant (x \vee z) \leftharpoondown y,$$
$$(x + y) \rightharpoonup y \leqslant x, \quad (y + x) \leftharpoondown y \leqslant x.$$

Let us recall some necessary facts concerning ideals of $DR\ell$-monoids [11], [12]. Let A be a $DR\ell$-monoid. Since A has no least element in general, we define the *absolute value of* $x \in A$ by $|x| = x \vee (0 \rightharpoonup x)$ (or equivalently, $|x| = x \vee (0 \leftharpoondown x)$). Of course, $x \geqslant 0$ iff $x = |x|$.

A non-empty subset I of A is said to be an *ideal* in A if (i) $x + y \in I$ for all $x, y \in I$, and (ii) $|y| \leqslant |x|$ entails $y \in I$ for all $x \in I$ and $y \in A$. Under the ordering by set-inclusion, the ideals of any $DR\ell$-monoid form an algebraic, distributive (and hence relatively pseudo-complemented) lattice, the *ideal lattice* $\mathrm{Id}(A)$. For every $X \subseteq A$, the set

$$I(X) = \{a \in A\colon\ |a| \leqslant |x_1| + \ldots + |x_n| \text{ for some } x_1, \ldots, x_n \in X, n \in \mathbb{N}\}$$

is the least ideal in A containing X, thus in particular for each $x \in A$ we have

$$I(x) = \{a \in A\colon\ |a| \leqslant n|x| \text{ for some } n \in \mathbb{N}\}.$$

In addition, $I(x) \cap I(y) = I(|x| \wedge |y|)$ and $I(x) \vee I(y) = I(|x| \vee |y|) = I(|x| + |y|)$, which obviously yields that every finitely generated ideal is principal, and therefore the compact elements of $\mathrm{Id}(A)$ are just the principal ideals.

An ideal I is called *normal* if $a + I^+ = I^+ + a$ for all $a \in A$, where $I^+ = \{x \in I\colon\ x \geqslant 0\}$. The normal ideals of A correspond one-to-one to its congruence relations. Under this correspondence, I corresponds to $\theta_I = \{\langle x, y\rangle\colon\ (x \rightharpoonup y) \vee (y \rightharpoonup x) \in I\}$. We shall write A/I for A/θ_I and x/I as an abbreviation of $[x]_{\theta_I}$. It is worth adding that in A/I we have $x/I \leqslant y/I$ iff $(x \rightharpoonup y) \vee 0 \in I$.

We define an ideal I to be *prime* if for all $J, K \in \mathrm{Id}(A)$, if $J \cap K \subseteq I$ then $J \subseteq I$ or $K \subseteq I$. In view of the distributivity of $\mathrm{Id}(A)$, I is prime if and only if it is meet-irreducible in $\mathrm{Id}(A)$, i.e., $J \cap K = I$ implies $J = I$ or $K = I$ for all $J, K \in \mathrm{Id}(A)$. If, moreover, A fulfils the identities

$$(*)\qquad \begin{aligned}(x \rightharpoonup y) \wedge (y \rightharpoonup x) \leqslant 0,\\ (x \leftharpoondown y) \wedge (y \leftharpoondown x) \leqslant 0,\end{aligned}$$

then any ideal I is prime if and only if the set of all ideals exceeding I is totally ordered, and a normal ideal I is prime if and only if A/I is totally ordered. When regarded as $DR\ell$-monoids, all ℓ-groups, GMV-algebras and pseudo BL-algebras satisfy $(*)$.

Every ideal is the intersection of all primes including it since by [12], Theorem 2.2, for each proper ideal I and $a \notin I$, there exists a prime ideal P with $I \subseteq P$ and $a \notin P$. However, we shall need somewhat more:

Lemma 1. *Let A be a $DR\ell$-monoid and $\ell(A)$ the lattice reduct of A. If $I \in \mathrm{Id}(A)$ and D is a filter in $\ell(A)$ such that $I \cap D = \emptyset$, then $I \subseteq P$ and $P \cap D = \emptyset$ for some proper prime ideal P.*

Proof. By Zorn's lemma there exists a maximal ideal, say P, such that $I \subseteq P$ and $P \cap D = \emptyset$. It remains to show that P is prime. To do this assume that $P = J \cap K$ for some $J, K \in \mathrm{Id}(A) \setminus \{P\}$. Then I is contained in both J and K and there are $x \in J \cap D$ and $y \in K \cap D$. Therefore $|x| \wedge |y| \in J \cap K \cap D = P \cap D = \emptyset$, which is impossible. $\square$

Notation. Let A be a $DR\ell$-monoid, $X \subseteq A$ and $a \in A$. We use X^+ to denote the set comprising those $x \in X$ for which $x \geqslant 0$, and we write a^+ for $a \vee 0$. In any lattice L, $[X)$ denotes the filter generated by $X \subseteq L$. If L is algebraic, then $\mathrm{Com}(L)$ stands for the join-subsemilattice of all compact elements in L.

Let A be a $DR\ell$-monoid. The *spectrum* of A, denoted by $\mathrm{Spec}(A)$, is the set of all proper prime ideals in A. For any $X \subseteq A$, let

$$\mathcal{S}(X) = \{P \in \mathrm{Spec}(A)\colon\ X \nsubseteq P\}$$

and

$$\mathcal{H}(X) = \{P \in \mathrm{Spec}(A)\colon\ X \subseteq P\}.$$

We write simply $\mathcal{S}(x)$ and $\mathcal{H}(x)$ instead of $\mathcal{S}(\{x\})$ and $\mathcal{H}(\{x\})$, respectively.

