SIGNED 2-DOMINATION IN CATERPILLARS

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(Received December 19, 2003)

Abstract. A caterpillar is a tree with the property that after deleting all its vertices of degree 1 a simple path is obtained. The signed 2-domination number $\gamma_{\rm s}^2(G)$ and the signed total 2-domination number $\gamma_{\rm st}^2(G)$ of a graph G are variants of the signed domination number $\gamma_{\rm s}(G)$ and the signed total domination number $\gamma_{\rm st}(G)$. Their values for caterpillars are studied.

Keywords: caterpillar, signed 2-domination number, signed total 2-domination number

MSC 2000: 05C69, 05C05

This paper concerns caterpillars. A caterpillar [1] is a tree with the property that after deleting all its vertices of degree 1 a simple path is obtained. According to this definition a caterpillar has at least three vertices. But we need not care about graphs with one or two vertices. For such graphs our considerations are trivial.

Let G be a caterpillar. The mentioned simple path will be denoted by B and called the body of the caterpillar G. Let the number of vertices of B be m. Let $a_1,\ldots,a_m$ be these vertices and let a_ia_{i+1} for $i=1,\ldots,m-1$ be the edges of B. By $[m]$ we shall denote the set of integers i such that $1\leqslant i\leqslant m$. For each $i\in[m]$ let s_i be the degree of a_i in G. The vector $\vec{s}=(s_1,\ldots,s_m)$ will be called the degree vector of the caterpillar G.

Now we shall define variants of the signed domination number and of the signed total domination number [2] of a graph. For a vertex $u\in V(G)$ the symbol $N(u)$ denotes the open neighbourhood of u in G, i.e. the set of all vertices which are adjacent to u in G. The closed neighbourhood of u is $N[u]=N(u)\cup\{u\}$. Similarly the open 2-neighbourhood $N^2(u)$ is the set of all vertices having the distance 2 from u in G. The closed 2-neighbourhood of u is $N^2[u]=N[u]\cup N^2(u)$. If f is a mapping of $V(G)$ into some set of numbers and $S \subseteq V(G)$, then $f(S) = \sum\limits_{x \in s} f(x)$ and the weight of f is $w(f) = f(V(G)) = \sum\limits_{x \in V(G)} f(x)$.

Let $f\colon V(G) \to \{-1, 1\}$. If $f(N^2[u]) \geqslant 1$ (or $f(N^2(u)) \geqslant 1$) for each $u \in V(G)$, then f is called a signed 2-dominating (or signed total 2-dominating, respectively) function on G. The minimum of weights $w(f)$ taken over all signed 2-dominating (or all signed total 2-dominating) functions f is the signed 2-dominating number $\gamma^2_{\mathrm{s}}(G)$ (or the signed total 2-domination number $\gamma^2_{\mathrm{st}}(G)$, respectively) of G.

For each $i \in [m]$ let $t_i \in \{1, 2\}$ and $t_i \equiv s_i + 1 \pmod 2$.

We shall prove a theorem concerning $\gamma^2_{\mathrm{s}}(G)$.

Theorem 1. *Let G be a caterpillar with the degree vector $\vec{s} = (s_1, \dots, s_m)$ such that $n \geqslant 2$ and $s_i \geqslant 3$ for all $i \in [m]$. Then*

$$\gamma^2_{\mathrm{s}}(G) = \sum_{i=1}^{m} t_i - 2m + 2.$$

Proof. Consider a vertex a_i with $i \in [m]$. As $s_i \geqslant 3$, there exists at least one vertex $u \in N(a_i)$ which does not belong to B and has degree 1. Then $N^2[u] = N[a_i]$. Let f be a signed 2-dominating function on G. Then $f(N^2[u]) = f(N[a_i]) \geqslant 1$. The set $N[a_i]$ has s_i+1 vertices. If s_i is even, then s_i+1 is odd. At least $\frac{1}{2}(s_i+2) = \frac{1}{2}s_i+1$ vertices of $N[a_i]$ must have the value 1 in f and at most $\frac{1}{2}s_i$ of them may have the value -1. Then $f(N^2[u]) \geqslant (\frac{1}{2}s_i + 1) - \frac{1}{2}s_i = 1 = t_i$. If s_i is odd, then s_{i+1} is even and at least $\frac{1}{2}(s_i + 1) + 1$ vertices of $N[a_i]$ must have the value 1 in f and at most $\frac{1}{2}(s_i + 1) - 1$ of them may have the value -1. Then $f(N^2[u]) \geqslant 2 = t_i$. We may easily construct the function f such that it has the value -1 in exactly $\frac{1}{2}s_i$ vertices of degree 1 in $N[a_i]$ with i even and in exactly $\frac{1}{2}(s_i + 1) - 1 = \frac{1}{2}(s_i - 1)$ vertices of degree 1 in $N[a_i]$ with i odd. In all other vertices (including all vertices of the body) the function f has the value 1.