Lemma 2. *For each $X \subseteq A$, $\mathcal{S}(X) = \mathcal{S}(I(X))$ and $\mathcal{H}(X) = \mathcal{H}(I(X))$.*

Proof. It is easily seen that if $X \nsubseteq P$ for $P \in \mathrm{Spec}(A)$, then $I(X) \nsubseteq P$; thus $\mathcal{S}(X) \subseteq \mathcal{S}(I(X))$. Conversely, $X \subseteq P$ yields $I(X) \subseteq P$ for any $P \in \mathrm{Spec}(A)$, and so $\mathcal{S}(I(X)) \subseteq \mathcal{S}(X)$. One analogously verifies $\mathcal{H}(X) = \mathcal{H}(I(X))$. $\square$

Consequently, it will be enough to study $\mathcal{S}(X)$ and $\mathcal{H}(X)$ for $X \in \mathrm{Id}(A)$. In particular, we will observe $\mathcal{S}(x) = \mathcal{S}(I(x))$ and $\mathcal{H}(x) = \mathcal{H}(I(x))$ for $x \in A^+$, since $I(x) = I(|x|)$ for each x.

Let us recall the concept of a polar. We say that $x, y \in A$ are *orthogonal* and write $x \perp y$ if $|x| \wedge |y| = 0$. The *polar* of an ideal X is the set

$$X^\perp = \{a \in A\colon\ a \perp x \text{ for all } x \in X\}.$$

The polar $X^\perp$ is the pseudo-complement of an ideal X in the lattice $\mathrm{Id}(A)$. Moreover, by [12], Corollary 2.3 and Corollary 3.4, we have

Lemma 3.

$$X = \bigcap \mathcal{H}(X) \quad \text{and} \quad X^\perp = \bigcap \mathcal{S}(X)$$

for any $X \in \mathrm{Id}(A)$.

Lemma 4. *The following properties hold in any $DR\ell$-monoid A:*
(1) *$\mathcal{S}(0) = \emptyset$ and $\mathcal{S}(A) = \mathrm{Spec}(A)$;*
(2) *$\mathcal{S}(x \wedge y) = \mathcal{S}(x) \cap \mathcal{S}(y)$ for all $x, y \in A^+$;*
(3) *$\mathcal{S}(x \vee y) = \mathcal{S}(x) \cup \mathcal{S}(y)$ for all $x, y \in A^+$;*
(4) *$\mathcal{S}(X \cap Y) = \mathcal{S}(X) \cap \mathcal{S}(Y)$ for all $X, Y \in \mathrm{Id}(A)$;*
(5) *for any collection $\{X_i\}_{i \in I}$ of ideals in A,*

$$\mathcal{S}\Bigl(\bigvee_{i \in I} X_i\Bigr) = \bigcup_{i \in I} \mathcal{S}(X_i).$$

Proof. The property (1) is obvious as 0 belongs to any ideal and neither of the proper prime ideals includes A.

We have only to prove (4) and (5) as by [11], Proposition 12, $I(x\wedge y)=I(x)\cap I(y)$ and $I(x\vee y)=I(x)\vee I(y)$ for all $x,y\in A^+$. But both (4) and (5) are almost evident:

It is clear that for any $P\in\mathrm{Spec}(A)$, $X\cap Y\nsubseteq P$ entails $X\nsubseteq P$ and $Y\nsubseteq P$. Conversely, if $X\cap Y\subseteq P$ then $X\subseteq P$ or $Y\subseteq P$ since P is prime. Thus $\mathcal{S}(X\cap Y)=\mathcal{S}(X)\cap\mathcal{S}(Y)$. Similarly (5). $\square$

Corollary 5. *Given any $DR\ell$-monoid A, $\mathfrak{S}=\{\mathcal{S}(X)\colon\ X\in\mathrm{Id}(A)\}$ is a topology on* $\mathrm{Spec}(A)$.

Thus $\mathrm{Spec}(A)$ endowed with $\mathfrak{S}$ is a topological space that will be called the *(prime) spectrum of* A; $\mathfrak{S}$ is said to be the *spectral topology of* A.

Proposition 6. (1) *If $P\in\mathrm{Spec}(A)$ then*

$$\mathfrak{B}(P)=\{\mathcal{S}(x)\colon\ x\in A^+\setminus P\}$$

is a basis for $\mathrm{Spec}(A)$ *at* P.

(2) *The set*

$$\mathfrak{B}=\{\mathcal{S}(x)\colon\ x\in A^+\}$$

is a basis for the topology $\mathrm{Spec}(A)$.

Proof. To see (1), suppose $P\in\mathrm{Spec}(A)$ and $P\in\mathcal{S}(X)$ for some $\mathcal{S}(X)\in\mathfrak{S}$, i.e. $X\nsubseteq P$. Then there is $x\in X^+\setminus P$ and hence $P\in\mathcal{S}(x)\subseteq\mathcal{S}(X)$ as desired. The statement (2) has a similar proof. $\square$

Proposition 7. *Let A be a $DR\ell$-monoid. Then the mapping $X\mapsto\mathcal{S}(X)$ is an isomorphism between the ideal lattice* $\mathrm{Id}(A)$ *and the lattice of open sets of* $\mathrm{Spec}(A)$.

Proof. Obviously $X\mapsto\mathcal{S}(X)$ is onto and by Lemma 4 it is a homomorphism. In addition, $X=\bigcap\mathcal{H}(X)$ for each $X\in\mathrm{Id}(A)$ and therefore, if $\mathcal{S}(X)=\mathcal{S}(Y)$ then $X=Y$ as $\mathcal{H}(X)=\mathcal{H}(Y)$ iff $\mathcal{S}(X)=\mathcal{S}(Y)$. $\square$

A positive element $u\in A^+$ is called a *strong order unit* if for any $x\in A$ there exists a positive integer k such that $x\leqslant ku$. It is easy to see that $u\in A$ is a strong order unit if and only if $I(u)=A$, so $DR\ell$-monoids with a strong order unit generalize bounded $DR\ell$-monoids.

Since the ideal lattice $\mathrm{Id}(A)$ is an algebraic lattice whose compact elements are exactly principal ideals, we immediately obtain

Corollary 8. *For any $x \in A$, $\mathcal{S}(x)$ is a compact subset in* $\operatorname{Spec}(A)$*. The topological space* $\operatorname{Spec}(A)$ *is compact if and only if A possesses a strong order unit.*

Proposition 9. *Given a $DR\ell$-monoid A,* $\operatorname{Spec}(A)$ *is a T_0-space.*

Proof. If $P, Q \in \operatorname{Spec}(A)$ are distinct, then $P \nsubseteq Q$ or $Q \nsubseteq P$, say $P \nsubseteq Q$. Then $Q \in \mathcal{S}(P)$ yet $P \notin \mathcal{S}(P)$. □

Let us denote by $\operatorname{Spec}_{\mathrm{m}}(A)$ the set of all minimal prime ideals in a $DR\ell$-monoid A and by $\operatorname{Max}(A)$ the set of all its maximal ideals provided it is non-empty. (In general, A may have no maximal ideal.) As pointed out before, every ideal equals the intersection of all prime ideals exceeding it, and hence every maximal ideal is prime [12], i.e. $\operatorname{Max}(A) \subseteq \operatorname{Spec}(A)$. In what follows, we shall investigate some properties of the subspaces $\operatorname{Spec}_{\mathrm{m}}(A)$ and $\operatorname{Max}(A)$.

Proposition 10. *For any $DR\ell$-monoid A, both* $\operatorname{Spec}_{\mathrm{m}}(A)$ *and* $\operatorname{Max}(A)$ *are T_1-spaces.*

Proof. If P, Q belong to $\operatorname{Spec}_{\mathrm{m}}(A)$ or $\operatorname{Max}(A)$, then P and Q are incomparable and hence $P \in \mathcal{S}(Q)$ and $Q \in \mathcal{S}(P)$. □

Proposition 11. *Let A be a $DR\ell$-monoid satisfying the identities* $(*)$*. If P and Q are incomparable proper prime ideals of A, then there exist disjoint neighbourhoods of P and Q in* $\operatorname{Spec}(A)$*.*

Proof. Since $P \nsubseteq Q$ and $Q \nsubseteq P$, one can find $x \in P^+ \setminus Q$ and $y \in Q^+ \setminus P$, thus $P \in \mathcal{S}(y)$ and $Q \in \mathcal{S}(x)$. By replacing x by $(x \rightharpoonup y)^+$ and y by $(y \rightharpoonup x)^+$ we may assume that $x \perp y$, which yields $\mathcal{S}(x) \cap \mathcal{S}(y) = \emptyset$ by Lemma 4. □

Corollary 12. *If A satisfies* $(*)$ *then both* $\operatorname{Spec}_{\mathrm{m}}(A)$ *and* $\operatorname{Max}(A)$ *are T_2-spaces.*

Let $a \in A \setminus \{0\}$. By Zorn's lemma there exists an ideal M, called a *value* of a in A, which is maximal not containing a. The set of all values of a in A is denoted by $\operatorname{Val}_A(a)$. Observe that every $M \in \operatorname{Val}_A(a)$ is a prime ideal. Indeed, if $M = J \cap K$ for some $J, K \in \operatorname{Id}(A) \setminus \{M\}$, then certainly $a \in J$ and $a \in K$, whence $a \in J \cap K = M$, a contradiction. Therefore, $\operatorname{Val}_A(a) \subseteq \mathcal{S}(a)$ for all $a \in A \setminus \{0\}$. In addition, if A satisfies $(*)$ then for each $P \in \mathcal{S}(a)$, the set of all ideals exceeding P is totally ordered by set-inclusion [12], so there is a unique $M \in \operatorname{Val}_A(a)$ such that $P \subseteq M$.

Proposition 13. *If A satisfies $(*)$ then the mapping $\varphi\colon \mathcal{S}(a) \to \mathrm{Val}_A(a)$ which to each $P \in \mathcal{S}(a)$ assigns M, the unique value of a containing P, is continuous.*

Proof. Let $a \in A \setminus \{0\}$ and $P \in \mathcal{S}(a)$. Let $\mathcal{U}$ be a neighbourhood of $\varphi(P)$ in $\mathrm{Val}_A(a)$, a subspace of $\mathrm{Spec}(A)$. Since $\{\mathrm{Val}_A(a) \cap \mathcal{S}(x)\colon x \in A^+\}$ is a basis for $\mathrm{Val}_A(a)$, we may assume that $\mathcal{U} = \mathrm{Val}_A(a) \cap \mathcal{S}(x)$ for some $x \in A^+$.

Observe that if $Q \in \mathrm{Val}_A(a) \setminus \mathcal{S}(x)$, then Q and $\varphi(P)$ are incomparable, and hence by Proposition 11 there exists a neighbourhood $\mathcal{U}_Q$ of Q and a neigbourhood $\mathcal{V}_Q$ of $\varphi(P)$ (and hence of P) in $\mathcal{S}(a)$ such that $\mathcal{U}_Q \cap \mathcal{V}_Q = \emptyset$. The set of all such $\mathcal{U}_Q$ forms a covering of $\mathcal{S}(a) \setminus \mathcal{S}(x)$. Indeed, if $R \in \mathcal{S}(a) \setminus \mathcal{S}(x)$ then $\varphi(R) \in \mathrm{Val}_A(a) \setminus \mathcal{S}(x)$ and $R \in \mathcal{U}_{\varphi(R)}$ since $\mathcal{U}_{\varphi(R)} = \mathcal{S}(a) \cap \mathcal{S}(I)$ for some $I \in \mathrm{Id}(A)$, so $I \nsubseteq \varphi(R)$ entails $I \nsubseteq R$.