We have $\bigcup\limits_{i=1}^{m} N[a_i] = V(G)$. The vertex a_1 is contained in exactly two sets $N[a_i]$, namely in $N[a_1]$ and $N[a_2]$. Similarly a_m is contained in exactly two sets $N[a_{m-1}]$, $N[a_m]$. For $i \in [m] - \{1, m\}$ the vertex a_i is contained in exactly three sets $N[a_{i-1}]$, $N[a_i]$, $N[a_{i+1}]$. Each vertex outside the body is contained in exactly one of these sets. By the Inclusion-Exclusion Principle we have

$$\begin{aligned} w(f) = f(V(G)) &= \sum_{i=1}^{m} f(N[a_i]) - 2\sum_{i=2}^{m-1} f(a_i) - f(a_1) - f(a_m) \\ &= \sum_{i=1}^{m} t_i - 2(m-2) - 1 - 1 = \sum_{i=1}^{m} t_i - 2m + 2. \end{aligned}$$

As f is the minimum function satisfying the requirements, we have

$$\gamma_{\mathrm{s}}^2(G) = w(f) = \sum_{i=1}^{m} t_i - 2m + 2.$$

□

An analogous theorem concerns $\gamma_{\mathrm{st}}^2(G)$.

Theorem 2. *Let G be a caterpillar with the degree vector $\vec{s} = (s_1, \ldots, s_m)$ such that $m \geqslant 2$ and $s_i \geqslant 4$ for all $i \in [m]$. Then*

$$\gamma_{\mathrm{st}}^2(G) = \sum_{i=1}^{m} t_i + 2.$$

Proof. Consider a vertex a_i with $i \in [m]$. As $s_i \geqslant 5$, there exists at least one vertex $u \in N(a_i)$ which does not belong to B and has degree 1. Then $N^2(u) = N(a_i) - \{u\}$. Let f be a signed total 2-dominating function on G. Then $f(N^2(u)) = f(N(a_i) - \{u\}) \geqslant 1$. The set $N(a_i) - \{u\}$ has $s_i - 1$ vertices. If s_i is even, then $s_i - 1$ is odd. At least $\frac{1}{2}s_i$ vertices of $N(a_i) - \{u\}$ must have the value 1 in f and at most $\frac{1}{2}(s_i - 2) = \frac{1}{2}s_i - 1$ of them may have the value -1. Then $f(N^2(u)) \geqslant \frac{1}{2}s_i - (\frac{1}{2}s_i - 1) = 1 = t_i$. If s_i is odd, then $s_i - 1$ is even and at least $\frac{1}{2}(s_i - 1) + 1$ vertices of $N(a_i) - \{u\}$ must have the value 1 in f and at most $\frac{1}{2}(s_i - 1) - 1$ of them may have the value -1. Then $f(N^2(u)) \geqslant 2 = t_i$. As $s_i \geqslant 5$ for $i \in [m]$, in both these cases we must admit the possibility $f(u) = 1$. Then in the case of s_i even we have $f(N(a_i)) \geqslant 2 = t_i + 1$ and in the case of s_i odd we have $f(N(a_i)) \geqslant 3 = t_i + 1$. We may easily construct the function f such that it has the value -1 in $\frac{1}{2}s_i - 1$ vertices of degree 1 in $N(a_i)$ for s_i even, in $\frac{1}{2}(s_i - 1) - 1 = \frac{1}{2}(s_i - 3)$ vertices of degree 1 in $S(a_i)$ for s_i odd and the value 1 for all other vertices (including all vertices of B). Each vertex a_j for $j \in [m] - \{1, m\}$ is contained in two sets $N(a_i)$, namely in $N(a_{j-1})$ and $N(a_{j+1})$. Each other vertex is contained in exactly one set $N(a_i)$. Again by the Inclusion-Exclusion Principle we have

$$\begin{aligned} w(f) = f(V(G)) &= \sum_{i=1}^{m} f(N(a_i)) - \sum_{i=2}^{m-1} f(a_i) \\ &= \sum_{i=1}^{m} (t_i + 1) - (m - 2) = \sum_{i=1}^{m} t_i + m - (m - 2) = \sum_{i=1}^{m} t_i + 2. \end{aligned}$$

As f is the minimum function satisfying the requirements, we have

$$\gamma_{\mathrm{st}}^2(G) = w(f) = \sum_{i=1}^{m} t_i + 2.$$

□

In Figs. 1 and 2 a caterpillar G with the degree vector $(5,6,7)$ is depicted. We have $t_1 = t_3 = 2$, $t_2 = 1$ and therefore $\gamma^2_{\rm st}(G) = 7$ and $\gamma^2_{\rm s}(G) = 1$. In Fig. 1 the values of the corresponding signed total 2-dominating function are illustrated; in the vertices denoted by $+$ the value is 1 and in the vertices denoted by $-$ it is -1. Similarly in Fig. 2 the corresponding signed 2-dominating function is illustrated.