Furthermore, $\mathcal{S}(a)$ is a compact subspace of $\mathrm{Spec}(A)$, $\mathcal{S}(a) \setminus \mathcal{S}(x) = \mathcal{S}(a) \cap \mathcal{H}(x)$ is a closed subset in $\mathcal{S}(a)$, and therefore $\mathcal{S}(a) \setminus \mathcal{S}(x)$ is compact in $\mathcal{S}(a)$. Hence there are $Q_1, \ldots, Q_n \in \mathrm{Val}_A(a) \setminus \mathcal{S}(x)$ such that $\mathcal{S}(a) \setminus \mathcal{S}(x) \subseteq \mathcal{U}_{Q_1} \cup \ldots \cup \mathcal{U}_{Q_n}$. This yields $\mathcal{S}(a) \cap \mathcal{S}(x) = \mathcal{S}(a) \setminus (\mathcal{S}(a) \setminus \mathcal{S}(x)) \supseteq \mathcal{S}(a) \setminus (\mathcal{U}_{Q_1} \cup \ldots \cup \mathcal{U}_{Q_n}) = \mathcal{S}(a) \setminus \mathcal{U}_{Q_1} \cap \ldots \cap \mathcal{S}(a) \setminus \mathcal{U}_{Q_n} \supseteq \mathcal{V}_{Q_1} \cap \ldots \cap \mathcal{V}_{Q_n}$ (since $\mathcal{U}_{Q_i} \cap \mathcal{V}_{Q_i} = \emptyset$ implies $\mathcal{V}_{Q_i} \subseteq \mathcal{S}(a) \setminus \mathcal{U}_{Q_i}$). Let $\mathcal{C} = \mathcal{S}(a) \setminus (\mathcal{U}_{Q_1} \cup \ldots \cup \mathcal{U}_{Q_n})$ and $\mathcal{V} = \mathcal{V}_{Q_1} \cap \ldots \cap \mathcal{V}_{Q_n}$. Then $\mathcal{V} \cap \mathrm{Val}_A(a) \subseteq \mathcal{C} \cap \mathrm{Val}_A(a) \subseteq \mathcal{S}(a) \cap \mathcal{S}(x) \cap \mathrm{Val}_A(a) = \mathcal{U}$. Since $\mathcal{C}$ is closed in $\mathcal{S}(a)$, it follows that $\varphi(\mathcal{C}) \subseteq \mathcal{C} \cap \mathrm{Val}_A(a) \subseteq \mathcal{U}$. Indeed, $\mathcal{C}$ is closed iff $\mathcal{C} = \mathcal{S}(a) \cap \mathcal{H}(I)$ for some $I \in \mathrm{Id}(A)$; thus if $R \in \mathcal{C}$ then $I \subseteq R$ whence $I \subseteq \varphi(R)$, so $\varphi(R) \in \mathcal{C} \cap \mathrm{Val}_A(a)$. Therefore we obtain that $\varphi(\mathcal{V}) \subseteq \mathcal{U}$, where $\mathcal{V}$ is a neighbourhood of P in $\mathcal{S}(a)$. □

Corollary 14. *If A satisfies $(*)$ then $\mathrm{Val}_A(a)$ is a compact T_2-space for all $a \in A \setminus \{0\}$.*

Proof. This follows by a well-known fact that a continuous image of a compact space is still compact: by the previous proposition, $\mathrm{Val}_A(a)$ is a continuous image of a compact space $\mathcal{S}(a)$. □

It is obvious that if u is a strong order unit in A, then $\mathrm{Max}(A) = \mathrm{Val}_A(u) \neq \emptyset$.

Corollary 15. *If A satisfies $(*)$ and possesses a strong order unit u, then $\mathrm{Max}(A)$ is a compact T_2-space.*

We are now going to deal with the closure $\overline{\mathcal{X}}$ of $\mathcal{X} \subseteq \mathrm{Spec}(A)$, which will allow to find the condition for $\mathrm{Id}(A)$ to be a Stone lattice.

Propositin 16. *For any $\mathcal{X} \subseteq \mathrm{Spec}(A)$, $\overline{\mathcal{X}} = \mathcal{H}(\bigcap \mathcal{X})$.*

Proof. First, observe that closed sets in $\mathrm{Spec}(A)$ are just $\mathcal{H}(I)$ for $I \in \mathrm{Id}(A)$. It is easily seen that $\mathcal{X} \subseteq \mathcal{H}(\bigcap \mathcal{X})$. Let now $\mathcal{X} \subseteq \mathcal{H}(I)$ for some $I \in \mathrm{Id}(A)$. Then $I \subseteq P$ for all $P \in \mathcal{X}$, so $I \subseteq \bigcap \mathcal{X}$ which yields $\mathcal{H}(\bigcap \mathcal{X}) \subseteq \mathcal{H}(I)$. Hence $\mathcal{H}(\bigcap \mathcal{X})$ is the smallest closed set including $\mathcal{X}$. □

Corollary 17. *Let $\mathcal{X} \subseteq \mathrm{Spec}(A)$. Then $\mathcal{X}$ is dense if and only if $\bigcap \mathcal{X} = \{0\}$, and nowhere dense if and only if $(\bigcap \mathcal{X})^{\perp} = \{0\}$.*

P r o o f. Clearly, $\mathcal{X}$ is dense iff $\overline{\mathcal{X}} = \mathrm{Spec}(A)$ iff $\mathcal{H}(\bigcap \mathcal{X}) = \mathrm{Spec}(A)$ iff $\bigcap \mathcal{X} = \{0\}$. Further, $\mathcal{X}$ is nowhere dense iff $\mathrm{Spec}(A) \setminus \overline{\mathcal{X}}$ is dense, i.e. $\bigcap \mathrm{Spec}(A) \setminus \overline{\mathcal{X}} = \{0\}$. But $\overline{\mathcal{X}} = \mathcal{H}(\bigcap \mathcal{X})$, so that $P \notin \overline{\mathcal{X}}$ iff $P \in \mathcal{S}(\bigcap \mathcal{X})$ and this implies $\mathrm{Spec}(A) \setminus \overline{\mathcal{X}} = \mathcal{S}(\bigcap \mathcal{X})$, so $\bigcap \mathrm{Spec}(A) \setminus \overline{\mathcal{X}} = \bigcap \mathcal{S}(\bigcap \mathcal{X}) = (\bigcap \mathcal{X})^{\perp}$. □

Let us observe the case $\mathcal{X} = \mathcal{S}(I)$ for some ideal I in A. We have $\bigcap \mathcal{S}(I) = I^{\perp}$ by virtue of Lemma 3, and therefore we obtain the following:

Corollary 18. *Let X be an ideal of a $DR\ell$-monoid A. Then $\mathcal{S}(X)$ is dense if and only if $X^{\perp} = \{0\}$; $\mathcal{S}(X)$ is nowhere dense if and only if $X^{\perp\perp} = \{0\}$.*

Corollary 19. *For any $X \in \mathrm{Id}(A)$, $\mathcal{S}(X)$ is a clopen set if and only if $\mathcal{S}(X) = \mathcal{H}(X^{\perp})$.*

P r o o f. We have $\overline{\mathcal{S}(X)} = \mathcal{H}(X^{\perp})$. □

Proposition 20. *Let $X \in \mathrm{Id}(A)$. Then $\mathcal{S}(X)$ is a clopen set in $\mathrm{Spec}(A)$ if and only if $X \vee X^{\perp} = A$.*

P r o o f. Let $\mathcal{S}(X)$ be clopen, i.e. $\mathcal{S}(X) = \overline{\mathcal{S}(X)} = \mathcal{H}(X^{\perp})$. Assume that $X \vee X^{\perp} \neq A$. Then there is $a \in A \setminus (X \vee X^{\perp})$, and thus by [12], Theorem 2.2, one can find $P \in \mathrm{Spec}(A)$ such that $X \vee X^{\perp} \subseteq P$ and $a \notin P$. Hence $X \subseteq P$ and $X^{\perp} \subseteq P$, so $P \notin \mathcal{S}(X)$ yet $P \in \mathcal{H}(X^{\perp})$, a contradiction.

Suppose that $X \vee X^{\perp} = A$ and let $P \in \mathrm{Spec}(A)$. Since $X \cap X^{\perp} = \{0\}$ and P is prime, it follows that $X \subseteq P$ or $X^{\perp} \subseteq P$. Thus $X \nsubseteq P$ yields $X^{\perp} \subseteq P$. Conversely, if $X^{\perp} \subseteq P$ then $X \nsubseteq P$ because otherwise $A = X \vee X^{\perp} \subseteq P$ which is impossible. Summarizing, $X \nsubseteq P$ iff $X^{\perp} \subseteq P$ for all $P \in \mathrm{Spec}(A)$, so $\mathcal{S}(X) = \mathcal{H}(X^{\perp})$. □

Let $\mathrm{Pol}(A)$ be the set of all polars in A; again, it is partially ordered by set-inclusion. Since the polars are precisely the pseudo-complements in the ideal lattice $\mathrm{Id}(A)$, $\mathrm{Pol}(A)$ is a complete Boolean algebra which is a complete meet-subsemilattice of $\mathrm{Id}(A)$ [12]. Obviously, $\mathrm{Id}(A)$ is a Stone lattice, i.e., $\mathrm{Pol}(A)$ is a sublattice of $\mathrm{Id}(A)$ if and only if $X^{\perp} \vee X^{\perp\perp} = A$ for every ideal X. Hence by Proposition 20 we obtain

Theorem 21. *The ideal lattice $\mathrm{Id}(A)$ of any $DR\ell$-monoid A is a Stone lattice if and only if $\mathcal{S}(X^{\perp})$ is clopen for all $X \in \mathrm{Id}(A)$.*

Since any ideal X of a $DR\ell$-monoid A is a $DR\ell$-monoid again (as a matter of fact, X is a convex subalgebra of A [11]), we can focus the prime spectrum of X. Our next objective is to describe $\mathrm{Spec}(A/X)$, the spectrum of the quotient $DR\ell$-monoid over a normal ideal X. The following results generalize [17], Theorem 5 and Theorem 12, characterizing the spectra of quotient GMV-algebras and the spectra of principal ideals generated by idempotent elements.

Lemma 22 [12]. *If $X \in \mathrm{Id}(A)$ then the mappings*

$$\varphi\colon\ P \mapsto P \cap X \quad \text{and} \quad \psi\colon\ Q \mapsto X * Q,$$

where $X*Q = \{a \in A\colon\ |a| \wedge |x| \in Q$ for all $x \in X\}$ is the relative pseudo-complement of X with respect to Q in $\mathrm{Id}(A)$, are mutually inverse order-isomorphisms between the set $\mathcal{S}(X)$ consisting of all prime ideals in A not including X and the set $\mathrm{Spec}(X)$ of all proper prime ideals in X.