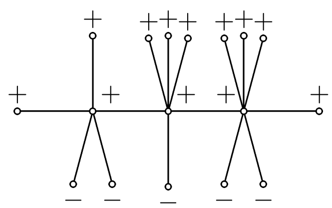

Fig. 1.

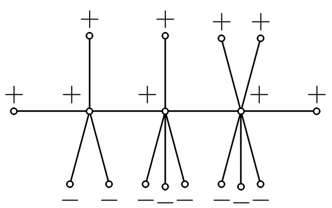

Fig. 2.

In Theorems 1 and 2 we had the assumption $m \geqslant 2$. The following proposition concerns the singular case $m = 1$.

Proposition 1. *Let G be a caterpillar with the body consisting of one vertex, i.e. a star with the central vertex a_1 and with $s_1 \geqslant 2$ vertices of degree 1. Then $\gamma^2_{\rm st}(G)$ is undefined and $\gamma^2_{\rm s}(G) = t_1$.*

Proof. The open 2-neighbourhood $N^2(a_1) = \emptyset$ and thus $f(N^2(a_1)) = 0$ for any function $f\colon V(G) \to \{-1,1\}$, hence none of such functions might be signed total 2-dominating in G. On the other hand, $N^2[a_1] = V(G)$ and $|V(G)| = s_1 + 1$. Analogously as in the proofs of Theorems 1 and 2 we prove that for s_1 even we have $\gamma^2_{\rm s}(G) = 1 = t_1$ and for s_1 odd we have $\gamma^2_{\rm s}(G) = 2 = t_1$. $\square$

Proposition 2. *Let G be a caterpillar with $m \equiv 2 \pmod 5$, $m \geqslant 5$, $s_i = 3$ for all $i \in [m]$. Then $\gamma^2_{\rm st}(G) \leqslant \frac{4}{3}(m+3) + 2$, while $\sum\limits_{i=1}^{m} t_i + 2 = 2(m+1)$.*

Proof. As $s_i = 3$ for each $i \in [m]$, we have $t_i = 2$ for each $i \in [m]$. Each vertex a_i for $i \in [m] - \{1,m\}$ is adjacent to exactly one vertex v_i of degree 1. The vertex a_1 is adjacent to two such vertices v_1, w_1 and similarly a_m to v_m, w_m. Let $f\colon V(G) \to \{-1,1\}$ be defined so that $f(v_i) = -1$ for $i \equiv 0 \pmod 3$ and $f(u) = 1$ for all other vertices u. This is a signed total 2-dominating function on G (this can be easily verified by the reader) and $w(f) = \frac{1}{3}(4m+10)$. Therefore $\gamma^2_{\rm st}(G) \leqslant \frac{1}{3}(4m+10)$, while $\sum\limits_{i=1}^{m} t_i + 2 = 2(m+1)$. For $m \geqslant 3$ we have $\frac{1}{3}(4m+10) < 2(m+1)$.

In Fig. 3 we see such a caterpillar for $m = 8$ with the corresponding function f. In this case $\gamma^2_{\rm st}(G) = 14$, $\sum\limits_{i=1}^{m} t_i + 2 = 18$. For the signed 2-domination number

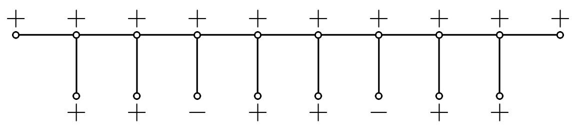

Fig. 3.

here Theorem 1 holds. In Fig. 4 the same caterpillar is depicted with the function f realizing the signed domination number $\gamma_{\mathrm{s}}^2(G) = \sum\limits_{i=1}^{m} t_i - 2m + 2 = 2$. □

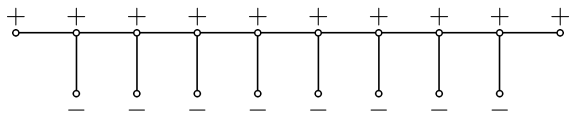

Fig. 4.