As a corollary we have

Theorem 23. *If X is an ideal in a $DR\ell$-monoid A then the spectrum $\mathrm{Spec}(X)$ is homeomorphic with the subspace $\mathcal{S}(X)$ of $\mathrm{Spec}(A)$.*

Proof. To prove the continuity of $\varphi\colon\ \mathcal{S}(X) \to \mathrm{Spec}(X)$ we make use of the simple observation that $J \subseteq X \cap P$ iff $X * J \subseteq P$ for any $J \in \mathrm{Id}(X)$. Indeed, $J \subseteq X \cap P$ implies $X * J \subseteq X * (X \cap P) = \psi(\varphi(P)) = P$, and conversely, $X * J \subseteq P$ entails $J = X \cap (X * J) \subseteq X \cap P$. Note that $X \cap (X * J) = J$ since $J \subseteq X$.

Let $\mathcal{U}$ be an open set in $\mathrm{Spec}(X)$, that is, $\mathcal{U} = \{Q \in \mathrm{Spec}(X)\colon\ J \nsubseteq Q\}$ for some $J \in \mathrm{Id}(X)$. Then $P \in \mathcal{S}(X)$ belongs to $\varphi^{-1}(\mathcal{U})$ iff $\varphi(P) = X \cap P \in \mathcal{U}$ iff $J \nsubseteq X \cap P$ iff $X * J \nsubseteq P$, whence $\varphi^{-1}(\mathcal{U}) = \{P \in \mathcal{S}(X)\colon\ X * J \nsubseteq P\}$. This shows that $\varphi\colon\ \mathcal{S}(X) \to \mathrm{Spec}(X)$ is continuous.

Let now $\mathcal{V}$ be an open set in $\mathcal{S}(X)$, thus $\mathcal{V} = \{P \in \mathcal{S}(X)\colon\ I \nsubseteq P\}$ for some $I \in \mathrm{Id}(A)$. Then for $Q \in \mathrm{Spec}(X)$ we have $Q \in \psi^{-1}(\mathcal{V})$ iff $\psi(Q) = X * Q \in \mathcal{V}$ iff $I \nsubseteq X * Q$ iff $I \cap X \nsubseteq Q$. Hence $\psi^{-1}(\mathcal{V}) = \{Q \in \mathrm{Spec}(X)\colon\ I \cap X \nsubseteq Q\}$, which is an open set in $\mathrm{Spec}(X)$ and so $\psi\colon\ \mathrm{Spec}(X) \to \mathcal{S}(X)$ is also a continuous mapping. $\square$

Lemma 24. *If X is a normal ideal in A then $\mathrm{Id}(A/X) \cong [X) \subseteq \mathrm{Id}(A)$.*

Proof. One readily verifies that if $J \in [X)$ then $J/X = \{a/X\colon\ a \in J\}$ is an ideal in A/X and each ideal K in A/X is obtained in this form. Indeed, $\widetilde{K} = \{a \in A\colon\ a/X \in K\}$ is an ideal in A with $X \subseteq \widetilde{K}$ and obviously $K = \widetilde{K}/X$. Therefore, it can be easily proved that $\varphi\colon\ J \mapsto J/X$ and $\psi\colon\ K \mapsto \widetilde{K}$ are mutually inverse bijections between $[X) = \{I \in \mathrm{Id}(A)\colon\ X \subseteq I\}$ and $\mathrm{Id}(A/X)$ that preserve set-inclusion. $\square$

Theorem 25. *For any normal ideal X in A,* $\mathrm{Spec}(A/X)$ *and* $\mathcal{H}(X)$ *are homeomorphic.*

P r o o f. Let $\mathcal{U}$ be an open subset in $\mathrm{Spec}(A/X)$ and let $\mathcal{V}$ be an open subset in $\mathcal{H}(X)$, i.e., $\mathcal{U} = \{Q \in \mathrm{Spec}(A/X)\colon K \nsubseteq Q\}$ for some $K \in \mathrm{Id}(A/X)$ and $\mathcal{V} = \{P \in \mathcal{H}(X)\colon J \nsubseteq P\}$ for some $J \in \mathrm{Id}(A)$. In view of the previous lemma we immediately obtain $\varphi^{-1}(\mathcal{U}) = \psi(\mathcal{U}) = \{P \in \mathcal{H}(X)\colon \widetilde{K} \nsubseteq P\}$ and $\psi^{-1}(\mathcal{V}) = \varphi(\mathcal{V}) = \{Q \in \mathrm{Spec}(A/X)\colon J/X \nsubseteq Q\}$, which are open sets in the respective topological spaces $\mathcal{H}(X)$ and $\mathrm{Spec}(A/X)$. $\square$

T. Kovář [10] proved that every $DR\ell$-monoid is isomorphic to the direct product of an ℓ-group and a lower-bounded $DR\ell$-monoid. Therefore we now turn to the spectrum of two factor direct products of $DR\ell$-monoids.

We say that a $DR\ell$-monoid A is the *direct product (cardinal sum) of ideals* $X_1, X_2 \in \mathrm{Id}(A)$ if the mapping $\langle x_1, x_2\rangle \mapsto x_1 + x_2$ is an isomorphism of $X_1 \times X_2$, the direct product of $DR\ell$-monoids X_1 and X_2, onto A. In other words, $A = X_1 \times X_2$, where the sets X_1 and X_2 are identified with the sets $\{\langle x_1, 0\rangle\colon x_1 \in X_1\}$ and $\{\langle 0, x_2\rangle\colon x_2 \in X_2\}$, respectively. It is easily seen that in this case $X_1 \cap X_2 = \{0\}$ and $X_1 \vee X_2 = A$, and moreover, both X_1 and X_2 are normal ideals of A.

Lemma 26. *Let A be the direct product of its ideals X_1 and X_2. Then every $I \in \mathrm{Id}(A)$ can be uniquely expressed in the form $I = I_1 \vee I_2$ for some $I_i \in \mathrm{Id}(X_i)$, $i = 1, 2$.*

P r o o f. Obviously, $I_i = I \cap X_i \in \mathrm{Id}(X_i)$ and $I = I_1 \vee I_2$ since the ideal lattice $\mathrm{Id}(A)$ is distributive. To prove the uniqueness, suppose that $I = J_1 \vee J_2$ for some $J_i \in \mathrm{Id}(X_i)$, $i = 1, 2$. Then $I_1 = (J_1 \vee J_2) \cap X_1 = (J_1 \cap X_1) \vee (J_2 \cap X_1) = J_1 \vee \{0\} = J_1$ and analogously $I_2 = J_2$. $\square$

Lemma 27. *Let A be the direct product of ideals X_1 and X_2, and let P be an ideal of A. Then $P \in \mathrm{Spec}(A)$ if and only if either $P = P_1 \vee X_2$ for some $P_1 \in \mathrm{Spec}(X_1)$, or $P = X_1 \vee P_2$ for some $P_2 \in \mathrm{Spec}(X_2)$.*

P r o o f. Assume that $P = P_1 \vee X_2$ for $P_1 \in \mathrm{Spec}(X_1)$. Let $P = J \cap K$ for $J, K \in \mathrm{Id}(A) \setminus \{P\}$. By the previous lemma we have $J = J_1 \vee J_2$ and $K = K_1 \vee K_2$, where J_i, K_i are ideals in X_i. Then $P = (J_1 \vee J_2) \cap (K_1 \vee K_2) = (J_1 \cap K_1) \vee (J_2 \cap K_2)$ as $\mathrm{Id}(A)$ is distributive. However, the expression $P = P_1 \vee X_2$ is unique, and thus $P_1 = J_1 \cap K_1$ and $X_2 = J_2 = K_2$. Moreover, $P_1 = J_1$ or $P_1 = K_1$ since P_1 is a prime ideal in X_1. Therefore $P = J_1 \vee J_2 = J$ or $P = K_1 \vee K_2 = K$, proving $P \in \mathrm{Spec}(A)$.

Conversely, let $P = P_1 \vee P_2$ for $P_i \in \mathrm{Id}(X_i)$ be a prime ideal in A. Then we have $(P_1 \vee X_2) \cap (X_1 \vee P_2) = (P_1 \cap X_1) \vee (X_2 \cap P_2) = P_1 \vee P_2 = P$. Since P is prime, it follows that $P = P_1 \vee X_2$ or $P = X_1 \vee P_2$, say $P = P_1 \vee X_2$. To see that $P_1 \in \mathrm{Spec}(X_1)$, suppose that $P_1 = J_1 \cap K_1$ for some $J_1, K_1 \in \mathrm{Id}(X_1)$. Then $P = (J_1 \cap K_1) \vee X_2 = (J_1 \vee X_2) \cap (K_1 \vee X_2)$. But $P \in \mathrm{Spec}(A)$, so $P = J_1 \vee X_2$ or $P = K_1 \vee X_2$, and hence $P_1 = J_1$ or $P_1 = K_1$ as P is assumed to be in the form $P = P_1 \vee X_2$. □

Corollary 28. *If A is the direct product of ideals $X_1, X_2 \in \mathrm{Id}(A)$, then* $\mathrm{Spec}(A)$ *is homeomorphic to* $\mathrm{Spec}(X_1) \oplus \mathrm{Spec}(X_2)$, *the sum of topological spaces* $\mathrm{Spec}(X_1)$ *and* $\mathrm{Spec}(X_2)$.

P r o o f. In view of Lemma 27 it is obvious that $\mathrm{Spec}(A) = \mathcal{S}(X_1) \cup \mathcal{S}(X_2)$ and $\mathcal{S}(X_1) \cap \mathcal{S}(X_2) = \emptyset$, so $\mathrm{Spec}(A)$ is homeomorphic to $\mathcal{S}(X_1) \oplus \mathcal{S}(X_2)$, and consequently to $\mathrm{Spec}(X_1) \oplus \mathrm{Spec}(X_2)$ by Theorem 23. □

A topological space S is said to be a *Stone space* if (i) S is a T_0-space, (ii) the family of all compact open subsets of S is a ring of sets and a basis for S, and (iii) if $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ are non-empty families of non-empty compact open sets and if

$$\bigcap_{i\in I} U_i \subseteq \bigcup_{j\in J} V_j$$

then

$$U_{i_1} \cap \ldots \cap U_{i_m} \subseteq V_{j_1} \cup \ldots \cup V_{j_n}$$

for some $i_1, \ldots, i_m \in I$ and $j_1, \ldots, j_n \in J$.

It is a well-known fact that Stone spaces arise from distributive lattices [1]. Let L be a distributive lattice. For any $x \in L$, let $\hat{x}$ be the set of all prime ideals in L that do not contain x. Then $\{\emptyset\} \cup \{\hat{x}\colon\ x \in L\}$ is a basis for a certain topology on $\Omega(L)$, the set of all proper prime ideals in L; $\Omega(L)$ endowed with this topology is called the *Stone space of* L. Any Stone space X is (up to homeomorphism) of the form $\Omega(L)$, where L is the lattice of all compact open subsets in X.