Proposition 3. *Let G be a caterpillar with $m \geqslant 2$ and $s_i = 2$ for each $i \in [m]$. Then $\sum\limits_{i=1}^{m} t_i - 2m + 2 < \gamma_{\mathrm{s}}^2(G)$, but $\sum\limits_{i=1}^{m} t_i + 2 = \gamma_{\mathrm{st}}^2(G)$.*

P r o o f. The caterpillar thus described is a simple path of length $m+1$. It has $m+2$ vertices. The inequality $\gamma_{\mathrm{s}}^2(G) \leqslant \sum\limits_{i=1}^{m} t_i - 2m + 2$ would imply that there exists a signed 2-dominating function f which has the value -1 in m vertices, while the value 1 only in two vertices. This is evidently impossible. On the other hand the open 2-neighbourhood of any vertex consists of at most two vertices and therefore the unique signed total 2-dominating function is the constant function equal to 1 in the whole set $V(G)$. Then

$$\gamma_{\mathrm{st}}^2(G) = w(f) = \sum_{i=1}^{m} t_i + 2 = m + 2.$$

□

Now we shall study the signed 2-domination number of a simple path P_n with n vertices (i.e. of length $n-1$). We shall not use the notation for caterpillars used above, but we shall denoted the vertices by $u_1, \ldots, u_n$ and edges by $u_i u_{i+1}$ for $i = 1, \ldots, n-1$.

Theorem 3. *Let P_n be a path with n vertices. If $n \equiv 0 \pmod 5$, then $\gamma_{\mathrm{s}}^2(P_n) = \frac{1}{5}n$. In general, asymptotically $\gamma_{\mathrm{s}}^2(P_n) \approx \lfloor \frac{1}{5}n \rfloor$.*

P r o o f. If $n \equiv 0 \pmod 5$, then the closed neighbourhood $N^2[u_i] = \{u_{i-2}, u_{i-1}, u_i, u_{i+1}, u_{i+2}\}$ for $i \equiv 3 \pmod 5$, $3 \leqslant i \leqslant n-2$, form a partition of $V(P_n)$. Let f

be a signed 2-dominating function on P_n. Then f must have the value 1 in at least three vertices and may have the value -1 in at most two vertices of each class of this partition. Then $w(f) \geqslant \frac{3}{5}n = \frac{1}{5}n$. A function f for which the equality occurs may be defined so that $f(u_i) = -1$ for $i \equiv 0 \pmod 5$ and $i \equiv 1 \pmod 5$ and $f(u_1) = 1$ for $i \equiv 2 \pmod 5$, $i \equiv 3 \pmod 5$ and $i \equiv 4 \pmod 5$. Therefore $\gamma_{\rm s}^2(P_n) = w(f) = \frac{1}{5}n$.

Now let $m \equiv r \pmod 5$, $r \leqslant 4$. Let $q = n - r$. We have $q \equiv 0 \pmod 5$ and thus $\gamma_{\rm s}^2(P_q) = \frac{1}{5}q$. The path P_n is obtained from P_q by adding a path with r vertices. Let g be a minimum signed 2-dominating function on P_n, let g_0 be its restriction to P_q. We have $w(g_0) = \frac{1}{5}q$. Now the vertices of P_n not in P_q may have values 1 or -1 in g and thus $\frac{1}{5}q - r \leqslant w(g) \leqslant \frac{1}{5}q + r$. In general, $\frac{1}{5}q - 4 \leqslant \gamma_{\rm s}^2(P_n) \leqslant \frac{1}{5}q + 4$. This implies

$$\frac{9}{5n} - \frac{4}{n} \leqslant \frac{\gamma_{\rm s}^2(P_n)}{n} \leqslant \frac{9}{5n} + \frac{4}{n}.$$

Therefore $\lim\limits_{n\to\infty} \frac{\gamma_{\rm s}^2(P_n)}{n} = \frac{9}{5m}$ and thus $\gamma_{\rm s}^2(P_n) \approx \frac{9}{5} = \lfloor \frac{n}{5} \rfloor$. $\square$

In Fig. 5 we see a path P_{15} (with $\gamma_{\rm s}^2(P_{15}) = 3$) in which the corresponding signed 2-dominating function is illustrated.

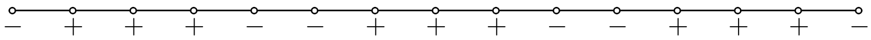

Fig. 5.

As has already been mentioned, $\gamma_{\rm st}^2(P_n) = n$ for each positive integer n.

Without a proof we shall state the values of $\gamma_{\rm s}^2(P_n)$ for $n \leqslant 4$. We have $\gamma_{\rm s}^2(P_1) = 1$, $\gamma_{\rm s}^2(P_2) = 2$, $\gamma_{\rm s}^2(P_3) = 1$, $\gamma_{\rm s}^2(P_4) = 2$.

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