Theorem 29. *For any $DR\ell$-monoid A, the spectrum* $\mathrm{Spec}(A)$ *is a Stone space.*

P r o o f. We already know that $\mathrm{Spec}(A)$ is a T_0-space whose compact open sets are precisely $\mathcal{S}(x)$ for $x \in A^+$, and the family of all such $\mathcal{S}(x)$ can serve as a basis for $\mathrm{Spec}(A)$. Therefore, we have only to verify the condition (iii) from the definition of a Stone space.

Let $\{\mathcal{S}(x_i)\}_{i\in I}$ and $\{\mathcal{S}(y_j)\}_{j\in J}$ be non-empty families of non-empty compact open sets such that $\bigcap\limits_{i\in I} \mathcal{S}(x_i) \subseteq \bigcup\limits_{j\in J} \mathcal{S}(y_j)$. Without loss of generality we may assume that all x_i and y_j are positive. Let us set $X = \{x_i\}_{i\in I}$ and $Y = \{y_j\}_{j\in J}$.

If $[X) \cap I(Y) = \emptyset$ then $I(Y) \subseteq P$ and $[X) \cap P = \emptyset$ for some $P \in \mathrm{Spec}(A)$ by Lemma 1. However, then $P \in \bigcap_{i \in I} \mathcal{S}(x_i)$ and $P \notin \bigcup_{j \in J} \mathcal{S}(y_j)$, which is impossible, and so there exists $0 \leqslant a \in [X) \cap I(Y) \neq \emptyset$. Hence $x_{i_1} \wedge \ldots \wedge x_{i_m} \leqslant a \leqslant y_{j_1} + \ldots + y_{j_n}$ for some $i_1, \ldots, i_m \in I$ and $j_1, \ldots, j_n \in J$. This yields $I(x_{i_1}) \cap \ldots \cap I(x_{i_m}) \subseteq I(y_{j_1}) \vee \ldots \vee I(y_{j_n})$, whence $\mathcal{S}(x_{i_1}) \cap \ldots \cap \mathcal{S}(x_{i_m}) \subseteq \mathcal{S}(y_{j_1}) \cup \ldots \cup \mathcal{S}(y_{j_n})$, since the ideal lattice $\mathrm{Id}(A)$ and the lattice of all open sets in $\mathrm{Spec}(A)$ are isomorphic. □

Let us denote by $\mathrm{Com}(\mathrm{Id}(A))$ the sublattice of all principal ideals in a $DR\ell$-monoid A. Since $\mathrm{Id}(A)$ is isomorphic with the lattice of open sets in $\mathrm{Spec}(A)$, it follows that $\mathrm{Com}(\mathrm{Id}(A))$ is isomorphic with the sublattice of all compact open sets in $\mathrm{Spec}(A)$.

Corollary 30. *Given a $DR\ell$-monoid A, the spectrum $\mathrm{Spec}(A)$ is homeomorphic with $\Omega(\mathrm{Com}(\mathrm{Id}(A)))$, the Stone space of the lattice $\mathrm{Com}(\mathrm{Id}(A))$.*

Let us recall that for a distributive lattice L, $\Omega(L)$ is (i) compact if and only if L has the greatest element, (ii) a T_1-space if and only if L is relatively complemented, and (iii) a T_2-space if and only if L is a generalized Boolean algebra.

Corollary 31. *Let A be a $DR\ell$-monoid. The following statements are equivalent:*

(1) $\mathrm{Spec}(A)$ *is a T_1-space;*

(2) $\mathrm{Spec}(A)$ *is a T_2-space;*

(3) $\mathrm{Com}(\mathrm{Id}(A))$ *is a generalized Boolean algebra.*

Furthermore, $\mathrm{Spec}(A)$ *is a Boolean space, that is, a compact T_2-space in which the clopen sets form a basis if and only if* $\mathrm{Com}(\mathrm{Id}(A))$ *is a Boolean algebra.*

R e m a r k . A lower-bounded distributive lattice L is said to be *relatively normal* if its set of all prime ideals is a root system. We shall denote by $\mathcal{IRN}$ the class consisting of the ideal lattices of relatively normal lattices. Actually, a lattice L is a member of $\mathcal{IRN}$ if and only if it is an algebraic distributive lattice whose meet-irreducible elements form a root system and $\mathrm{Com}(L)$ is a sublattice of L.

By [18], Lemma 2.1, a lower-bounded distributive lattice L is relatively normal if and only if any pair of incomparable prime ideals in L has disjoint open neighbourhoods in $\Omega(L)$, the Stone space of L. Further, by [18], Corollary 2.2, an algebraic, distributive lattice L belongs to $\mathcal{IRN}$ if and only if $\mathrm{Com}(L)$ is a relatively normal lattice. Hence the ideal lattice $\mathrm{Id}(A)$ of a $DR\ell$-monoid A is a member of $\mathcal{IRN}$ if and only if $\mathrm{Com}(\mathrm{Id}(A))$ is relatively normal. Since $\Omega(\mathrm{Com}(\mathrm{Id}(A)))$ and $\mathrm{Spec}(A)$ are homeomorphic, we obtain

Corollary 32. *For any $DR\ell$-monoid A, $\mathrm{Id}(A) \in \mathcal{IRN}$ if and only if any pair of incomparable prime ideals in A has disjoint neighbourhoods in $\mathrm{Spec}(A)$. Therefore, if A satisfies $(*)$ then $\mathrm{Id}(A) \in \mathcal{IRN}$.*

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